CUT RATIOS AND LAPLACIAN EIGENVALUES

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ABSTRACT. We introduce local cut ratios as a sampling scheme for the cut ratio of a vertex subset in a graph. We prove a bound for the sampling error, as well as a two-sided estimate for the local cut ratio, in terms of Laplacian eigenvalues. We derive an inequality for the vertex cut ratio in terms of the (edge) cut ratio.

1. PROLOGUE

Let X be a graph, and let U be a proper subset of vertices. The *edge boundary* of U, denoted ∂U, is the set of edges joining a vertex in U to a vertex in U^c, the complement of U. The *cut ratio* of U is given by

$$h(U) = \frac{|\partial U|}{|U|}.$$

The cut ratio, and its vertex relative, to be mentioned shortly, underpin expansion behaviours in graphs. See [8] for the broad picture, as well as Section 4 herein. The general problem we are interested in is that of estimating the cut ratio $h(U)$ as a function of the size of U. This relies, of course, on certain parameters of the ambient graph X as being given. We use the number of vertices, denoted by n, and the Laplacian eigenvalues

$$0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n.$$

Actually, the relevant eigenvalues are λ_2 and λ_n. These two Laplacian eigenvalues are, in effect, spectral ways of encoding the connectivity of X–an interpretation which originates in Fiedler's pioneering work [6]. Let us note, on the other hand, that the vertex degrees, recording pointwise connectivity, are not involved in what follows.

A good starting point that illustrates our perspective is the following estimate (cf. [10, Prop.2.1]): if $|U| = \varepsilon n$, where $0 < \varepsilon < 1$, then

$$\lambda_2(1-\varepsilon) \leq h(U) \leq \lambda_n(1-\varepsilon). \tag{1}$$

The lower estimate is a penetrating observation of Alon and Milman [2]. The idea is to relate the cut ratio $h(U)$ to a suitable Rayleigh quotient, for which λ_2 is a lower bound thanks to the Courant-Fischer principle. On the other hand, using λ_n as an upper bound for the Rayleigh quotient gives the upper estimate in (1). It is conceptually more interesting to view the two estimates in (1) as being dual to each other. Namely, consider the graph X', the complement of X. The cut ratios for X and X' are related by $h(U) + h'(U) = |U^c| = n(1-\varepsilon)$, and the Laplacian

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eigenvalues of X' are $0 = \lambda'_1 \leq \lambda'_2 = n - \lambda_n \leq \cdots \leq \lambda'_n = n - \lambda_2$. It follows that the double estimate (1) is invariant under graph complementation.

The question of extremal cuts, that is to say vertex subsets U attaining the lower or the upper bound in (1), is natural and interesting. As pointed out above, the two extremal cases are dual–a small cut in a graph is a large cut in its complement–and attention has been mostly given to equality in the lower bound. See [5] for a detailed study, and [11, p.117] for some examples.

2. Local cut ratios

Fix a vertex subset U in a graph X, and suppose we want to measure the cut ratio $h(U)$. Counting the boundary edges of U could be complicated if U is very large, so we might try to estimate $h(U)$ by sampling a relatively small subset of U. This natural idea leads us to the following notion.

Definition. The *local cut ratio* of a non-empty subset $S \subseteq U$ is given by

$$h_U(S) = \frac{e(S, U^c)}{|S|}$$

where $e(S, U^c)$ denotes the number of edges joining a vertex in S to a vertex in U^c, the complement of U.

The local cut ratio h_U is to be viewed as a function on subsets of U. Note that for $S = U$ we recover the cut ratio: $e(U, U^c) = |\partial U|$, and so $h_U(U) = h(U)$.

To what extent can the local cut ratio h_U, evaluated on small subsets of U, be used as a proxy for the cut ratio $h(U)$? The first pertinent fact is that, whenever we sample a fixed proportion, the expected value of the local cut ratio is the cut ratio.

Theorem 1. *The expected value of $h_U(S)$, subject to $S \subseteq U$ having a fixed size, is $h(U)$.*

The second fact, more interesting, is an estimate for the absolute deviation from the mean.

