THE UNREASONABLE SLIGHTNESS OF E_2 OVER IMAGINARY QUADRATIC RINGS

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ABSTRACT. It is almost always the case that the elementary matrices generate the special linear group SL_n over a ring of integers in a number field. The only exceptions to this rule occur for SL_2 over rings of integers in imaginary quadratic fields. The surprise is compounded by the fact that, in these cases when elementary generation fails, it actually fails rather badly: the group E_2 generated by the elementary 2-by-2 matrices turns out to be an infinite-index, non-normal subgroup of SL_2.

We give an elementary proof of this strong failure of elementary generation for SL_2 over imaginary quadratic rings.

1. INTRODUCTION

The group $\mathrm{SL}_n(\mathbb{Z})$ is generated by the elementary matrices; recall, these are the matrices which have 1 along the diagonal and at most one nonzero off-diagonal entry. The proof rests on the fact that $\mathbb{Z}$ is a Euclidean domain: row- and column-operations dictated by division with remainder will reduce any matrix in $\mathrm{SL}_n(\mathbb{Z})$ to the identity matrix.

What happens if we replace $\mathbb{Z}$ by another ring of integers? More precisely, if $\mathcal{O}_\mathbb{K}$ denotes the ring of integers in a number field $\mathbb{K}$, then is it still the case that $\mathrm{SL}_n(\mathcal{O}_\mathbb{K})$ is generated by the elementary matrices?

We owe to Cohn [3] the first result in this direction: when $n = 2$, the answer is negative for all but five imaginary quadratic fields.

Theorem 1.1 (Cohn). *Let $\mathcal{O}_D$ denote the ring of integers in $\mathbb{Q}(\sqrt{-D})$, where D is a square-free positive integer. If $D \neq 1, 2, 3, 7, 11$, then $\mathrm{SL}_2(\mathcal{O}_D)$ is not generated by the elementary matrices.*

We remind the reader that

$$\mathcal{O}_D = \begin{cases} \mathbb{Z}[\sqrt{-D}] & \text{if } D \equiv 1, 2 \bmod 4, \\ \mathbb{Z}\big[\frac{1}{2}(1+\sqrt{-D})\big] & \text{if } D \equiv 3 \bmod 4. \end{cases}$$

For $D = 1, 2, 3, 7, 11$ the ring of integers $\mathcal{O}_D$ is a Euclidean domain, and hence $\mathrm{SL}_2(\mathcal{O}_D)$ is generated by the elementary matrices. For no other value of D is $\mathcal{O}_D$ Euclidean (Motzkin [9]), and Theorem 1.1 could be viewed as a strong way of asserting this fact. The values $D = 19, 43, 67, 163$ are particularly interesting, as $\mathcal{O}_D$ is a principal ideal domain in these cases (for $D = 19$, see [12] and [2]).

Soon after Cohn's theorem, it became clear that he had uncovered a highly singular phenomenon: $\mathrm{SL}_n(\mathcal{O}_\mathbb{K})$ is generated by the elementary matrices in all *other* cases. For $n = 2$, this was shown in [11].

Theorem 1.2 (Vaserstein). *Let $\mathcal{O}_\mathbb{K}$ denote the ring of integers in a number field $\mathbb{K}$. If $\mathbb{K}$ is not an imaginary quadratic field, then $\mathrm{SL}_2(\mathcal{O}_\mathbb{K})$ is generated by the elementary matrices.*

For $n \geq 3$, the situation is pleasantly uniform. In the course of their solution to the congruence subgroup problem [1], Bass, Milnor, and Serre established the following result.

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Theorem 1.3 (Bass - Milnor - Serre)**.** *Let $\mathcal{O}_{\mathbb{K}}$ denote the ring of integers in a number field $\mathbb{K}$. If $n \geq 3$, then $\mathrm{SL}_n(\mathcal{O}_{\mathbb{K}})$ is generated by the elementary matrices.*

The failure of elementary generation for $\mathrm{SL}_2(\mathcal{O}_D)$, where $D \neq 1, 2, 3, 7, 11$, is not only surprising, but also dramatic. It turns out that $\mathrm{E}_2(\mathcal{O}_D)$, the subgroup of $\mathrm{SL}_2(\mathcal{O}_D)$ generated by the elementary matrices, is a non-normal, infinite-index subgroup of $\mathrm{SL}_2(\mathcal{O}_D)$. Our aim is to give an elementary proof of this fact. For the moment, let us point out that the conceptual explanation behind this situation is Murphy's law: "When it rains, it pours."

