Representation of categories

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1 Introduction

One of the earliest theorems in category theory stated that an abelian category could be represented faithfully by exact functors into the category **Ab** of abelian groups [Freyd, 1964], [Lubkin, 1961] and [Heron, unpublished]. Then Mitchell [1965] showed that every such category had a full exact embedding into a module category. An equivalent formulation is that every abelian category into a category of additive functors into **Ab** or even into a **Set**-valued functor category. Mitchell's argument was based on what was essentially the earliest theorem in category theory: Grothendieck's theorem that every AB5 category with a generator had an injective cogenerator [Grothendieck 1957].

Continuing in this vein, I showed in [Barr, 1971] that every regular category had a full, regular embedding into a category of set-valued functors. In doing this, I first tried to mimic Grothendieck's argument. Unfortunately, I never succeeded in demonstrating a non-abelian version of Grothendieck's theorem. There is a very good reason for that: it is false, see Corollary 12, below. Instead, the proof was based on showing that the obvious non-abelian adaptation of Lubkin's argument [Lubkin, 1960] not only continued to give a family of embeddings, but when the functors were put together into a category (with all natural transformations between them), the embedding was even full.

The proof was difficult, to say the least (it has been described as 'hermetic'), and the theorem has apparently had little impact although at least one better proof has been published since [Makkai, 1980]. Here we give yet another proof (Corollary 15). Surprisingly, it is based on Grothendieck's argument. It turns out that a weaker condition than injectivity is sufficient to make the proof work and the non-abelian version of Grothendieck's argument is sufficient to give that weaker condition. This argument ultimately goes back to Baer's proof that divisible abelian groups are injective.

Here is an outline of the new proof.

1. Show that when $\mathcal{C}$ is regular, so is $FL(\mathcal{C}, Set)^{\mathrm{op}}$. (FL is the full subcategory of finite limit preserving functors.

2. Adapt Grothendieck's transfinite induction proof [1957] of the existence of injectives in an AB5 abelian category to show that $Lex(\mathcal{C}, Set)^{\mathrm{op}}$ has enough $\mathcal{C}$-projectives = regular functors.

3. Adapt Mitchell's proof [1965] of the abelian category full embedding theorem to show that by taking a sufficiently large full subcategory $\mathcal{P}$ in $Lex(\mathcal{C}, Set)$ consisting of regular functors, then the evaluation functor $\mathcal{C} - -\rangle Func(\mathcal{P}^{\mathrm{op}}, Set)$ is full and faithful.

The theorem suggests a natural generalization to toposes; one might expect that a topos has a full embedding into a functor category that preserves the finitary part of the topos structure, i.e. finite limits, finite sums and epis (such a functor is called bf near exact in [Freyd, 1972] and we will stick to this usage). However, Makkai has given an example to show that such a result is false. In fact, we give a necessary condition for the existence of such an embedding—that the lattice of complemented subobjects of each object be a complete atomic boolean algebra—that makes it seem as though very few small toposes have such an embedding. We do give some sufficient condition for the existence of such an embedding, but a necessary and sufficient condition is still lacking. The necessary condition is very simple to state: any topos that has a full near exact embedding into a functor category has a complete atomic boolean algebra as its lattice of it complemented subobjects. Although there are some details to be checked, the argument is very simple: in any topos, that lattice is represented by 2 and 2 is preserved by near exact functors.

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2 Representations of regular categories

For a category $\mathcal{C}$, we let $\mathrm{FL}(\mathcal{C}, \mathbf{Set})$ denote the category of finite limit preserving functors into sets, with all natural transformations as morphisms. There is a Yoneda embedding $\mathcal{C} \longrightarrow \widetilde{\mathcal{C}} = \mathrm{FL}(\mathcal{C}, \mathbf{Set})$ and we will henceforth treat $\mathcal{C}$ as a full subcategory of $\widetilde{\mathcal{C}}$. We begin with some useful facts, which are given in the dual category because the functor category is more familiar than its opposite. Let $\mathcal{R} = \mathcal{C}^{\mathrm{op}}$ and $\mathcal{X} = \widetilde{\mathcal{C}}^{\mathrm{op}}$

2.1 Lemma. *When $\mathcal{C}$ is small,*

(i) *$\mathcal{X}$ is complete and cocomplete;*

(ii) *filtered colimits are exact;*

(iii) *the inclusion of* $\mathcal{R}$ *into* $\mathcal{X}$ *preserves all limits as well as finite colimits;*

(iv) *every object of* $\mathcal{X}$ *is a filtered colimit of objects of* $\mathcal{R}$*;*

(v) *for* R *in* $\mathcal{R}$*,* $\mathrm{Hom}(R,-)$ *commutes with filtered colimits;*

(vi) $\mathcal{R}$ *is coregular.*

Proof. (i) is well known; see [Barr-Wells, 1984], Exercise (LIM FUN) of Section 1.7. As for (ii), first observe that since a filtered colimit of left exact functors is left exact, the inclusion $\mathcal{X} \longrightarrow \mathrm{Func}(\mathcal{C}, \mathbf{Set})$ preserves filtered colimits. It also preserves limits, in particular, pullbacks. It follows that in $\mathcal{X}$, pullbacks commute with filtered colimits. It is clear that a colimit of a diagram, each of whose nodes is terminal is also terminal, if and only if the diagram is connected, which every filtered diagram is. To see (iii), the preservation of limits is a consequence of the Yoneda lemma, while the inclusion preserves the colimit of any diagram in $\mathcal{R}$ whose dual in $\mathcal{C}$ is preserved by every functor in $\mathcal{X}$, essentially by definition. (iv) is well known; see [Barr-Wells, 1984], Exercise (FILT) of Section 4.4. (v) follows from the fact that the representables commute with all colimits in the functor category (Yoneda, again) and in the subcategory commute with all those whose colimit is preserved by the inclusion. (vi) is obvious. □

2.2 Theorem. *Let* $\mathcal{C}$ *be a regular category. Then so is* $\widetilde{\mathcal{C}}$*.*

Proof. Since this is the one new idea in this paper, we will do it carefully. We must show that in $\mathcal{X}$, if the square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ Z \arrow[r] & W \end{tikzcd}$$

is a pushout and if the top row is regular mono, so is the bottom row. From (iv) above each of the objects in the diagram is the colimit of the filtered diagram of all the representable objects that map to it. Suppose we begin with arrows from representables $R \longrightarrow X$, $S \longrightarrow Y$, and $T \longrightarrow Z$. Since the diagram of representables is filtered, we can in fact suppose the existence of an $S' \longrightarrow Y$ that factors both $R \longrightarrow X \longrightarrow Y$ and $S \longrightarrow Y$. In $\mathcal{R}$, factor the arrow $R \longrightarrow S'$ as $R \twoheadrightarrow R' \rightarrowtail S'$ with the first arrow an epic and the second a regular monic. It follows from (iii) that this is also an epic/regular monic factorization in $\mathcal{X}$ as well. Then from the diagram

$$\begin{tikzcd} R \arrow[r, two heads] \arrow[dd] & R' \arrow[d] \\ & S' \arrow[d] \\ X \arrow[r, tail] & Y \end{tikzcd}$$

we get an arrow $R' \longrightarrow X$. The composite $R' \longrightarrow X \longrightarrow Z$ and replace $R \longrightarrow X$ by $R_0 \longrightarrow X$, a node later in the diagram. We may and do replace R by R_0. Similarly, we can suppose that there is a $T' \longrightarrow Z$ that factors both $R' \longrightarrow X \longrightarrow Z$ and $T \longrightarrow Z$.

Then given $R \longrightarrow X$, $S \longrightarrow Y$ and $T \longrightarrow Z$ and having made the above replacements, we may consider the following diagram, in which the outer square is a pushout and the map $U' \longrightarrow W$ is the unique one making all the squares commute.

$$\begin{tikzcd} R' \arrow[rrr, tail] \arrow[dr] \arrow[ddd] & & & S' \arrow[dl] \arrow[ddd] \\ & X \arrow[r, tail] \arrow[d] & Y \arrow[d] & \\ & Z \arrow[r] & W & \\ T' \arrow[rrr, tail] \arrow[ur] & & & U' \arrow[ul] \end{tikzcd}$$

Since colimits commute with pushouts and a filtered colimit of regular monos is a regular mono, the conclusion follows. □

2.3 Proposition. *Suppose every epi in $\mathcal{C}$ is regular. Then $\widetilde{\mathcal{C}}$ has the same property.*

Proof. We must show that every mono in $\mathrm{FL}(\mathcal{C}, \mathbf{Set})$ is regular. If $X \rightarrowtail Y$, we saw in the proof above that it is a colimit of monos in $\mathcal{R} \cong \mathcal{C}^{\mathrm{op}}$. But in that category, all monos are regular and it is evident that a filtered colimit of regular monos is a regular mono. □

2.4 Theorem. *Let $\mathcal{C}$ be a pretopos. Then so is $\widetilde{\mathcal{C}}$.*

Proof. The regularity follows from the preceding. We must show that if for $i = 1, \ldots, n$,

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y_i \arrow[d] \\ Z \arrow[r] & W_i \end{tikzcd}$$

is a pushout, then so is

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \prod Y_i \arrow[d] \\ Z \arrow[r] & \prod W_i \end{tikzcd}$$

The conclusions of Lemma 2 are still valid and (vi) may now be strengthened to

(vi) it if $\mathcal{C}$ is a pretopos, then $\mathcal{R}$ is a co-pretopos.