Theorem 2. *Let $|U| = \varepsilon n$, and let $S \subseteq U$ be non-empty. Then:*

$$\big|h_U(S) - h(U)\big| \leq \frac{\lambda_n - \lambda_2}{2} \sqrt{(1-\varepsilon)\Big(\frac{|U|}{|S|} - 1\Big)}.$$

The above inequality controls the discrepancy between $h_U(S)$, thought of as a partial measurement of $h(U)$, and the true value of $h(U)$. For instance, it tells us what proportion of nodes in U should be sampled–in the sense that a node is accessed, and the connectivity of the node is used for counting the number of boundary edges based at that node–in order to ascertain the value of the cut ratio $h(U)$ up to a given precision. To give a quantitative example: sampling more than $1/(2-\varepsilon)$ of the nodes in U pins the value of $h(U)$ in a narrower interval than the interval defined by (1).

On the other hand, we can also give a stand-alone estimate for the local cut ratio $h_U(S)$, in terms of the size of U and the size of the sampling subset $S \subseteq U$. The following two-sided estimate is a generalization of (1), which arises in the extremal case $S = U$.

Theorem 3. *Let $|U| = \varepsilon n$, and let $S \subseteq U$ be non-empty. Then:*

$$\lambda_2(1-\varepsilon) - \frac{\lambda_n - \lambda_2}{2}\left(\sqrt{\frac{|U|}{|S|}} - 1\right) \leq h_U(S) \leq \lambda_n(1-\varepsilon) + \frac{\lambda_n - \lambda_2}{2}\left(\sqrt{\frac{|U|}{|S|}} - 1\right).$$

Note that Theorems 2 and 3 can be related by using the bound (1). Combining Theorem 2 and (1), we get a double bound for the local cut ratio $h_U(S)$ that is quite similar to the one of Theorem 3. This combined bound beats the bound of Theorem 3 when the sampling set S is sufficiently small; for instance, $|S| \leq \varepsilon^2|U|/4$ suffices. Conversely, we might combine Theorem 3 and (1) to get bounds for the deviation $|h_U(S) - h(U)|$; it can be checked, however, that the bound of Theorem 2 is always sharper.

We now consider the notion of vertex cut ratio. Again, let U be a proper vertex subset. The *vertex boundary* of U, denoted δU, is the set of all vertices in U^c that are adjacent to some vertex in U. Correspondingly, we have a *vertex cut ratio* for U:

$$g(U) = \frac{|\delta U|}{|U|}.$$

Alternative notations are in circulation; we are more or less following Chung [3, Sec.2.2].

The vertex and edge cut ratios are related by the inequality $g(U) \leq h(U) \leq \Delta g(U)$, where Δ denotes the maximal degree of X. As an application of Theorem 2, we prove another lower bound for $g(U)$ in terms of $h(U)$. In line with the general philosophy of the paper, it is degree-free.

Theorem 4. *Assume X is connected. Let $|U| = \varepsilon n$. Then:*

$$\frac{1}{g(U)} \leq \frac{\varepsilon}{1-\varepsilon} + \left(\frac{\lambda_n - \lambda_2}{2}\right)^2 \frac{1-\varepsilon}{h(U)^2}.$$

The spectral quantity $\lambda_n - \lambda_2$, which appears in Theorems 2, 3, and 4, is known as the *Laplacian spread* of X. It measures the distance between λ_2 and the top of the spectrum, so it serves as the counterpart of the *Laplacian gap* λ_2, which measures the distance from λ_2 to the bottom of the spectrum.

The Laplacian spread is invariant under graph complementation. In fact, invariance under complementation–a principle we first highlighted when discussing (1)–is a feature of the bounds involving the local cut ratio, Theorems 2 and 3 herein. This owes to the functional relation $h_U + h'_U = |U^c|$, where h_U and h'_U are the local cut ratios defined by U in X, respectively in its complement X'. By way of contrast, the vertex cut ratio, and the bound of Theorem 4 in particular, is poorly behaved under graph complementation. So there is no harm in assuming X is connected, as we did.

3. Proofs

Let us first recall the bare essentials on the Laplacian. Given a finite simple graph X, the Laplacian is a symmetric linear operator on the space of real-valued functions defined on the vertex set V of X. This is a finite-dimensional space, endowed with the inner product

$$\langle \phi, \psi \rangle = \sum_{v \in V} \phi(v)\, \psi(v).$$

The Laplacian, denoted L, is described by its matrix coefficients as follows: diagonally, $\langle L\mathbb{1}_v, \mathbb{1}_v\rangle$ is the degree of the vertex v; off-diagonally,

$$\langle L\mathbb{1}_v, \mathbb{1}_w\rangle = \begin{cases} -1 & \text{if } v \text{ is adjacent to } w \\ 0 & \text{if } v \text{ is not adjacent to } w \end{cases}$$

for any two distinct vertices v and w.