Instead of rings of integers in imaginary quadratic fields, we consider, more generally, imaginary quadratic rings. A *quadratic ring* is a subring of $\mathbb{C}$ of the form $\mathbb{Z} \oplus \mathbb{Z}\omega$. Then ω must satisfy a quadratic relation which, up to an integral shift, is either $\omega^2 \pm D = 0$ or $\omega^2 - \omega \pm D = 0$, with D a positive integer. The quadratic rings are split by the choice of sign into real quadratic rings, $\mathbb{Z}[\sqrt{D}]$ and $\mathbb{Z}\big[\frac{1}{2}(1+\sqrt{1+4D})\big]$, and imaginary quadratic rings, $\mathbb{Z}[\sqrt{-D}]$ and $\mathbb{Z}\big[\frac{1}{2}(1+\sqrt{1-4D})\big]$.

Using a criterion of Cohn, Dennis [5] extended Theorem 1.1 to imaginary quadratic rings.

Theorem 1.4 (Dennis)**.** *Let $A = \mathbb{Z}[\sqrt{-D}]$ or $A = \mathbb{Z}\big[\frac{1}{2}(1+\sqrt{1-4D})\big]$, where $D \geq 4$. Then $\mathrm{SL}_2(A)$ is not generated by the elementary matrices.*

In Theorem 1.4, the discarded values of D correspond to the following imaginary quadratic rings:

$$\mathbb{Z}[\sqrt{-1}],\ \ \mathbb{Z}[\sqrt{-2}],\ \ \mathbb{Z}[\sqrt{-3}],\ \ \mathbb{Z}\big[\tfrac{1}{2}(1+\sqrt{-3})\big],\ \ \mathbb{Z}\big[\tfrac{1}{2}(1+\sqrt{-7})\big],\ \ \mathbb{Z}\big[\tfrac{1}{2}(1+\sqrt{-11})\big]$$

These are the five Euclidean rings of integers that we ruled out in Theorem 1.1, together with $\mathbb{Z}[\sqrt{-3}]$. Although $\mathbb{Z}[\sqrt{-3}]$ is not a Euclidean domain (in fact, it even fails to be a unique factorization domain), it is still the case that $\mathrm{SL}_2(\mathbb{Z}[\sqrt{-3}])$ is generated by the elementary matrices (see [5]).

The stage is now set for stating the theorem we are interested in. Namely, we prove the following strong failure of elementary generation for SL_2 over imaginary quadratic rings.

Theorem 1.5. *Let $A = \mathbb{Z}[\sqrt{-D}]$ or $A = \mathbb{Z}\big[\frac{1}{2}(1+\sqrt{1-4D})\big]$, where $D \geq 4$. Then the group $\mathrm{E}_2(A)$ generated by the elementary matrices is a non-normal, infinite-index subgroup of $\mathrm{SL}_2(A)$.*

Theorem 1.5 significantly strengthens Theorem 1.4; furthermore, our proof is more elementary and more explicit than the argument in [5]. It should be mentioned, however, that Theorem 1.5 is "known," in the sense that experts in this topic would immediately see it as a consequence of two other results from the literature. Firstly, the presentations obtained by Cohn in [4] show that, for A as in Theorem 1.5, $\mathrm{E}_2(A)$ is in fact independent of A. Secondly, a theorem of Frohman and Fine [6] says that, for A a ring of integers as in Theorem 1.1, $\mathrm{SL}_2(A)$ is an amalgamated product having $\mathrm{E}_2(A)$ as one of the factors. Alas, both results have complicated proofs. What we offer here is a new proof for a fact that is interesting enough to be considered on its own. Our approach descends directly from Cohn's [3].

