Enhanced hyperbolicity: why and how

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Enhanced hyperbolicity. It is a well-known fact that CAT(-1) spaces are Gromov hyperbolic. Both are metric notions of negative curvature, the difference being that the CAT(-1) condition is sharp whereas hyperbolicity is coarse. A simple illustration of this difference is the following: in CAT(-1) spaces, pairs of points are joined by unique geodesics; in hyperbolic spaces, there may be several geodesics but they are uniformly close.

Enhanced hyperbolicity is an informal term denoting a middle ground between the sharp and the coarse. More specifically, though still not entirely precise, we would like hyperbolic groups to admit geometric actions on hyperbolic spaces that have additional CAT(-1) features. Our concrete motivations come from the analytic theory of hyperbolic groups, though somewhere in the background is the old and still unresolved foundational question whether hyperbolic groups admit geometric actions on CAT(-1) spaces. Below, we illustrate the analytic uses of enhanced hyperbolicity in two instances. But before we get to imposing additional demands, we should be ready to make some concessions. Namely, we give ourselves the flexibility of working with *roughly geodesic* hyperbolic spaces. A metric space X is said to be roughly geodesic if there is a constant $C \geq 0$ so that, for any pair of points $x, y \in X$, there is a (not necessarily continuous) map $\gamma : [a, b] \to X$ satisfying $\gamma(a) = x$, $\gamma(b) = y$, and $|s - s'| - C \leq |\gamma(s), \gamma(s')| \leq |s - s'| + C$ for all $s, s' \in [a, b]$.

Why? After the deep and groundbreaking work of Vincent Lafforgue, the resolution of the Baum - Connes conjecture for hyperbolic groups hinged on the following geometric ingredient: every hyperbolic group Γ admits a geometric action on a roughly geodesic, *strongly bolic* hyperbolic space. A roughly geodesic hyperbolic space is said to be strongly bolic if for every $\eta, r > 0$ there exists $R > 0$ such that $|x, y|+|z, t| \leq r$ and $|x, z|+|y, t| \geq R$ imply $|x, t|+|y, z| \leq |x, z|+|y, t|+\eta$. Mineyev and Yu [5] show that, indeed, every hyperbolic group Γ can be endowed with an 'admissible' metric - that is, a metric which is Γ-invariant, quasi-isometric to the word metric, and roughly geodesic - which is furthermore strongly bolic.

In a different direction, the geometric ingredient needed in [6] is the following: every hyperbolic group Γ admits a geometric action on a roughly geodesic, *good* hyperbolic space X. One can then obtain a proper affine isometric action of Γ on an L^p-space associated to the double boundary $\partial X \times \partial X$. Here, we say that a hyperbolic space X is good if the following two properties hold: i) the Gromov product $(\cdot, \cdot)_o$ extends continuously from X to the bordification $X \cup \partial X$ for each basepoint $o \in X$, and ii) there is some $\epsilon > 0$ such that $\exp(-\epsilon\,(\cdot, \cdot)_o)$ is a metric on the boundary ∂X, again for each basepoint $o \in X$. The concrete X used in [6] is Γ itself, equipped with an 'admissible' metric which is furthermore good. Such a metric was constructed by Mineyev in [3, 4], and it is a slightly improved version of the metric used by Mineyev and Yu in [5].

We note that CAT(-1) spaces are strongly bolic (this can be checked directly, and it even holds for CAT(0) spaces) and good (this is a theorem of Bourdon [1]).

How? Our main goal in [7] was to find a metric notion, weaker than the CAT(-1) condition, that guarantees enhanced hyperbolicity, and such that every hyperbolic group has a *natural* 'admissible' metric satisfying it. Here is our metric notion, and the results that fulfill our wishes.

Definition 1. A metric space X is *superbolic* if, for some $\epsilon > 0$, we have

$$\exp(-\epsilon\,(x,y)_o) \leq \exp(-\epsilon\,(x,z)_o) + \exp(-\epsilon\,(z,y)_o)$$

for all $x, y, z, o \in X$.

Theorem 2. *A roughly geodesic superbolic space is a good, strongly bolic hyperbolic space.*

Theorem 3. CAT(-1) *spaces are superbolic.*

Theorem 4. *The Green metric arising from a random walk on a hyperbolic group is superbolic.*

Theorem 2 is quantitative: ϵ-superbolic implies ϵ-good, $(\log 2)/\epsilon$-hyperbolic, and strongly bolic with exponential control. In Theorem 3, we show that CAT(-1) spaces are 1-superbolic; as a corollary, we recover Bourdon's theorem that CAT(-1) spaces are 1-good. We also find the best constant of hyperbolicity, in the sense of Gromov's original definition, for the hyperbolic plane $\mathbb{H}^2$. Quite surprisingly, this was not known before.

Corollary 5. $\mathbb{H}^2$ *is* $\log 2$*-hyperbolic, and this is optimal.*

In Theorem 4, the random walk is assumed to be symmetric and supported on a finite generating subset of the hyperbolic group. The random walk metric, or the *Green metric*, on a hyperbolic group is given by the formula $|x,y|_G = -\log F(x,y)$, where $F(x,y)$ is the probability that the random walk started at x ever hits y. It turns out that the Green metric is 'admissible'. A corollary of Theorem 4 is the fact that the Green metric is good, and this is a positive answer to a question raised in [6]. As another corollary, we recover the result of Haïssinsky and Mathieu [2] that the Green metric is strongly bolic. A third corollary is the following.

Corollary 6. *On the boundary of a hyperbolic group, the harmonic measure defined by a random walk equals the Hausdorff probability measure defined by any Green visual metric.*

We find the Green metric to be a simple and natural alternative to the metrics constructed in [5, 3, 4].

References

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