The Laplacian has non-negative eigenvalues. The smallest one is $\lambda_1 = 0$, and it admits the constant function $\mathbb{1}$ as an eigenvector; that is, $L\mathbb{1} = 0$. The interested reader is referred to, say, [3] or [11], for details and further information on Laplacian eigenvalues.

A key fact for our purposes is the following relation between the Laplacian and the local cut ratio:

$$h_U(S) = \Big\langle L\mathbb{1}_U, \frac{1}{|S|}\,\mathbb{1}_S\Big\rangle \tag{2}$$

for any non-empty subset $S \subseteq U$. As usual, $\mathbb{1}_S$ denotes the characteristic function of S. To justify the relation (2), we start by writing

$$e(S, U^c) = -\sum_{v\in S, w\in U^c} \langle L\mathbb{1}_v, \mathbb{1}_w\rangle = -\langle L\mathbb{1}_S, \mathbb{1}_{U^c}\rangle = -\langle L\mathbb{1}_{U^c}, \mathbb{1}_S\rangle.$$

On the other hand,

$$\langle L\mathbb{1}_{U^c}, \mathbb{1}_S\rangle + \langle L\mathbb{1}_U, \mathbb{1}_S\rangle = \langle L\mathbb{1}, \mathbb{1}_S\rangle = 0.$$

We deduce that $e(S, U^c) = \langle L\mathbb{1}_U, \mathbb{1}_S\rangle$; dividing through by $|S|$ yields (2).

3.1. **Proof of Theorem 1.** Let $s \in \{1, \ldots, |U|\}$ be the fixed size for the variable subset $S \subseteq U$. Then

$$\mathbb{E}\Big[\frac{1}{|S|}\mathbb{1}_S \,\Big|\, S \subseteq U, |S| = s\Big] = \frac{1}{\binom{|U|}{s}} \sum_{S\subseteq U, |S|=s} \frac{1}{|S|}\,\mathbb{1}_S$$

$$= \frac{1}{s\,\binom{|U|}{s}} \binom{|U|-1}{s-1}\,\mathbb{1}_U = \frac{1}{|U|}\,\mathbb{1}_U.$$

Using (2), we deduce that

$$\mathbb{E}\big[h_U(S) \,\big|\, S \subseteq U, |S| = s\big] = \Big\langle L\mathbb{1}_U, \frac{1}{|U|}\,\mathbb{1}_U\Big\rangle = h_U(U) = h(U).$$

3.2. **Proof of Theorem 2.** We actually prove a more general estimate. We let S and T be two non-empty vertex subsets of U, and we seek an upper bound for $|h_U(S) - h_U(T)|$. In light of (2), we can write

$$h_U(S) - h_U(T) = \langle L\mathbb{1}_U, f\rangle, \qquad f = \frac{1}{|S|}\,\mathbb{1}_S - \frac{1}{|T|}\,\mathbb{1}_T.$$

We note, for later use, that

$$\langle \mathbb{1}, f\rangle = \langle \mathbb{1}_U, f\rangle = 0.$$

Now let $\varphi_1, \varphi_2, \ldots, \varphi_n$ be an orthonormal basis consisting of Laplacian eigenfunctions. They correspond to the Laplacian eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, and we set

$$\varphi_1 = \frac{1}{\sqrt{n}}\,\mathbb{1}$$

to be the eigenfunction corresponding to $\lambda_1 = 0$. Expand the characteristic function of U, as well as f, in the above eigenbasis:

$$\mathbb{1}_U = \sum_{k=1}^n u_k \varphi_k, \qquad f = \sum_{k=1}^n f_k \varphi_k.$$

The first Fourier coefficients are, respectively

$$u_1 = \langle \mathbb{1}_U, \varphi_1 \rangle = \frac{|U|}{\sqrt{n}}, \qquad f_1 = \langle f, \varphi_1 \rangle = 0.$$