To put Theorem 1.5 into perspective, notice that the imaginary quadratic rings are precisely the orders in imaginary quadratic fields. We remind the reader that an order in a number field $\mathbb{K}$ is a subring of $\mathbb{K}$ which has maximal possible rank, namely equal to the degree $[\mathbb{K} : \mathbb{Q}]$, when viewed as an abelian group. The ring of integers $\mathcal{O}_{\mathbb{K}}$ is an order, in fact the maximal order in the sense that all orders in $\mathbb{K}$ are contained in $\mathcal{O}_{\mathbb{K}}$.

The following result, extracted from [8], shows that the behavior described by Theorem 1.5 is exceptional among orders in number fields.

Theorem 1.6 (Liehl)**.** *Let A be an order in a number field which is not imaginary quadratic. Then $\mathrm{E}_2(A)$ is a normal, finite-index subgroup in $\mathrm{SL}_2(A)$. Moreover, if the number field has a real embedding then $\mathrm{SL}_2(A)$ is generated by the elementary matrices.*

The failure of normality in Theorem 1.5 should also be considered against the following important, and somewhat mysterious, result from [10].

Theorem 1.7 (Suslin). *Let $n \geq 3$, and let A be a commutative ring with identity. Then $\mathrm{E}_n(A)$ is normal in $\mathrm{SL}_n(A)$.*

2. Preliminaries

In this section, A denotes a commutative ring with identity.

2.1. **Unimodular pairs.** A pair of elements from A is said to be *unimodular* if it forms the first row of a matrix in $\mathrm{SL}_2(A)$. Observe that $\mathrm{SL}_2(A)$ acts by right-multiplication on unimodular pairs, and that this action is transitive.

Moreover, we have the following fact:

(†) $\mathrm{SL}_2(A)$ is generated by the elementary matrices if and only if the elementary group $\mathrm{E}_2(A)$ acts transitively on the set of unimodular pairs of A.

The forward implication in (†) is obvious. For the converse, let $S \in \mathrm{SL}_2(A)$. As $\mathrm{E}_2(A)$ acts transitively on unimodular pairs, we can right-multiply the first row of S by some $E \in \mathrm{E}_2(A)$ to obtain $(1,0)$. In other words, we have

$$SE = \begin{pmatrix} 1 & 0 \\ * & * \end{pmatrix}.$$

Taking determinants, we see that the $(2,2)$-entry of the right-hand side is 1, so the right-hand side is in fact an elementary matrix. Hence $S \in \mathrm{E}_2(A)$, and we conclude that $\mathrm{SL}_2(A) = \mathrm{E}_2(A)$.

We do not use (†) per se in what follows, but rather its message: the way $\mathrm{E}_2(A)$ sits in $\mathrm{SL}_2(A)$ can be understood through the action of $\mathrm{E}_2(A)$ on unimodular pairs.

2.2. **Cohn's standard form.** Instead of the elementary matrices

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \qquad (a \in A)$$

we use the following matrices:

$$E(a) = \begin{pmatrix} a & 1 \\ -1 & 0 \end{pmatrix} \quad (a \in A).$$

Note that

$$E(a) = \begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \tag{1}$$

for all $a \in A$. Since

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \mathrm{E}_2(A),$$

we deduce from (1) that each $E(a)$ is in $\mathrm{E}_2(A)$. On the other hand, (1) shows that we can express the elementary matrices as follows:

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} = -E(-a)E(0), \quad \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} = -E(0)E(a).$$

Consequently, every matrix in $\mathrm{E}_2(A)$ has the form $\pm E(a_1) \cdots E(a_r)$. Furthermore, such a form can be "standardized."

Lemma 2.1 (Cohn). *Every matrix in $\mathrm{E}_2(A)$ can be written as $\pm E(a_1) \cdots E(a_r)$ where all a_i's but a_1 and a_r are different from $0, \pm 1$.*

Proof. Given a matrix in $\mathrm{E}_2(A)$, consider its shortest expression as $\pm E(a_1) \cdots E(a_r)$. The relations

$$E(a)E(0)E(a') = -E(a+a'), \qquad E(a)E(\pm 1)E(a') = \pm E(a \mp 1)E(a' \mp 1)$$

show that only a_1 and a_r may take on the values $0, \pm 1$. $\square$

3. Proof of Theorem 1.5

Let $A = \mathbb{Z}[\sqrt{-D}]$ or $A = \mathbb{Z}\big[\frac{1}{2}(1+\sqrt{1-4D})\big]$, with $D \geq 4$. The verification of the next lemma is left to the reader.