The argument is similar. Given $R \longrightarrow X$, for $i = 1, \ldots, n$, $S_i \longrightarrow Y$ and $T \longrightarrow Z$, we may, after suitable replacement, suppose that each $R \longrightarrow X \longrightarrow Y_i$ factors through the corresponding S_i and that $R \longrightarrow X \longrightarrow Z$ factors through T. Thus we can form pushout diagrams

$$\begin{tikzcd} R \arrow[r] \arrow[d] & S_i \arrow[d] \\ T \arrow[r] & U_i \end{tikzcd}$$

and the fact that $\mathcal{R}$ is a co-pretopos implies that

$$\begin{tikzcd} R \arrow[r] \arrow[d] & \prod S_i \arrow[d] \\ T \arrow[r] & \prod U_i \end{tikzcd}$$

is a pushout. Taking the colimit over all such diagrams and using the fact that filtered colimits commute with finite products, we draw the desired conclusion. □

2.5 Proposition. *Any regular category (resp. pretopos) can be fully embedded in a pretopos in which all epis (resp. finite epi families) are universal and regular.*

Proof. Simply take the category of sheaves for the topology of regular epis (resp. finite regular epi families). Then the least exact subcategory of the sheaf category which contains the original category will do. □

We therefore will suppose, whenever it is convenient, that every epi in a regular category (resp. every finite epi family in a pretopos) is universal and regular.

2.6 Definition. Let $\mathcal{C}$ be a full subcategory of $\widetilde{\mathcal{C}}$. An object P is said to be $\mathcal{C}$**-projective** if whenever $A \twoheadrightarrow B$ is an arrow in $\mathcal{C}$, then $\mathrm{Hom}(P, A) \longrightarrow \mathrm{Hom}(P, B)$ is surjective. An object is said to be $\mathcal{C}$-bf injective if it is $\mathcal{C}^{\mathrm{op}}$-projective in $\widetilde{\mathcal{C}}^{\mathrm{op}}$.

We make the trivial observation that when $\widetilde{\mathcal{C}} = \mathrm{FL}(\mathcal{C}, \mathbf{Set})^{\mathrm{op}}$ an object P is $\mathcal{C}$-projective if and only if as a functor it preserves regular epimorphisms. In fact, taking the variance into account, the Yoneda lemma says that $\mathrm{Hom}_{\widetilde{\mathcal{C}}}(P, C) = P(C)$.

2.7 Theorem. *Suppose $\mathcal{C}$ is a small, full subcategory of $\widetilde{\mathcal{C}}$ and the latter is complete, with finite colimits and filtered limits commute with finite colimits. Then each object of $\widetilde{\mathcal{C}}$ is covered by a $\mathcal{C}$-projective.*

Proof. We will prove this in the dual category, the formulation being more familiar. So we assume a category $\mathcal{X}$ and a full subcategory $\mathcal{R}$ with filtered colimits and show the existence of $\mathcal{R}$-injectives. We systematically use capital letters R, S, T to denote objects of $\mathcal{R}$ and X, Y, Z to denote those of $\mathcal{X}$.

2.8 Lemma. *Every object X of $\mathcal{X}$ has an embedding $X \rightarrowtail X^{\#}$ with the property that for every diagram*

$$\begin{tikzcd} R \arrow[r, tail] \arrow[d] & S \\ X \arrow[r] & X^{\#} \end{tikzcd}$$

there is an arrow $S \longrightarrow X^{\#}$ rendering the square commutative.

Proof. We define an ordinal sequence

$$X_0 \rightarrowtail X_1 \rightarrowtail X_2 \rightarrowtail \cdots X_\omega \rightarrowtail \cdots$$

as follows. Well order the diagrams of $\mathcal{R}$ of the form:

$$\begin{tikzcd} R \arrow[r, tail] \arrow[d] & S \\ X & \end{tikzcd}$$

Let $X_0 = X$; at a limit ordinal, let $X_\alpha = \mathrm{colim}\{X_\beta | \beta < \alpha\}$. To define $X_{\alpha+1}$, let $R \rightarrowtail S$ be the least element of the well ordering such that there is no arrow $S \longrightarrow X_\alpha$ for which

$$\begin{tikzcd} R \arrow[r, tail] \arrow[d] & S \\ X \arrow[r, tail] & X_\alpha \end{tikzcd}$$

can be made to commute. Then define $X_{\alpha+1}$ so that the square

$$\begin{tikzcd} R \arrow[r, tail] \arrow[d] & S \arrow[dd] \\ X \arrow[d] & \\ X_\alpha \arrow[r, tail] & X_{\alpha+1} \end{tikzcd}$$

is a pushout. Since $\mathcal{R}$ is small, the process must eventually stop and when it does, the final object clearly satisfies the conclusion. Of course the coregularity and exactness of filtered colimits insure that all the required maps remain mono. $\square$

Now we may return to the proof of Theorem 6. Define a sequence

$$X^0 \rightarrowtail X^1 \rightarrowtail X^2 \rightarrowtail \cdots X^*$$

by letting $X^0 = X$, and $X^{n+1} = X^{n\#}$ and $X^* = \operatorname{colim} X^n$. To see that X^* has the required property, it is clearly sufficient to show that if $f: R \longrightarrow X^*$ is a morphism whose domain lies in $\mathcal{R}$, then f factors through some X^n. But the colimit along a chain – or any filtered colimit – is preserved by the embedding of the left exact functor category into the category of all functors. Thus X^* is the colimit of the X^n even in the category of all functors. But in that category, Hom from representable functors commutes with arbitrary colimits. This is what is meant when one says that colimits (and limits) in a functor category are computed 'pointwise'. $\square$

This construction does more than what was promised. We use it to simplify the proof of the main embedding, although it is possible to avoid it.

2.9 Proposition. *If $X \longrightarrow X^*$ is as described in the proof above and if Y is an $\mathcal{R}$-injective, then any map $X \longrightarrow Y$ has an extension to X^*.*

Proof. We first prove that any such map can be extended to $X^\#$. But $X^\#$ is constructed from colimits of X_β along ordinal chains, so it is sufficient to extend to each link. At limit ordinals, X_β is constructed as a colimit, while the diagram

$$\begin{tikzcd} R \arrow[r, tail] \arrow[d] & S \arrow[dd] \\ X \arrow[d] & \\ X_\alpha \arrow[r, tail] & X_{\alpha+1} \end{tikzcd}$$

is defined to be a pushout. Assuming we have a map $X_\alpha \longrightarrow Y$ and its restriction to R can, by the $\mathcal{R}$-injectivity of Y, be extended to S, the universal mapping property of the pushout gives us a map defined on $X_{\alpha+1}$. □

The simplest way to think of this construction is as being a generalization of the construction of the algebraic closure of a field. The algebraic closure is injective with respect to algebraic extensions, but no other. And in fact, this observation will lead in Section 4 to an example that shows that the existence of injectives in such categories cannot be expected in general.

3 The embedding theorem

3.1 Theorem. *Every small regular category has a full embedding into a set-valued functor category that preserves finite limits and regular epimorphisms.*

Proof. We will describe a full subcategory $\mathcal{P} \subseteq \widetilde{\mathcal{C}}$ with the property that the functor $\Phi{:}\,\mathcal{C} \longrightarrow \mathrm{Func}(\mathcal{P}^{\mathrm{op}}, \mathbf{Set})$ defined by $\Phi(C)(P) = P(C) = \mathrm{Hom}_{\widetilde{\mathcal{C}}}(P, C)$ has the required properties. First, the fact that all functors in $\widetilde{\mathcal{C}}$ preserve finite limits implies that Φ does. A necessary and sufficient condition that Φ preserve regular epis is that every functor in $\mathcal{P}$ be $\mathcal{C}$-projective, so that we will allow only such functors into $\mathcal{P}$. To get faithfullness, it will be sufficient that each object of $\mathcal{C}$ is the target of a regular epimorphism from at least one object of $\mathcal{P}$. For in that case, given a monic $C' \rightarrowtail C$ in $\mathcal{C}$ that is not an isomorphism, any regular epimorphism $P \twoheadrightarrow C$ cannot factor through C' so that $P(C') \longrightarrow P(C)$ is not an isomorphism.