We now have

$$L\mathbb{1}_U = \sum_{k=1}^n \lambda_k u_k \varphi_k = \sum_{k=2}^n \lambda_k u_k \varphi_k$$

and so

$$h_U(S) - h_U(T) = \langle L\mathbb{1}_U, f \rangle = \langle L\mathbb{1}_U, f \rangle - C\langle \mathbb{1}_U, f \rangle = \sum_{k=2}^n (\lambda_k - C)\, u_k f_k$$

for any real constant C. An optimal choice of C will be specified shortly. We deduce that

$$\big| h_U(S) - h_U(T) \big| \le \sum_{k=2}^n |\lambda_k - C| |u_k| |f_k| \le \Big(\max_{k=2,\dots,n} |\lambda_k - C| \Big) \sum_{k=2}^n |u_k| |f_k|.$$

We now choose C so as to minimize the above maximum: $C = \frac{1}{2}(\lambda_2 + \lambda_n)$. As for the right-most sum, it can be bounded by using the Cauchy-Schwarz inequality. To that end, we compute

$$\sum_{k=2}^n |u_k|^2 = \langle \mathbb{1}_U, \mathbb{1}_U \rangle - |u_1|^2 = \frac{|U||U^c|}{n}, \qquad \sum_{k=2}^n |f_k|^2 = \langle f, f \rangle = \frac{|S \triangle T|}{|S||T|}.$$

Summarizing, we have shown that

$$\big| h_U(S) - h_U(T) \big| \le \frac{\lambda_n - \lambda_2}{2} \sqrt{\frac{|U||U^c|}{n}} \sqrt{\frac{|S \triangle T|}{|S||T|}}. \tag{3}$$

The bound stated in the theorem follows by specializing $T = U$.

Remark. The strategy of the previous proof is similar to the one used in proving any version of the Expander Mixing Lemma.

Remark. The quantity

$$\sqrt{\frac{|S \triangle T|}{|S||T|}} = \left\| \frac{1}{|S|} \mathbb{1}_S - \frac{1}{|T|} \mathbb{1}_T \right\|_2$$

defines a metric on the non-empty subsets of U. From this viewpoint, the bound (3) is a Lipschitz inequality for the local cut ratio h_U.

3.3. **Proof of Theorem 3.** Recall the following Laplacian version of the Expander Mixing Lemma, due to Chung [4, Thm.7].

(†) Let S and T be disjoint vertex subsets in X, and let $e(S,T)$ denote the number of edges joining a vertex in S to a vertex in T. Then
$$\left|e(S,T)-\frac{\lambda_2+\lambda_n}{2n}\,|S||T|\right|\leq\frac{\lambda_n-\lambda_2}{2n}\sqrt{|S|(n-|S|)|T|(n-|T|)}.$$

Now let S be a subset of U. We apply (†) to the disjoint vertex subsets S and $T=U^c$. Note that $\sqrt{(n-|S|)(n-|U|)}\leq n-\sqrt{|S||U|}$, with equality if and only if $S=U$. Hence
$$\sqrt{|S|(n-|S|)|U|(n-|U|)}\leq n\sqrt{|S||U|}-|S||U|.$$
Using the above estimate, the bound of (†) gives the following:
$$\left|e(S,U^c)-\frac{\lambda_2+\lambda_n}{2n}\,|S||U^c|\right|\leq\frac{\lambda_n-\lambda_2}{2n}\left(n\sqrt{|S||U|}-|S||U|\right).$$
Dividing through by $|S|$, and using $|U|=\varepsilon n$, we get
$$\begin{aligned}\left|h_U(S)-\frac{\lambda_2+\lambda_n}{2}\,(1-\varepsilon)\right|&\leq\frac{\lambda_n-\lambda_2}{2}\left(\sqrt{\frac{|U|}{|S|}}-\varepsilon\right)\\&=\frac{\lambda_n-\lambda_2}{2}\,(1-\varepsilon)+\frac{\lambda_n-\lambda_2}{2}\left(\sqrt{\frac{|U|}{|S|}}-1\right)\end{aligned}$$
The desired two-sided bound for $h_U(S)$ immediately follows.

3.4. **Proof of Theorem 4.** Let U be a proper vertex subset, and partition U as follows: the *interior* of U, denoted $\mathrm{int}(U)$, is the set of vertices in U all of whose neighbours are still in U; the *inner boundary* of U, denoted $\delta_{\mathrm{in}}(U)$, is the set of vertices in U that are adjacent to some vertex in the complement U^c.