Lemma 3.1. *Let $a \in A$. Then $|a| \geq 2$ whenever a is different from $0, \pm 1$.*

We call a unimodular pair (a, b) *special* if $|a| = |b| < |a \pm b|$. Geometrically, this means that 0, a, b, and $a+b$ are the vertices of a "fat" rhombus in which both diagonals are longer than the sides. The key observation is that special unimodular pairs are equivalent under the action of the elementary group $\mathrm{E}_2(A)$ only in a very restricted circumstance.

Lemma 3.2. *Let (a, b) and (a', b') be special unimodular pairs which are $\mathrm{E}_2(A)$-equivalent. Then (a', b') is one of (a, b), $(-b, a)$, $(-a, -b)$, $(b, -a)$.*

Note that the set $\{(a, b), (-b, a), (-a, -b), (b, -a)\}$ is the orbit of (a, b) under the 4-element subgroup of $\mathrm{E}_2(A)$ generated by $E(0)$, and that it consists of special unimodular pairs whenever (a, b) is a special unimodular pair. Lemma 3.2 can then be re-stated as follows: two special unimodular pairs are $\mathrm{E}_2(A)$-equivalent if and only if they are $\langle E(0)\rangle$-equivalent.

The proof of Lemma 3.2 is based on the following analysis.

Lemma 3.3. *Let (a, b) be a unimodular pair. Denote by (a', b') the unimodular pair $(a, b)E(c) = (ca - b, a)$, where $c \in A$.*
i) If $|a| > |b|$, then $|a'| > |b'|$ whenever $c \neq 0, \pm 1$.
ii) If (a, b) is special, then $|a'| > |b'|$ whenever $c \neq 0$.

Proof. i) Let $c \neq 0, \pm 1$, so $|c| \geq 2$. Then $|ca - b| \geq |ca| - |b| \geq 2|a| - |b| > |a|$.

ii) Let $c \neq 0$. For $c = \pm 1$, we have $|ca - b| = |a \mp b| > |a|$ because (a, b) is special. Now let $c \neq 0, \pm 1$, so $|c| \geq 2$. As in part i), we have

$$|ca - b| \geq |ca| - |b| \geq 2|a| - |b| = |a|. \tag{2}$$

Assume, by way of contradiction, that $|ca - b| = |a|$. Consequently, we must have equalities throughout (2): $|ca - b| = |ca| - |b|$, and $|c| = 2$. The first equality says that 0, b, and ca are colinear in this order; as $|ca| = 2|a| = 2|b|$, we deduce that $b = ca/2$.

If $c = \pm 2$, then $b = \pm a$, which contradicts the fact that (a, b) is special. As an addendum to Lemma 3.1, the reader may check that $|c| = 2$ admits solutions different from ± 2 only when $D = 4$; namely, $c = \pm\omega$ or $c = \pm\overline{\omega}$ in $\mathbb{Z}[\omega]$, where $\omega = \sqrt{-4}$ or $\omega = \frac{1}{2}(1+\sqrt{-15})$.

We have $ax + by = 1$ for some $x, y \in \mathbb{Z}[\omega]$, since (a, b) is unimodular. Using $b = ca/2$, we get

$$a(2x + cy) = 2. \tag{3}$$

If both a and $2x + cy$ are different from ± 1, then $|a| \geq 2$ and $|2x + cy| \geq 2$—by Lemma 3.1—and (3) fails. Also, if $a = \pm 1$, then $b = \pm c/2$ is no longer in $\mathbb{Z}[\omega]$. Therefore $2x + cy = \pm 1$. By conjugating or changing the signs of x or y if necessary, we may assume that $2x + \omega y = 1$. Putting $x = x_1 + x_2\omega$ and $y = y_1 + y_2\omega$ (where $x_1, x_2, y_1, y_2 \in \mathbb{Z}$), we obtain $2x_1 + 2x_2\omega + y_1\omega + y_2\omega^2 = 1$. But $\omega^2 = -4$ (when $\omega = \sqrt{-4}$) or $\omega^2 = \omega - 4$ (when $\omega = \frac{1}{2}(1+\sqrt{-15})$), and we reach the contradiction $2x_1 - 4y_2 = 1$ in either case. $\square$

Lemma 3.2 follows readily from Lemma 3.3.