For each object C of $\mathcal{C}$, let $P_C \xrightarrow{e_C} C$ be a $\mathcal{C}$-projective cover of C as in Theorem 2.7 and let $Q_C \underset{b_C}{\overset{a_C}{\rightrightarrows}} P_C$ be a $\mathcal{C}$-projective cover of the kernel pair of $P_C \longrightarrow C$. Thus there is a coequalizer $Q_C \rightrightarrows P_C \longrightarrow C$ in $\mathcal{C}$. Let $\mathcal{P}$ be the full subcategory consisting of all the P_C and Q_C. Then we need show only that the embedding $\mathcal{C} \longrightarrow \mathrm{Func}(\mathcal{P}, \mathbf{Set})$ is full. Suppose that $\phi{:}\,\Phi(C) \longrightarrow \Phi(C')$ is a natural transformation. This means that there is given, for each object P of $\mathcal{P}$ a function $\phi P{:}\,\mathrm{Hom}(P, C) \longrightarrow \mathrm{Hom}(P, C')$. Naturality means that for $g{:}\,P' \longrightarrow P$

$$\begin{tikzcd}
\mathrm{Hom}(P,C) \arrow[r, "\mathrm{Hom}(g,P)"] \arrow[d, "\phi P"'] & \mathrm{Hom}(P',C) \arrow[d, "\mathrm{Hom}(g,P')"] \\
\mathrm{Hom}(P,C') \arrow[r, "\phi P'"'] & \mathrm{Hom}(P',C')
\end{tikzcd}$$

commutes. Aplying this to an $h: P \longrightarrow C$, this says that $\phi(h \circ g) = \phi(h) \circ g$. We apply this to the diagram $Q_C \underset{b_C}{\overset{a_C}{\rightrightarrows}} P_C \xrightarrow{e_C} C$, which tells us that

$$\phi(e_C) \circ a_C = \phi(e_C \circ a_C) = \phi(e_C \circ b_C) = \phi(e_C) \circ b_C$$

The coequalizer then implies the existence of a unique arrow $f: C \longrightarrow C'$ such that $\phi(e_c) = f \circ e_C$. Now suppose $k: P \longrightarrow C$ is arbitrary with P an object of $\mathcal{P}$. From Proposition 2.9, it follows that there is an $l: P \longrightarrow P_C$ such that $e_C \circ l = k$ and then $\phi(k) = \phi(e_C \circ l) = \phi(e_C) \circ l = f \circ e_C \circ l = f \circ k$. Thus ϕ is just composition with f. □

4 Example

Let $\mathcal{C}$ be the category of those rings which are finite products of fields of characteristic 0 generated by a finite number of elements, i.e. simple extensions of fields of finite transcendence degree over the rational numbers. We denote the sum in this category by $\widetilde{\otimes}$. The first thing we must do is to see how it relates to the ordinary tensor product, which is the sum in the category of commutative rings.

4.1 Lemma. *If $B \longleftarrow A \longrightarrow C$ are morphisms of $\mathcal{C}$, then $B \otimes_A C \longrightarrow B \widetilde{\otimes}_A C$ is monic.*

Proof. We first observe that if we write $A = k_1 \times k_2 \times \cdots \times k_n$, where the k_i are fields, then each of B and C splits up into a product of k_i algebras and the tensor product commutes with that decomposition. As a matter of fact, it will follow from this lemma that $\mathcal{C}^{\mathrm{op}}$ is a pretopos and this procedure dualizes what happens to a map into a finite sum in a pretopos. Thus we can reduce the question to the case in which $A = k$ is a field. Then we may suppose that

$$B = k(x_1, x_2, \ldots, x_n)[\alpha] \text{ and } C = k(y_1, y_2, \ldots, y_m)[\beta]$$

with $x_1, \ldots, x_n$ and $y_1, \ldots, y_m$ independent transcendentals, while α and β are algebraic, resp., over the preceding transcendentals.

The argument can now be reduced, using associativity of tensor product and the fact that tensoring over a field is exact, to the following observations:

1. $k(x_1, x_2, \ldots, x_n) \widetilde{\otimes}_k k(y_1, y_2, \ldots, y_m) \cong k(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m)$;
2. $k[\alpha] \widetilde{\otimes}_k k(y_1, y_2, \ldots, y_m) \cong k(y_1, y_2, \ldots, y_m)[\alpha]$;
3. $k[\alpha] \widetilde{\otimes}_k k[\beta]$ is the cartesian product of a finite number of field extensions of k.

This last observation is standard in the theory of separable field extensions. Its failure for inseparable extensions is the reason we have restricted ourselves to characteristic 0. □

Note that in cases 2 and 3 above, $\widetilde{\otimes} = \otimes$.

4.2 Proposition. *Every monomorphism of $\mathcal{C}$ rings is universal and regular.*

Proof. Given $A \rightarrowtail B$, form the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & B/A \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B \arrow[r] \arrow[d] & B \otimes B \arrow[r] \arrow[d] & B \otimes B/A \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B/A \arrow[r] \arrow[d] & B/A \otimes B \arrow[r] \arrow[d] & B/A \otimes B/A \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

in which the second and third row are formed by tensoring the top row with B and B/A, respectively and similarly for the columns. The top row is exact by definition and flatness insures that the second and third rows are. A diagram chase shows that then the upper left corner is a pullback, from which it is clear that

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d] \\
B \arrow[r] & B \otimes B
\end{tikzcd}$$

is a pullback as well, which means that $A \rightarrowtail B$ is regular. Finally, if $A \longrightarrow C$ is an arbitrary map of $\mathcal{C}$, the flatness of C as an A module forces $C \rightarrowtail B \otimes C$ as well. $\square$

The following theorem was found by John Kennison, to whom many thanks.

4.3 Theorem. $\mathrm{FL}(\mathcal{C}^{\mathrm{op}}, \mathbf{Set})$ *is equivalent to a full subcategory of von Neumann regular rings of characteristic* 0 *which contains all fields of characteristic* 0*.*

Proof. Every left exact functor $T{:}\,\mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ is given by a filtered colimit of representable functors. So let $T = \operatorname{colim}(\operatorname{Hom}(-, R_i))$, where each R_i is a finite product of finitely generated fields. Let $R = \operatorname{colim} R_i$ in the category of regular rings. Since this is a finitary equational category and the diagram is filtered, this colimit is simply the union and is, in fact, the colimit in the category of rings. I claim that for F a product of finitely generated fields,

$$\operatorname{colim}(\operatorname{Hom}(F, R_i)) \longrightarrow \operatorname{Hom}(F, R)$$

is an isomorphism. If $F = \mathbf{Q}(x_1, \ldots, x_n)[\alpha]$ is a field, this is a standard argument since every homomorphism to R takes each of the x_i to some R_j and by directedness there is some R_j that contains the image of all of them, along with the image of α. But a regular ring that contains the image of an invertible element also contains its inverse. This remark applies not only to the x_i and α, but to all rational functions in these elements. If now F is a finite product, repeat the above argument with each of the finitely many primitive idempotents. This shows that each functor is represented by a commutative regular ring. As for natural transformations between functors, it is clear that each ring homomorphism induces one. For the converse, it is evident that if R and S are two von Neumann regular rings, each of which is a filtered union of subrings which are products of finitely generated fields, then a coherent family of homomorphisms on those subrings extends to a unique homomorphism between the rings. Finally every field of characteristic 0 is in the category, since it is the union of its finitely generated subfields. $\square$

A commutative von Neumann regular ring not in the category is given by an infinite power of a field, say $\mathbf{Q}^{\mathbf{N}}$. Only the subset of functions $\mathbf{N} \longrightarrow \mathbf{Q}$ of finite range belong to finitely generated extensions. At any rate, we can now conclude,

4.4 Corollary. *The category* $\mathrm{FL}(\mathcal{C}^{\mathrm{op}}, \mathbf{Set})$ *has no non-zero injective.*

Proof. For if $k \longrightarrow K$ is an inclusion of fields, no map $k \longrightarrow P$ can be extended to K unless the latter is smaller than P. By first taking a putative injective P, we then take $\mathbf{Q} \longrightarrow K$ where K is a field larger than P. Since there is always a map $\mathbf{Q} \longrightarrow K$ ($\mathbf{Q}$ is initial in the category), this shows that P cannot be injective. $\square$

5 Embedding conditions

Although the embedding has already been established, it is worth exploring more general conditions that allow one to infer that a restricted Yoneda embedding is full and faithful. We begin with faithfulness.