Consider the subset $S=\mathrm{int}(U)$ of U. Then $h_U(S)=0$; in fact, the interior of U is the largest subset of U with this property. The bound of Theorem 2 gives
$$\left(\frac{2h(U)}{\lambda_n-\lambda_2}\right)^2\leq(1-\varepsilon)\left(\frac{|U|}{|S|}-1\right)=\frac{|U^c|}{n}\,\frac{|\delta_{\mathrm{in}}(U)|}{|U|-|\delta_{\mathrm{in}}(U)|}.$$
Rearranging, we get
$$\frac{|U|-|\delta_{\mathrm{in}}(U)|}{|\delta_{\mathrm{in}}(U)|}=\frac{|U|}{|\delta_{\mathrm{in}}(U)|}-1\leq\left(\frac{\lambda_n-\lambda_2}{2}\right)^2\frac{|U^c|}{n\,h(U)^2}.$$
We pass to the usual, outer, vertex boundary by the relation $\delta U=\delta_{\mathrm{in}}(U^c)$. Replacing U by U^c, the above inequality turns into
$$\frac{|U^c|}{|\delta U|}-1\leq\left(\frac{\lambda_n-\lambda_2}{2}\right)^2\frac{|U|}{n\,h(U^c)^2}.$$
As $|U^c|\,h(U^c)=e(U^c,U)=h(U)\,|U|$, the above inequality can be brought to
$$\frac{|U|}{|\delta U|}-\frac{|U|}{|U^c|}\leq\left(\frac{\lambda_n-\lambda_2}{2}\right)^2\frac{|U^c|}{n\,h(U)^2}.$$
That is
$$\frac{1}{g(U)}-\frac{\varepsilon}{1-\varepsilon}\leq\left(\frac{\lambda_n-\lambda_2}{2}\right)^2\frac{1-\varepsilon}{h(U)^2},$$

as claimed.

Remark. For the vertex cut ratio, the following spectral bound holds: if $|U| = \varepsilon n$, then

$$\frac{1}{g(U)} \leq \frac{\varepsilon}{1-\varepsilon} + \frac{(\lambda_n - \lambda_2)^2}{4\lambda_n\lambda_2}\,\frac{1}{1-\varepsilon}. \tag{4}$$

This is a slight variation on a result of Chung [4, Thm.8]. It can be deduced from the Expander Mixing Lemma (†), as follows. Firstly, let us note that a consequence of (†) is the following inequality due to Haemers. The original approach [7, Lem.6.1] used a different spectral technique.

(∗) Assume S and T are disjoint vertex subsets in X, such that there are no edges between S and T. Then

$$\frac{|S||T|}{(n-|S|)(n-|T|)} \leq \left(\frac{\lambda_n - \lambda_2}{\lambda_n + \lambda_2}\right)^2.$$

Now fix a proper vertex subset U. Then $S = \mathrm{int}(U)$ and $T = U^c$ are disjoint, and there are no edges between the two sets. Applying the Haemers inequality (∗), and rearranging, leads to the bound

$$|\mathrm{int}(U)| \leq \frac{n|U|}{n + (\kappa - 1)|U^c|}, \qquad \kappa := \left(\frac{\lambda_n + \lambda_2}{\lambda_n - \lambda_2}\right)^2$$

In inner boundary terms:

$$|\delta_{\mathrm{in}}(U)| = |U| - |\mathrm{int}(U)| \geq |U| - \frac{n|U|}{n + (\kappa-1)|U^c|} = \frac{(\kappa-1)|U||U^c|}{n + (\kappa-1)|U^c|}.$$

We use U^c in place of U, in view of the relation $\delta U = \delta_{\mathrm{in}}(U^c)$, and we obtain

$$|\delta U| \geq \frac{(\kappa-1)|U||U^c|}{n + (\kappa-1)|U|}.$$

Thus

$$\frac{1}{g(U)} = \frac{|U|}{|\delta U|} \leq \frac{n + (\kappa-1)|U|}{(\kappa-1)|U^c|} = \frac{|U|}{|U^c|} + \frac{1}{\kappa-1}\,\frac{n}{|U^c|}.$$

Upon replacing, on the right-hand side, $|U| = \varepsilon n$, $|U^c| = (1-\varepsilon)n$, and κ, we arrive at (4).