Proof of Lemma 3.2. According to Lemma 2.1, we have

$$(a', b') = \pm(a, b)E(c_1)\cdots E(c_r),$$

where only c_1 and c_r can take on the values $0, \pm 1$. Assume that (a', b') is not $\langle E(0)\rangle$-equivalent to (a, b). By replacing (a, b) by $(a, b)E(0)$, and (a', b') by $(a', b')E(0)$ if necessary, we may furthermore impose $c_1, c_r \neq 0$. For $r = 1$, part ii) of Lemma 3.3 leads to a contradiction. For $r \geq 2$, parts i) and ii) of Lemma 3.3 show that

$$(a'', b'') := \pm(a, b)E(c_1) \cdots E(c_{r-1})$$

has $|a''| > |b''|$ so necessarily $c_r = \pm 1$. But then

$$(a'', b'') = (a', b')E(\pm 1)^{-1} = (b', \pm b' - a')$$

has $|b'| < |\pm b' - a'|$, that is, $|a''| < |b''|$. This contradiction ends the proof. □

We now attack Theorem 1.5.

The case $A = \mathbb{Z}[\sqrt{-D}]$. The general strategy is captured by the following claim:

(‡) Let k be a positive integer for which the Pell-type equation $X^2 - DY^2 = k^2 + 1$ has integral solutions. To each positive integral solution (x, y) of $X^2 - DY^2 = k^2 + 1$ we associate a matrix

$$S_{x,y} := \begin{pmatrix} -k + y\sqrt{-D} & -x \\ x & k + y\sqrt{-D} \end{pmatrix} \in \mathrm{SL}_2(A).$$

Then, for sufficiently large solutions (x, y), the following assertions hold:

i) the conjugate $S_{x,y}^{-1}E(0)S_{x,y}$ is not in $\mathrm{E}_2(A)$, and in particular $S_{x,y}$ is not in $\mathrm{E}_2(A)$;

ii) matrices $S_{x,y}$ corresponding to distinct sufficiently large solutions (x, y) lie in distinct left cosets of $\mathrm{E}_2(A)$.

We prove the claim (‡). Consider the unimodular pairs

$$u_{x,y} := (1,1)S_{x,y} = (x - k + y\sqrt{-D}, -x + k + y\sqrt{-D}),$$
$$v_{x,y} := (-1,1)S_{x,y} = (x + k - y\sqrt{-D}, x + k + y\sqrt{-D}).$$

Note that a unimodular pair of the form $(a, \pm\bar{a})$ is special if and only if $|\mathrm{Im}\, a| < \sqrt{3}\,|\mathrm{Re}\, a|$ and $|\mathrm{Re}\, a| < \sqrt{3}\,|\mathrm{Im}\, a|$. Therefore, $u_{x,y}$ and $v_{x,y}$ are special unimodular pairs precisely when

$$\frac{1}{\sqrt{3}} < \frac{x \pm k}{y\sqrt{D}} < \sqrt{3}. \tag{4}$$

The relation $x^2 - Dy^2 = k^2 + 1$ implies that

$$\frac{x \pm k}{y\sqrt{D}} \longrightarrow 1 \text{ as } x, y \longrightarrow \infty;$$

consequently, sufficiently large solutions (x, y) of $X^2 - DY^2 = k^2 + 1$ fulfill (4).