5.1 Theorem. *Let $\mathcal{X}$ be a category and $\mathcal{P}$ be a small full subcategory of $\mathcal{X}$. Then the 'restricted' Yoneda embedding* $\mathcal{X}^{\mathrm{op}} \longrightarrow \mathrm{Func}(\mathcal{P}, \mathbf{Set})$ *is faithful if and only if every object of $\mathcal{X}$ is the target of an epimorphic sieve whose domains are in $\mathcal{P}$.*

Proof. Consider, for each object X of $\mathcal{X}$ the largest sieve: the family $\{P \longrightarrow X\}$ of all maps to X with domain in $\mathcal{P}$. This is an epi family if and only if for any two distinct maps $f, g\colon X \longrightarrow Y$, there is at least one $h\colon R \longrightarrow X$ with $fh \neq gh$. But this is exactly the same condition as that the images of f and g remain distinct in $\mathrm{Func}(\mathcal{P}, \mathbf{Set})$. $\square$

In any category $\mathcal{D}$, we say that a sieve $\{f_i\colon D_i \longrightarrow D\}$ is a **regular epimorphic sieve** if, given any object D' and family of arrows $\{g_i\colon D_i \longrightarrow D'\}$ such that for any object E and any pair of arrows $h\colon E \longrightarrow D_i$ and $k\colon E \longrightarrow D_j$, $f_i \circ h = f_j \circ k$ implies that $g_i \circ h = g_j \circ k$, then there is a unique $g\colon D \longrightarrow D'$ for which $g \circ f_i = g_i$. We say it is **universal** if given any $D' \longrightarrow D$ there is a family of commutative squares

$$\begin{tikzcd} D'_i \arrow[r] \arrow[d] & D' \arrow[d] \\ D_i \arrow[r] & D \end{tikzcd}$$

for which $\{D'_i \longrightarrow D'\}$ is a regular epimorphic sieve.

This amounts to the statement that

$$\mathrm{Hom}(D, D') \longrightarrow \prod \mathrm{Hom}(D_i, D') \rightrightarrows \prod \mathrm{Hom}(E, D')$$

is an equalizer, where the second product is indexed by the (possibly large) family of all pairs (h, k) as in the definition. If $\mathcal{G}$ is a generating family in $\mathcal{D}$, so that for every object E, there is an epimorphic family $\{G_k \longrightarrow E\}$, which implies that $\mathrm{Hom}(E, D') \rightarrowtail \prod \mathrm{Hom}(G_k, D')$ is monic, then we conclude that

$$\mathrm{Hom}(D, D') \longrightarrow \prod \mathrm{Hom}(D_i, D') \rightrightarrows \prod \mathrm{Hom}(G, D')$$

is also an equalizer, where the second product is indexed by all pairs (h, k) whose common domain lies in $\mathcal{G}$. Thus we conclude:

5.2 Proposition. *Suppose $\mathcal{D}$ is a category and $\mathcal{G}$ is a generating set. Then in order that $\{D_i \longrightarrow D\}$ be a regular epimorphic sieve it is sufficient that given any object D' and family of arrows $\{g_i\colon D_i \longrightarrow D'\}$ such that for any object $G \in \mathcal{G}$ and any pair of arrows $h\colon G \longrightarrow D_i$ and $k\colon G \longrightarrow D_j$, $f_i \circ h = f_j \circ k$ implies that $g_i \circ h = g_j \circ k$ then there is a unique $g\colon D \longrightarrow D'$ for which $g \circ f_i = g_i$.*

5.3 Theorem. *Let $\mathcal{D}$ be a category with pullbacks and $\mathcal{G}$ be a full subcategory of $\mathcal{D}$. Then the 'restricted' Yoneda embedding $\Phi\colon \mathcal{D} \longrightarrow \mathrm{Func}(\mathcal{G}^{\mathrm{op}}, \mathbf{Set})$ is full and faithful if every object of $\mathcal{D}$ is the target of a universal regular epimorphic sieve whose domains are in $\mathcal{G}$.*

Proof. Let $\phi\colon \Phi D \longrightarrow \Phi D'$ be a natural transformation. This means that for all $g\colon G \longrightarrow D$ with G an object of $\mathcal{G}$, we have $\phi(g)\colon G \longrightarrow D'$. Naturality means that for any $h\colon G' \longrightarrow G$ with G' also an object of $\mathcal{G}$, $\phi(g \circ h) = \phi(g) \circ h$. Now suppose $\{g_i\colon G_i$

$\longrightarrow D\}$ is a universal regular epic sieve. Then we have a family $\{\phi(g_i): G_i \longrightarrow D'\}$ and for any object G of $\mathcal{G}$, if $h: G \longrightarrow G_i$, $k: G \longrightarrow G_j$ is any pair of morphisms,

$$\phi(g_i) \circ h = \phi(g_i \circ h) = \phi(g_j \circ k) = \phi(g_j) \circ k$$

so that there is a unique $f: D \longrightarrow D'$ such that $\phi(g_i) = f \circ g_i$.

We must still show that $\phi(g) = fg$ for all $h: G \longrightarrow D$. According to the definition of universality, there is a family of squares

$$\begin{tikzcd} G_i' \arrow[r, "g_i'"] \arrow[d, "h_i"'] & G \arrow[d, "h"] \\ G_i \arrow[r, "g_i"'] & D \end{tikzcd}$$

in which the family $\{G_i' \longrightarrow G\}$ is a (regular) epimorphic family. Then for each i,

$$\phi(h) \circ g_i' = \phi(h \circ g_i') = \phi(g_i \circ h_i) = \phi(g_i) \circ h = f \circ g_i \circ h = f \circ h \circ g_i'$$

from which we conclude that $\phi(h) = f \circ h$ as required. $\square$

This gives an alternate proof of Theorem 3.1 that does not make use of 2.9

6 Intersections

One of the interesting, but heretofore unutilized properties of the full embedding of [Barr, 1971] is the fact that the functor preserved arbitrary intersections. In this section, we explore this condition.

A natural monomorphism $\alpha: F \rightarrowtail G$ of left exact functors is said to be an elementary embedding if whenever $A \rightarrowtail B$,

$$\begin{tikzcd} FA \arrow[r] \arrow[d] & FB \arrow[d] \\ GA \arrow[r] & GB \end{tikzcd}$$

is a pullback.

6.1 Example. If $F = h^D$ and $G = h^C$ are representable, then a natural transformation $F \longrightarrow G$ is induced by a map $C \longrightarrow D$. The transformation is mono if and only if the inducing map is epi. We claim the transformation is an elementary embedding if and only if the inducing map is a strong epi. For the definition of strong epi is that $C \longrightarrow D$ is a strong epi if and only if any square

$$\begin{tikzcd} C \arrow[r] \arrow[d] & D \arrow[d] \\ A \arrow[r, tail] & B \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{Hom}(D,A) \arrow[r] \arrow[d] & \mathrm{Hom}(D,B) \arrow[d] \\ \mathrm{Hom}(C,A) \arrow[r, tail] & \mathrm{Hom}(C,B) \end{tikzcd}$$

is a pullback.

Let F be a left exact functor on the left exact category $\mathcal{C}$ and A be an object of $\mathcal{C}$. If $a \in FA$, and A_0 is a subobject of A, then we say that A_0 bf admits a if there is an element $a_0 \in FA_0$ which maps to a under the function $FA_0 \longrightarrow FA$ induced by the inclusion. Since F is left exact, it preserves monos, and hence a_0 is unique if it exists. If one distinguishes monos form subobjects (a mono represents a subobject), we can legitimately say that $a \in FA_0$. Consider the set of all subobjects of A which admit a. If that collection of subobjects has an intersection then we say that intersection is the bf support of a.

If A_0 is the support of a, we do not usually expect A_0 to admit a.

6.2 Theorem. *Let $\mathcal{C}$ be a left exact category and $F{:}\,\mathcal{C} \longrightarrow \mathbf{Set}$ a left exact functor. Then of the following conditions,*

(i) *F is a filtered colimit of elementarily embedded representable functors;*

(ii) *F is a filtered colimit of representable functors in which the transition morphisms are elementary embeddings;*

(iii) *for every object A of $\mathcal{C}$, every element of $a \in FA$ has a support and that support admits a;*

(iv) *F preserves all intersections.*

(i) $\Rightarrow$ (ii); (ii) $\Rightarrow$ (iii) provided every morphism can be factored as a strict epi followed by a mono; (iii) $\Rightarrow$ (i) and (iv); and, if subobject lattices are complete, (iv) $\Rightarrow$ (iii).