The bound of Theorem 4 beats the bound (4) precisely when the cut ratio satisfies

$$h(U) \geq \sqrt{\lambda_n\lambda_2}\,(1-\varepsilon). \tag{5}$$

From the perspective of the spectral squeeze (1), the inequality (5) says that the cut ratio $h(U)$ is somewhat large–larger than the geometric average of the endpoints. This is, certainly, likely to happen. In fact, we claim that the cut ratio defined by any vertex subset U satisfies (5) either in X, or in the complement X', possibly in both. For otherwise we would have

$$\begin{aligned} h(U) &< \sqrt{\lambda_n\lambda_2}\,(1-\varepsilon),\\ h'(U) &< \sqrt{\lambda'_n\lambda'_2}\,(1-\varepsilon) = \sqrt{(n-\lambda_2)(n-\lambda_n)}\,(1-\varepsilon). \end{aligned}$$

However, $h(U) + h'(U) = n(1-\varepsilon)$, and $n - \sqrt{\lambda_n\lambda_2} < \sqrt{(n-\lambda_2)(n-\lambda_n)}$ would follow, a contradiction.

4. Isoperimetric constants and profiles

Isoperimetry in graphs is concerned with minimizing cut ratios. This can mean several things, according to which cut ratio we use, and what constraints we respect while minimizing. The following are the most common ones. They are based on the edge and vertex cut ratios for proper vertex subsets, $h(U)$ and $g(U)$ as previously defined.

Definition. (cf. [9], [8, Def.4.2, Def.4.3]) Let X be a connected graph on n vertices.

The *edge isoperimetric profile* of X and the *vertex isoperimetric profile* of X are the functions

$$h(\varepsilon) = \min_{|U|=\varepsilon n} h(U), \qquad g(\varepsilon) = \min_{|U|=\varepsilon n} g(U)$$

defined for $\varepsilon \in (0, 1)$.

The *edge isoperimetric number*, or the *edge expansion*, of X and the *vertex isoperimetric number*, or the *vertex expansion*, of X are the numbers

$$h = \min_{|U|\leq n/2} h(U) = \min_{\varepsilon\leq 1/2} h(\varepsilon), \qquad g = \min_{|U|\leq n/2} g(U) = \min_{\varepsilon\leq 1/2} g(\varepsilon).$$

Recall that $g(U) \leq h(U)$, whence $g(\varepsilon) \leq h(\varepsilon)$ for all $\varepsilon \in (0, 1)$, and $g \leq h$.

The thrust of this paper is has to do with spectral bounds for cut ratios. We now interpret them as bounds for isoperimetric profiles and isoperimetric constants. To begin with, (1) gives the following well-known lower bounds for edge isoperimetry:

$$h(\varepsilon) \geq \lambda_2(1 - \varepsilon), \qquad h \geq \frac{\lambda_2}{2}. \tag{6}$$

Next, we derive bounds for vertex isoperimetry from Theorem 4, and the bound (4).

Corollary 5. *With the above notations, the following hold:*

$$\frac{1}{g(\varepsilon)} \leq \frac{\varepsilon}{1-\varepsilon} + \Big(\frac{\lambda_n - \lambda_2}{2}\Big)^2 \frac{1-\varepsilon}{h(\varepsilon)^2}, \qquad \frac{1}{g} \leq 1 + \Big(\frac{\lambda_n - \lambda_2}{2}\Big)^2 \frac{1}{h^2}; \tag{7}$$

$$\frac{1}{g(\varepsilon)} \leq \frac{\varepsilon}{1-\varepsilon} + \frac{(\lambda_n - \lambda_2)^2}{4\lambda_n\lambda_2} \frac{1}{1-\varepsilon}, \qquad \frac{2}{g} \leq \frac{\lambda_n}{\lambda_2} + \frac{\lambda_2}{\lambda_n}. \tag{8}$$

Comparing, again, the two bounds, we see that (7) beats (8) precisely when $h(\varepsilon) \geq \sqrt{\lambda_n\lambda_2}\,(1-\varepsilon)$, respectively $h \geq \frac{1}{2}\sqrt{\lambda_n\lambda_2}$. These requirements are more stringent than (5), and they indicate a high connectivity–for instance, complete graphs satisfy them.

We end by mentioning another spectral bound for the vertex isoperimetric constant g which, just like (6), only involves λ_2:

$$\frac{4}{g^2} \geq \frac{1}{\lambda_2} - 2.$$

This was proved by Alon in [1], a seminal work on the close relationship between eigenvalues and expansion in graphs.

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