Let (x, y) be a sufficiently large solution so that $u_{x,y}$ and $v_{x,y}$ are special unimodular pairs. Clearly, $u_{x,y}$ and $v_{x,y}$ are not $\langle E(0)\rangle$-equivalent. As

$$u_{x,y}\big(S_{x,y}^{-1}E(0)S_{x,y}\big) = (1,1)E(0)S_{x,y} = (-1,1)S_{x,y} = v_{x,y},$$

it follows by Lemma 3.2 that $S_{x,y}^{-1}E(0)S_{x,y}$ is not in $\mathrm{E}_2(A)$. This justifies part i) of (‡). For part ii), let (x, y) and (x', y') be distinct, sufficiently large solutions and assume that $S_{x',y'} = S_{x,y}\, E$ for some $E \in \mathrm{E}_2(A)$. Right-acting on $(1, 1)$, we obtain $u_{x',y'} = u_{x,y}\, E$. Now Lemma 3.2 implies that $u_{x',y'}$ and $u_{x,y}$ are in fact $\langle E(0)\rangle$-equivalent, a contradiction.

We can infer from (‡) that $\mathrm{E}_2(A)$ is a non-normal, infinite-index subgroup of $\mathrm{SL}_2(A)$ as soon as we dispel the doubts surrounding the following two points: that positive integers k for which $X^2 - DY^2 = k^2 + 1$ has integral solutions do exist, and that $X^2 - DY^2 = k^2 + 1$ has, indeed, sufficiently large integral solutions as soon as it is solvable at all. Both points will be clarified by the following concrete implementation of (‡).

For $k = 2D$, the equation $X^2 - DY^2 = k^2 + 1$ admits the solution $x = 2D + 1$, $y = 2$. The identity

$$(x^2 - Dy^2)(p^2 - Dq^2) = (xp + Dyq)^2 - D(xq + yp)^2$$

tells us how to generate more integral solutions for $X^2 - DY^2 = k^2 + 1$. Namely, let (p_n, q_n) be the nth solution of the Pell equation $X^2 - DY^2 = 1$; it is given by the recurrence

$$p_{n+1} = p_1 p_n + Dq_1 q_n, \quad q_{n+1} = p_1 q_n + q_1 p_n,$$

where (p_1, q_1) is the smallest positive solution—the *fundamental solution*—of $X^2 - DY^2 = 1$. (See the first section of [7] for a quick reminder.) Setting

$$x_n = (2D+1)p_n + 2Dq_n, \quad y_n = (2D+1)q_n + 2p_n$$

we obtain a sequence of solutions for $X^2 - DY^2 = k^2 + 1$. Clearly $x_n, y_n \to \infty$ as $n \to \infty$.

We have already noticed that (x_n, y_n) satisfies the key condition (4) for large enough n. In this particular realization of (‡), however, it turns out that (x_n, y_n) satisfies (4) for all $n \geq 1$. Indeed, start by observing that $x_n \geq 2k + 1$. Then a straightforward manipulation shows that

$$\frac{1}{\sqrt{3}} < \frac{x \pm k}{\sqrt{x^2 - k^2 - 1}} < \sqrt{3}$$

whenever $x \geq 2k + 1$.

Thus, if we let

$$\begin{aligned} S_n \quad &:= \quad \begin{pmatrix} -2D + y_n\sqrt{-D} & -x_n \\ x_n & 2D + y_n\sqrt{-D} \end{pmatrix} \\ &= \quad \begin{pmatrix} -2D & 0 \\ 0 & 2D \end{pmatrix} + \begin{pmatrix} 2\sqrt{-D} & -(2D+1) \\ 2D+1 & 2\sqrt{-D} \end{pmatrix} \begin{pmatrix} p_n & q_n\sqrt{-D} \\ -q_n\sqrt{-D} & p_n \end{pmatrix} \end{aligned}$$

then we have:

- $S_n \in \mathrm{SL}_2(A)$;
- $S_n^{-1} E(0) S_n \notin \mathrm{E}_2(A)$, and in particular $S_n \notin \mathrm{E}_2(A)$;
- the S_n's lie in distinct left cosets of $\mathrm{E}_2(A)$.

The case $A = \mathbb{Z}\big[\frac{1}{2}(1 + \sqrt{1-4D})\big]$ requires no extra work. A moment's thought will convince the reader that, if we perform the above construction over $\mathbb{Z}[\sqrt{1-4D}]$, then we can still conclude that $\mathrm{E}_2(A)$ is a non-normal, infinite-index subgroup in $\mathrm{SL}_2(A)$.

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