Note that if subobject lattices are complete, then strict epi/mono factorizations exists and all four conditions are equivalent. Simply take the intersection of all subobjects through which the map factors.

Proof. (i)$\Rightarrow$(ii): This follows easily from the fact that if the outer square and right hand square of

$$\begin{tikzcd}
\mathrm{Hom}(D,A) \arrow[r, tail] \arrow[d] & \mathrm{Hom}(C,A) \arrow[r, tail] \arrow[d] & FA \arrow[d] \\
\mathrm{Hom}(D,B) \arrow[r, tail] & \mathrm{Hom}(C,B) \arrow[r, tail] & FB
\end{tikzcd}$$

are pullbacks, so is the left hand square.

(ii)$\Rightarrow$(iii) in the presence of the factorization: Let F be a colimit as described in the statement. Consider an element $a \in F(A)$, represented by a morphism $A_i \longrightarrow A$, where A_i is one of the nodes in the colimit. The map $A_i \longrightarrow A$ factors through a least subobject $A_0 \subseteq A$. If $A_j \longrightarrow A_i$ is a map in the colimit diagram, the induced map on the representable functors is an elementary embedding, which implies, as already observed, that the map is a strict epi. But then $A_j \longrightarrow A_i \longrightarrow A$ has the same image as $A_i \longrightarrow A$, which means that A_0 is the least subobject of A which admits a.

(iii)$\Rightarrow$(i): Let $h^A \longrightarrow F$ be a node in a diagram of which F is the colimit. This represents an element of $a \in F(A)$ which has a support A_0. I claim that the induced $h^{A_0} \longrightarrow F$ is an elementary embedding. In fact, if $g\colon B \rightarrowtail C$ is a mono, we must show that

$$\begin{tikzcd}
\mathrm{Hom}(A_0,B) \arrow[r, tail, "F(-)(a)"] \arrow[d, tail, "\mathrm{Hom}(A_0,g)"'] & FB \arrow[d, tail, "F(g)"] \\
\mathrm{Hom}(A_0,C) \arrow[r, tail, "F(-)(a)"'] & FC
\end{tikzcd}$$

is a pullback. Let $f \in \mathrm{Hom}(A_0,C)$ and $b \in F(B)$ such that $F(f)(a) = F(g)(b)$. Form the pullback

$$\begin{tikzcd}
A_1 \arrow[r] \arrow[d] & B \arrow[d, "g"] \\
A_0 \arrow[r, "f"'] & C
\end{tikzcd}$$

and apply F to get a pullback

$$\begin{tikzcd}
FA_1 \arrow[r] \arrow[d] & FB \arrow[d, "F(g)"] \\
FA_0 \arrow[r, "F(f)"'] & FC
\end{tikzcd}$$

But then the existence of the elements $a \in A_0$ and $b \in B$ with $F(f)(a) = F(g)(b)$ implies that $a \in FA_1$. Since we assumed that A_0 was the support of a, this implies that $A_0 = A_1$ which means that f factors through B, just what is needed.

(iii)$\Rightarrow$(iv): Consider an intersection $A_0 = \bigwedge(A_i)$ of subobjects of A. The map $FA_0 \longrightarrow \bigwedge F(A_i)$ is clearly monic. If $a \in FA_i$ for each i, then the support of a is included in each A_i, hence in their intersection A_0. But then $a \in FA_0$.

(iv)$\Rightarrow$(i) if the subobject lattices are complete: Consider an element $a \in FA$. Since subobject lattices are complete, we can form $A_0 = \bigwedge\{A_i \mid a \in A_i\}$. Since F preserves intersections, $a \in FA_0$. It is clear that $a \notin FA_1$ for any proper subobject $A_1 \subseteq A_0$, so that A_0 is the support of a. $\square$

6.3 Theorem. *Let $\mathcal{C}$ be a regular category, and $\mathcal{R}$ be the image of $\mathcal{C}^{\mathrm{op}}$ in $\mathcal{X} = \mathrm{FL}(\mathcal{C}, \mathbf{Set})$. Then if the object X of $\mathcal{X}$ preserves intersections, so does X^*.*

Proof. We begin by assuming, as we may from Proposition 5, that in $\mathcal{C}$ all epis are regular. It follows from Theorem 3 that all monos in $\mathcal{X}$ are regular. But regular monos are strict, so all monos are elementary embeddings. It is sufficient to show that the property of preserving intersections is preserved by the passage from X_α to $X_{\alpha+1}$ and by colimits along monomorphic chains. The latter condition will do both for the passage to X_α and the one to X^α when α is a limit ordinal. As for the first step, let that $X_\alpha = \operatorname{colim} R_i$, with each R_i an elementarily embedded subobject from $\mathcal{R}$. Let

$$\begin{tikzcd}
R \arrow[r, tail] \arrow[d] & S \arrow[dd] \\
X \arrow[d] & \\
X_\alpha \arrow[r, tail] & X_{\alpha+1}
\end{tikzcd}$$

be the pushout that defines $X_{\alpha+1}$. The map $R \longrightarrow X_\alpha$ factors through some R_i. The diagram may be replaced by the subdiagram consisting of all nodes beyond R_i. Let

$$\begin{tikzcd}
R \arrow[r, tail] \arrow[d] & S \arrow[d] \\
R_i \arrow[r, tail] & S_i
\end{tikzcd}$$

be a pushout. For $i < j$, let $S_i \longrightarrow S_j$ be defined so that the lower square in

$$\begin{tikzcd}
R \arrow[r, tail] \arrow[d] & S \arrow[d] \\
R_i \arrow[r, tail] \arrow[d] & S_i \arrow[d] \\
R_j \arrow[r, tail] & S_j
\end{tikzcd}$$

commutes. Since the outer and upper squares in this diagram are pushouts, so is the lower square. Since $R_i \rightarrowtail R_j$, the same is true of $S_i \longrightarrow S_j$. This verifies the finite step. Now let $X_\alpha = \lim_{\beta\langle\alpha} X_\beta$ in which $\beta < \gamma$ implies that $X_\beta \rightarrowtail X_\gamma$, which means it is an elementary embedding. Let $\{R_i\}$ be a set of subobjects of R and R_0 be their intersection. It is immediate that $X_\alpha(R_0) \rightarrowtail \bigwedge_i X_\alpha(R_i)$. Let $a \in X_\alpha(R)$ such that $a \in X_\alpha(R_i)$ for all i. Then fix an i and choose $\beta < \alpha$ such that $a \in X_\beta(R_i)$. For any $R_j \subseteq R_i$, there is a γ, which may be assumed less than β such that $a \in X_\gamma(R_j)$. Since

$$\begin{tikzcd}
X_\beta(R_j) \arrow[r] \arrow[d] & X_\gamma(R_j) \arrow[d] \\
X_\beta(R_j) \arrow[r] & X_\gamma(R_j)
\end{tikzcd}$$

is a pullback, it follows that $a \in X_\beta(R_j)$ for all j for which $R_j \subseteq R_i$. Thus $a \in \bigwedge_i X_\alpha(R_i)$. $\square$

7 Pretoposes

There would seem to be a regular progression: a small abelian category has a full, exact embedding into a module category and a small regular category has a full regular embedding into a functor category. The next step would seem to be a Theorem that embeds a pretopos near exactly into a functor category. No such theorem is possible, as shown by examples below. The first such example is the one credited to Makkai. We do show is that every pretopos $\mathcal{E}$ has a near exact embedding into a category in which there is a regular epimorphic cover by a family of $\mathcal{E}$ projectives. Only the universality is lacking.

There seems to be some confusion as to whether 'pretopos' includes the hypothesis of effective equivalence relations. One source of this confusion is in [Makkai-Reyes, 1977] in which, on page 122, a pretopos is defined to have quotients of equivalence relations. On page 117, a quotient of an equivalence relation is defined so that the equivalence relation is required to be the kernel pair, but the definition on page 122 does not point out this non-standard usage, so potential for confusion is evident. Thus to set the record straight, a pretopos is required to have effective equivalence relations. Nonetheless, none of the results of this paper depend on this hypothesis. I know that to be the case because I wrote it under the misapprehension engendered by the Makkai-Reyes paper.

Let $\mathcal{E}$ be a pretopos and $\mathcal{A}$ be the opposite of $\mathrm{FL}(\mathcal{E}, \mathbf{Set})$. Then $\mathcal{A}$ is also a pretopos by Theorem 4.

7.1 Lemma. *For A an object of $\mathcal{A}$, $\mathrm{Hom}(A, -)$ preserves finite sums if and only if A is not decomposable into a sum of two subobjects.*

Proof. If $f: A \longrightarrow B_1 + B_2$, then the universality of sums allows us to write $A = A_1 + A_2$, when A_i is defined by letting

$$\begin{tikzcd} A_i \arrow[r] \arrow[d] & A \arrow[d] \\ B_i \arrow[r] & B_1 + B_2 \end{tikzcd}$$

be a pullback for $i = 1,\ 2$. If A is indecomposable, it must be that one the A_i is 0 and the other is A which means that f factors through one of the summands and that

$$\mathrm{Hom}(A, B_1 + B_2) \cong \mathrm{Hom}(A, B_1) + \mathrm{Hom}(A, B_2)$$

To go the other way, let $A \cong A_1 + A_2$ with neither summand 0. A non-zero object of a functor category has an element defined over at least one representable functor, so there is, for $i = 1,\ 2$, and object Y_i and a non-trivial morphism $e_i: A_i \longrightarrow E_i$. Then $e = e_1 + e_2$ belongs to $\mathrm{Hom}(A, B_1 + B_2)$, but not to either of $\mathrm{Hom}(A, B_1)$ or $\mathrm{Hom}(A, B_2)$. $\square$

Remark. It is important to observe that A may be indecomposable even when $\mathrm{Hom}(A, -)$ does not commute with infinite sums. For an example in the dual of the category of commutative rings, observe that a map from a product of fields to a field factors through one of the direct factors, but there is no no need for this happen with an infinite product. To make this argument work with infinite products as well, we would have to suppose that, in addition, infinite sums were universal.

7.2 Lemma. *If $P = P_1 + P_2$, then P is $\mathcal{E}$-projective if and only if both P_1 and P_2 are.*

Proof. Let $P = P_1 + P_2$ be $\mathcal{E}$-projective. Consider a diagram

$$\begin{tikzcd} & P_1 \arrow[d] \\ A \arrow[r] & B \end{tikzcd}$$

with A and B in $\mathcal{E}$. Unless $P_2 = 0$, in which there is nothing to prove, there is an object C of $\mathcal{E}$ and a morphism $h\colon P_2 \longrightarrow C$. Then in the diagram

$$\begin{tikzcd} & P_1 + P_2 \arrow[d] \\ A + C \arrow[r] & B + C \end{tikzcd}$$

the projectivity of $P = P_1 + P_2$ guarantees the existence of a morphism $k\colon P_1 + P_2 \longrightarrow A + C$ that makes the triangle commute. Now in the diagram

$$\begin{tikzcd} P_1 \arrow[d] \arrow[rrd, "f"] & & \\ P_1 + P_2 \arrow[rd, "k"'] & A \arrow[r] \arrow[d] & B \arrow[d] \\ & A + C \arrow[r] & B + C \end{tikzcd}$$

the fact that the square is a pullback (in a pretopos) gives the required map $P_1 \longrightarrow A$. The converse is trivial. □

Now for an $\mathcal{E}$-projective object P, let $\mathrm{Bool}(P)$ denote the poset of complemented subobjects of P.

7.3 Lemma. $\mathrm{Bool}(P)$ *is a boolean algebra.*

Proof. If $P = P_1 + P_2 = P_3 + P_4$, then the universality of sums implies that also

$$P = (P_1 \wedge P_3) + (P_1 \wedge P_4) + (P_2 \wedge P_3) + (P_2 \wedge P_4)$$

from which it is easily seen that both $P_1 \wedge P_3$ and $P_1 \vee P_3$ are complemented. Since the complement of a complemented object as well as the least and greatest subobjects are evidently complemented, the conclusion follows. □

Now let $\mathbf{u}$ be an ultrafilter on $\mathrm{Bool}(P)$. Let $P_{\mathbf{u}} = \lim\{P_i \mid P_i \in \mathbf{u}\}$.

7.4 Theorem. *For any $\mathcal{E}$-projective P, and any ultrafilter $\mathbf{u}$ in $\mathrm{Bool}(P)$, $P_{\mathbf{u}}$ is an $\mathcal{E}$-projective indecomposable.*

Proof. It follows from the dual of Lemma 2(v) that when E is an object of $\mathcal{E}$, and $D: \mathcal{I} \longrightarrow X$ is a cofiltered diagram, then

$$\mathrm{colim}(\mathrm{Hom}(Di, E)) \cong \mathrm{Hom}(\lim Di, E)$$

Consequently, for E in $\mathcal{E}$, $\mathrm{Hom}(P_{\mathbf{u}}, E) \cong \mathrm{colim}(\mathrm{Hom}(P_i, E))$, the colimit taken over the $P_i \in \mathbf{u}$. Since from Lemma 19 each P_i is $\mathcal{E}$-projective and evidently a colimit of epis is epi, it is evident that $P_{\mathbf{u}}$ is also $\mathcal{E}$-projective. To show that $P_{\mathbf{u}}$ is indecomposable, consider a morphism $f: P_{\mathbf{u}} \longrightarrow A + B$ where A and B are objects of $\mathcal{E}$. From the above, it is represented by an arrow $P_i \longrightarrow A + B$, for some $P_i \in \mathbf{u}$. Then as in the proof of Lemma 18, we can decompose $P_i = P_1 + P_2$ where $f|P_1$ factors through A and $f|P_2$ factors through B. But the characteristic property of ultrafilters is that exactly one of P_1 and P_2 belongs to $\mathbf{u}$. If it is P_1 that belongs, then in the colimit f and $f|P_1$ represent the same element of $\mathrm{Hom}(P_{\mathbf{u}}, A + B)$ and the latter belongs to $\mathrm{Hom}(P_{\mathbf{u}}, A)$. Thus by the converse of Lemma TK, P is indecomposable. $\square$

7.5 Theorem. *The canonical map $\sum P_{\mathbf{u}} \longrightarrow P$, the sum taken over all the ultrafilters in $\mathrm{Bool}(P)$, is epic.*

Proof. Since the objects of $\mathcal{E}$ cogenerate, it is sufficient to show that given two maps $f, g: P \longrightarrow A$, with A an object of $\mathcal{E}$, there is an ultrafilter $\mathbf{u}$ on $\mathrm{Bool}(P)$ such that $\mathrm{Hom}(P_{\mathbf{u}}, f) \neq \mathrm{Hom}(P_{\mathbf{u}}, g)$. To see this, observe that for any decomposition $P = P_1 + P_2$, either $f|P_1 \neq g|P_1$ or $f|P_2 \neq g|P_2$ (or both). $\{i \mid f|P_i \neq g|P_i\}$ is clearly the dual of an ideal in $\mathrm{Bool}(P)$ and hence contains an ultrafilter $\mathbf{u}$ with the property that whenever $P_i \in \mathbf{u}$, $f|P_i \neq g|P_i$. Since two morphisms in a filtered colimit are equal if and only if they become equal at some stage, it follows that $\mathrm{Hom}(P_{\mathbf{u}}, f) \neq \mathrm{Hom}(P_{\mathbf{u}}, g)$. $\square$

From Propositions 3 and 5 above, we may suppose that this epi is, in fact, regular from which it follows that if P is a cover of an object X, so is $\{P_{\mathbf{u}} \mid \mathbf{u} \in \mathrm{Bool}(P)\}$. This epi is not universal, however, as we see in the next section.

8 Bounded pretoposes

We say that a pretopos is bf bounded is it has a full, near exact embedding into a set-valued functor category. In this section, we will investigate some of the properties of bounded pretoposes. In particular, we will show that if the pretopos has countable sums, then such an embedding is not only exact, but in fact preserves all colimits. If the pretopos is a Grothendieck topos, the embedding has a right adjoint and is therefore the left adjoint part of a geometric morphism.

In this section, we suppose that $\mathcal{E}$ is a bounded pretopos and that $\Phi\colon \mathcal{E} \longrightarrow \mathbf{Set}^{\mathcal{C}}$ is a full, near exact embedding. We begin with an exercise in boolean algebras which is left to the reader.

8.1 Proposition. *Let B' and B be lattices and $f\colon B' \longrightarrow B$ be a bijective increasing function. Suppose that B is a (complete) boolean algebra. Then so is B' and f is an isomorphism of those algebras.*

8.2 Proposition. *Let $\mathcal{E}$ be a bounded topos. Then for every object E of $\mathcal{E}$,* $\mathrm{Bool}(E)$ *is a complete atomic boolean algebra. If $f\colon E' \longrightarrow E$ is a morphism of $\mathcal{E}$, then* $\mathrm{Bool}(f)$ *is a morphism of complete atomic boolean algebras.*

Proof. It follows from the preceding proposition and

$$\mathrm{Hom}(E, 2) \cong \mathrm{Hom}(\Phi E, \Phi 2) \cong \mathrm{Hom}(\Phi E, 2)$$

that $\mathrm{Bool}(E) \cong \mathrm{Bool}(\Phi E)$. Similarly,

$$\begin{tikzcd}
\mathrm{Hom}(E,2) \arrow[r, "\cong"] \arrow[d, "\mathrm{Hom}(f,2)"'] & \mathrm{Hom}(\Phi E,2) \arrow[d, "\mathrm{Hom}(\Phi f,2)"] \\
\mathrm{Hom}(E',2) \arrow[r, "\cong"'] & \mathrm{Hom}(\Phi E',2)
\end{tikzcd}$$

commutes. Hence it is sufficient to prove that in a functor category $\mathbf{Set}^{\mathcal{C}}$, the complemented subobject lattice is a complete atomic boolean algebra. But the forgetful functor $\mathbf{Set}^{\mathcal{C}} \longrightarrow \mathbf{Set}^{\mathrm{Ob}(\mathcal{C})}$ creates all limits and colimits and may easily be seen to preserve the lattice operations in the subfunctor lattices. A subfunctor is complemented if and only if its complement in the latter lattice is a subfunctor. Thus any inf or sup of complemented subfunctors is complemented. □

8.3 Proposition. *For any object E of $\mathcal{E}$,* $\mathrm{Bool}(E)$ *is a complete sublattice of* $\mathrm{Sub}(E)$.

Proof. The argument above shows that the assertion is true when $\mathcal{E}$ is a functor category. That is, the union and intersection of complemented subobjects it is complemented. Let $\bigvee$ and $\bigcup$ denote the supremum operation in subobject lattices in $\mathcal{E}$ and $\mathbf{Set}^{\mathcal{C}}$, respectively and temporarily let Sup denote the operation in the complemented subobject lattices in $\mathcal{E}$. Then we have, for an object E of $\mathcal{E}$ and a collection $\{E_i\}$ of subobjects of E,

$$\bigcup \Phi E_i \subseteq \Phi(\bigvee E_i) \subseteq \Phi(\mathrm{Sup} E_i) = \bigcup \Phi E_i$$

The last equality is from Proposition 23. □

We say that a pretopos is bf molecular if every object is the union of indecomposable objects and the sum is universal. We say it is bf effectively molecular if every object is the sum of its indecomposable subobjects and that it is bf universally effectively molecular if those sums are universal. It is clear that in a topos, the last two concepts coincide and that in a Grothendieck topos, all three do.

8.4 Corollary. *Every bounded topos is universally effectively molecular.*

Proof. We must show that if E is an object of $\mathcal{E}$ and $\{E_i\}$ is the set of atoms of $\mathrm{Bool}(E)$, then $E \cong \sum E_i$. Let $\Phi: \mathcal{E} \longrightarrow \mathbf{Set}^{\mathcal{C}}$ be a full embedding. Then from the construction above it is clear that $\Phi E \cong \sum \Phi E_i$. If for each i, $f_i: E_i \longrightarrow F$ is given, there is a unique map $g: \Phi E \longrightarrow \Phi F$ such that $g|\Phi E_i = \Phi f_i$. Since Φ is full and faithful, there is a unique map $h: E \longrightarrow F$ such that $\Phi h = g$. The universality also follows immediately from that of the functor category. □

8.5 Theorem. *Let $\mathcal{E}$ be a bounded topos. Then a near exact functor $\Phi: \mathcal{E} \longrightarrow \mathbf{Set}^{\mathcal{C}}$ preserves all sums. If $\mathcal{E}$ has countable sums, then Φ is exact. If $\mathcal{E}$ is a Grothendieck topos, then Φ is a left adjoint of a geometric morphism $\mathbf{Set}^{\mathcal{C}} \longrightarrow \mathbf{Set}$.*

Proof. Let $E = \sum E_i$. If the E_i are atoms, the preceding corollary gives the conclusion. For the general case, write $E_i = \sum E_{ij}$. Then $E = \sum E_{ij}$ and this sum is preserved by Φ. Since also $\Phi E_i = \sum_j \Phi E_{ij}$, it follows easily that $\sum_{i,j} \Phi E_{ij} = \Phi E$. It is well known [Freyd, 1972] that a near exact functor is exact as soon as countable sums exist and are preserved. Finally, the special adjoint functor theorem gives a right adjoint as soon as all colimits are preserved. □

Any small pretopos can be fully embedded in a Grothendieck topos in which all universal sums are preserved. Simply form the category of sheaves for the least topology that includes all finite epi families and in which universal sums are covered by their summands. Although the Grothendieck topos is no longer small, it is bf essentially small [Barr-Wells, 1984], Exercise (UNIV) of Section 7.3, and we can work with it as though it were small. In particular, it is bounded as soon as the original category is. For any functor into a topos that preserves the covers will extend into a left adjoint of a geometric morphism.

9 Sufficient conditions for boundedness

A lattice is called bf noetherian if every ascending chain is finite. Such a lattice is evidently sup and hence inf complete. A lattice is bf co-heyting if finite sups distribute over arbitrary infs.

9.1 Theorem. *A small pretopos in which the subobject lattices are noetherian and co-heyting is bounded.*

Proof. We begin the proof with:

9.2 Lemma. *A noetherian co-heyting lattice has a complete embedding into a power set. In particular, such a lattice is completely distributive.*

Proof. Let L be such a lattice and consider two elements a and b of L for which $b \not\le a$. Among all ideals of $I \subseteq L$ with $a \in L$ and $b \notin L$—there is at least one, namely $a \wedge L$—let I be maximal. The noetherian condition on L insures that all ideals are principal, so that I is the principal ideal generated by an element p. Like every principal ideal it is closed under arbitrary sups. I claim its complement is closed under arbitrary infs. In fact, if $\bigwedge x_i \in I$, then from the co-heyting hypothesis,

$$\bigwedge x_i = p \vee \bigwedge x_i = \bigwedge (p \vee x_i)$$

But maximality of I means that if none of the x_i belong to I, $b \in p \vee x_i$ for all i, whence $b \in \bigwedge x_i$. a contradiction. Thus both I and its complement are complete, so that the 2-valued homomorphism of which I is the kernel is a morphism of complete lattices. Since such an ideal exists whenever $b \not\le a$, the set of such morphisms gives a complete embedding of L into a power set from which complete distributivity follows. □

9.3 Proposition. *Let $\mathcal{C}$ satisfy the hypotheses of the theorem. Then every cover in $\widetilde{\mathcal{C}}$ of an object of $\mathcal{C}$ has a finite refinement.*

Proof. Let A be an object of $\mathcal{C}$ and $\{P_i \longrightarrow A\}$ be a cover, i.e. a regular epimorphic family. Let $P_i = \lim B_{ij}$, a limit of representables taken over a filtered index category J_i. By replacing, if necessary, the index category by final segments, we can suppose that for each i, j there is given a map $g_{ij}\colon B_{ij} \longrightarrow A$ which represents $P_i \longrightarrow A$. Let A_{ij} denote the image of g_{ij}. Let c be a 'choice function' which chooses for each i an object $c(i)$ in J_i. We must have $\bigvee_i A_{i,c(i)} = A$. For otherwise that union would be a subobject of A which evidently contains the image of every $P_i \longrightarrow A$. Thus,

$$A = \bigwedge_c \bigvee_i A_{i,c(i)} = \bigvee_i \bigwedge_{j \in J_i} A_{ij}$$

, the latter equality being the complete distributive law. But the noetherian condition implies that there is a finite set of indices, say $i = 1, 2, \ldots, n$ such that

$$A = \bigvee_{i=1}^{n} \bigwedge_{j \in J_i} A_{ij}$$

from which it is evident that $P_1, P_2, \ldots, P_n$ cover A. □

We can now return to the proof of Theorem 28. From Theorem 6, there is a cover $P \twoheadrightarrow A$, with P $\mathcal{C}$-projective. Form Theorem 14, this can be replaced be a cover $\sum P_{\mathbf{u}} \twoheadrightarrow A$ with each $P_{\mathbf{u}}$ $\mathcal{C}$ projective and indecomposable. From the preceding proposition, it follows that the sum can be replaced by a finite sum. From Theorem 4 and Proposition 5, the finite family $\{P_{\mathbf{u}} \longrightarrow A\}$ is universal and it then follows from Theorem 14 that if $\mathcal{A}$ is the small exact subcategory generated by $\mathcal{C}$ and by enough $\mathcal{C}$-projective indecomposable functors to resolve the objects of $\mathcal{C}$, then the induced functor $\mathcal{C} \longrightarrow \mathbf{Set}^{\mathcal{A}}$ is full and faithful. □

9.4 Corollary. *A pretopos in which subobject lattices are finite is bounded.*

Proof. A finite, distributive lattice is completely distributive. □

10 Examples

Let X be any topological space which is T1, but not discrete. Then it is known that the points of $\mathrm{Sh}(X)$, the topos of sheaves on X, are exactly the stalks at the points of X. But being T1, there are no natural transformations between stalks at different points and no non-trivial endomorphisms of the individual stalk functors. In other words, the category of points is discrete. But the category of sheaves on a non-discrete space cannot be fully embedded into a discrete functor category by a functor that preserves sums. For one thing, the commutative diagram

$$\begin{tikzcd} \mathrm{Bool}(Y) \arrow[r] \arrow[d] & \mathrm{Sub}(Y) \arrow[d] \\ \mathrm{Bool}(\Phi Y) \arrow[r] & \mathrm{Sub}(\Phi Y) \end{tikzcd}$$

in which Φ is the functor and Y is a sheaf, consists of all monos and both the left hand and bottom arrows would be isomorphisms, whence the other two would be as well. But then the subobject lattices would be boolean, contradicting the assumption that the space is not discrete. Note that the space may be locally connected, thus showing that a molecular topos need not be bounded.

Here is another interesting example of the same thing which is instructive in other ways as well. Consider the category of sheaves on the open unit interval (0,1) (or equivalently, on the real line, but the open interval is a bit more convenient). Suppose we take for $\mathcal{E}$ the category of sheaves for which there is a uniform finite upper bound on the number of elements in each stalk. This is evidently the least exact full subcategory containing the space itself. Here is a projective over the space. Take the sequence of spaces of which the first is the interval (0,1), the second is the sum of the two intervals (0,2/3) and (1/3,1), the third is the sum of four intervals (0,4/9), (2/9,2/3), (1/3,7/9) and (5/9,1), etc. At each stage, divide each interval of the preceding stage into two overlapping intervals, each 2/3 the length of the previous ones. If these spaces are denoted

$$X_1 \longleftarrow X_2 \longleftarrow X_3 \longleftarrow \cdots$$

then $P = \mathrm{colim}(\mathrm{Hom}(X_i, -)$ is a projective functor since if $Y \twoheadrightarrow Z$ is a surjection of covers and some $X_i \longrightarrow X$ is given, it will, after suitable subdivision lift to $X_j \longrightarrow X$. On the other hand, an ultrafilter on $\mathrm{Bool}(P)$ is determined a point of (0,1) and the corresponding limit is the stalk at the point. It is known from sheaf theory that the only points of the category $\mathrm{Sh}(X)$ are given by the stalks at a point, when the space is sober.

Any near exact functor into **Set** is actually exact, since every relation generates an equivalence relation after a finite number of steps. For a set of n elements has only 2^{n^2} relations on it and hence every reflexive symmetric relation on such a set generates an equivalence relation after at most that many steps and the same is true of a sheaf in which each stalk has at most n elements. In any case, such a functor preserves covers and thus extends to the left adjoint of a geometric morphism on the category of sheaves, i.e. a point.

But the category of stalks is discrete (there are no morphisms when the space is hausdorff) and if the category of sheaves were bounded, and we would have a full embedding Φ of $\mathrm{Sh}(X)$ into a power of **Set**. In the latter category, every subobject is complemented. We have a commutative diagram,

$$\begin{tikzcd}
\mathrm{Bool}(Y) \arrow[r] \arrow[d] & \mathrm{Sub}(Y) \arrow[d] \\
\mathrm{Bool}(\Phi Y) \arrow[r] & \mathrm{Sub}(\Phi Y)
\end{tikzcd}$$

in which every arrow is mono and the left hand and bottom arrows are isomorphisms, which implies that the other two are as well. But then every subobject in $\mathrm{Sh}(X)$ would be complemented, which is not the case. On the other hand, $\mathrm{Sh}(X)$ is molecular, since X is locally connected [Barr-Paré, 1980].

Here is an example due to Makkai. It was given to show that not every pretopos is bounded at a time when that seemed like a plausible conjecture. Let $\mathcal{E}$ be a countable model of set theory with the axiom of choice, e.g. the standard model. If $\mathcal{E}$ were bounded, $\mathcal{E}$ would have to be molecular. But the only molecules are singletons, so that $\mathbf{N}$, for example, would have to be the sum of countably many copies of 1. But there are uncountably many ways of mapping such a sum to 1, so that is impossible.

11 Prime generated pretoposes

We say that an object in a pretopos is a prime if it is not the union of two proper subobjects. We say that a pretopos is prime generated if every object has a regular cover by primes.

11.1 Theorem. *A coherent prime generated pretopos is bounded.*

Proof. Every coherent object has a finite cover by primes. Hence it is sufficient to show that for every prime in $\mathcal{C}$, there is a $\mathcal{C}$-projective indecomposable P for which there is a regular epi $P \twoheadrightarrow A$. Begin by finding a $P \twoheadrightarrow A$ with a $\mathcal{C}$-projective P. In any decomposition $P = P_1 + P_2$, I claim that either $P_1 \twoheadrightarrow A$ or $P_2 \twoheadrightarrow A$. For suppose neither of these holds. Write $P_i = \lim B_{ij}$ for $i = 1,\ 2$. Then for any object C of $\mathcal{C}$,

$$\mathrm{Hom}(A, C) \longrightarrow \mathrm{Hom}(P_i, C) \cong \operatorname{colim} \mathrm{Hom}(B_{ij}, C)$$

is not monic, which means at least one $B_{ij} \longrightarrow A$ is not epi. Since this can be done for $i = 1,\ 2$, the result is that for some indices $j,\ k$, neither B_{1j} nor B_{2k} is a regular epi, which means, since A is prime, that $B_{1j} + B_{2k} \longrightarrow A$ factors through some proper subobject of A which means that $P \longrightarrow A$ does as well. Now we can apply the method of proof of Theorem 22. $\{P_i \in \mathrm{Bool}(P) \mid P_i \twoheadrightarrow A\}$ contains an ultrafilter $\mathbf{u}$ with the property that when $P_i \in \mathbf{u}$, $P_i \twoheadrightarrow A$. The result is that since filtered limits in $\mathcal{X}$ preserve finite colimits, $P_{\mathbf{u}} \twoheadrightarrow A$ as well. □

11.2 Corollary. *If $\mathcal{C}$ is a pretopos in which every object is the finite union of prime subobjects, then $\mathcal{C}$ is bounded.*

An bf atomic topos [Barr-Diaconescu, 1980] is a topos in which every object is a sum of irreducible subobjects. It is not entirely clear what the definition of atomic pretopos should be, but in the coherent case there is no doubt that every object should be the finite sum of such subobjects.

11.3 Corollary. *A small coherent atomic pretopos is bounded.*

Proof. Since an object in a coherent atomic topos is the finite universal sum of its atoms and its atoms are evidently primes, the conclusion follows from Theorem 33. □

11.4 Corollary. *A coherent Grothendieck atomic topos is bounded.*

Proof. We can always take a small subcategory which is a pretopos and contains all the atoms. An embedding into a functor category can be extended in a unique way to all sums of atoms, which is what the atomic topos consists of. □

11.5 Remark. All the toposes shown by the theorems above to be bounded have completely distributive subobject lattices. This raises the question of whether all bounded toposes do. There does not seem to be any obvious reason to expect this, but I have not found any counterexample either. One approach to finding a counter-example comes down to this: Find a left exact idempotent cotriple on a functor category that does not preserve unions. For if $\mathbf{G}$ is such a cotriple and it does turn out to preserve unions, we have the following computation in which the E_{ij} are all $\mathbf{G}$-coalgebras,

$$\bigvee_i \bigwedge_j E_{ij} = \bigcup_i G(\bigcap_j E_{ij}) = G(\bigcup_i \bigcap_j E_{ij}) = G(\bigcap_c \bigcup_i E_{i,c(i)}) = \bigwedge_c \bigvee_i E_{i,c(i)}$$

which means the subobject lattices are completely distributive.

However, the only examples I can think of of idempotent cotriples involve all functors that take a class of cocones (directed to get left exactness) to colimits. But then the cotriple evidently preserves unions.

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