Top$^{op}$ is a quasi-variety

Michael Barr*and M. Cristina Pedicchio

June 11, 1999

Abstract

We show that the opposite of the category of topological spaces is a quasi-variety, that is a subobject and product closed subcategory of a varietal category. We identify the varietal category as well as the simple Horn clause that determines the objects of the subcategory.

1 Introduction

The category of topological spaces is usually thought rather poorly of qua category. Although it is complete and cocomplete and the underlying set functor even has both adjoints, the free and cofree functors produce spaces without interesting structure and the triple and cotriple on Set produced by the adjoints are the identity. The category is far from exact, or even regular. Thus the properties of topological spaces seem rather far removed from those involved in the usual equational theories.

Thus it came as some surprise to us to discover that the situation is quite different when it comes to the dual category. It turns out that that category is at least regular (not that hard to prove, once you suspect it), although not exact and is, in fact quasi-varietal. In this paper we show that it is quasi-varietal, identify the variety and also the simple Horn clause that distinguishes the algebras that are the duals of topological spaces.

2 Quasi-varieties

Definition. Recall that a varietal category or variety is one that is tripleable over the category of sets. A quasi-variety is a full subcate-

*In the preparation of this paper I have been assisted by grants from the NSERC of Canada and the FCAR du Québec
category of a varietal category that is closed under subobjects and products. Equivalently, it is a surjective-reflective, or regular-epi-reflective subcategory of a variety.

The following theorem is found in [Pedicchio, to appear].

2.1 Theorem  A category $\mathcal{E}$ is quasi-varietal if and only if

QV–1. $\mathcal{E}$ is regular;
QV–2. $\mathcal{E}$ has coequalizers of equivalence relations;
QV–3. $\mathcal{E}$ has a regular projective generator $P$ such that arbitrary (small) sums of copies of $P$ exist in $\mathcal{E}$.

A regular projective generator is understood to be both regular projective and a regular generator. It is shown in [Barr, 1989] that such a quasi-variety is a full subcategory of a variety consisting of those objects that satisfy a class of generalized Horn clauses that take the form

$$\left( \bigwedge (\phi_i(x) = \psi_i(x)) \Rightarrow (\phi(x) = \psi(x)) \right)$$

where the $\phi_i$, $\psi_i$, $\phi$ and $\psi$ are operations in the theory and the conjunction may be infinite. These clauses are found by imposing the antecedent equations as an equivalence relation on a free algebra and the consequent is an additional equation necessary to reflect the quotient algebra into the subcategory (of which there may be many; each such gives an additional Horn clause).

As an application, we can see that opposite of the category of topological spaces satisfies these conditions and is thus a quasi-variety. We need a regular injective cogenerator in the category of topological spaces. The simplest such space is the space we call $P$ that has three points, say $a$, $b$ and $c$. Aside from $P$ and $\emptyset$, the only open set is $\{a,b\}$. If $X$ is a topological space, a map $f : X \rightarrow P$ is determined by three subsets $A = f^{-1}\{a\}$, $B = f^{-1}\{b\}$ and $C = f^{-1}\{c\}$. We must have $A \cup B \cup C = X$ so that such a map is uniquely determined by giving two sets $A$ and $B$. Continuity is equivalent to $U = A \cup B$ being open. Thus a continuous map $X \rightarrow P$ determines a pair $(U, A)$, where $U \subseteq X$ is open and $A \subseteq X$ is arbitrary. Conversely, it is clear that such a pair determines a unique continuous map that takes points of $A$ to $a$, points of $U - A$ to $b$ and all others to $c$.

There is no problem in proving directly that $P$ is a regular injective cogenerator, but it will also follow from the results of the next section.

3 Grids

Definition. Recall that a frame is a lattice with arbitrary supremums that are preserved by finite infimums. By a grid, we mean a frame with
an additional unary operation we denote $'$ satisfying some equations. The equations are best expressed using derived unary operations $x^\uparrow = x \lor x'$ and $x_1 = x \land x'$. The equations (in addition to the frame equations) are:

Gr–1. $u'' = u$.
Gr–2. $\uparrow$ and $\downarrow$ are $\lor$ homomorphisms.
Gr–3. $\uparrow$ is a $\land$ homomorphism, while $\downarrow$ satisfies $(u \land v)_{\downarrow} = u \land v_{\downarrow}$.
Gr–4. The interval $[u_1, u^\uparrow]$ is a complete atomic boolean algebra with the operations of $\lor$ and $'$.

This last requires some explanation to see why it is equational. First, if $v$ is arbitrary, let $v^u$ denote $(u_{\downarrow} \lor v)_{\land} u^\uparrow = u_{\downarrow} \lor (v \land u^\uparrow)$. Then $v^u \in [u_1, u^\uparrow]$ and $v^u = v$ if $v \in [u_1, u^\uparrow]$. It follows immediately that any sentence of the form $v \in [u_1, u^\uparrow] \Rightarrow \phi(v) = \psi(v)$ is equivalent to the equation $\phi(v^u) = \psi(v^u)$ and that is true for operations of any arity by replacing $v$ by any string of elements. Finally, a complete atomic boolean algebra is characterized by satisfying the complete distributive law which can be stated, if awkwardly, in terms of $\lor$ and $'$ as follows.

Let us denote by $\biglor^u$ and $\bigland^u$ the operations of the form

$$\biglor^u \left( \biglor_{i \in I} v_i \right) = \biglor_{i \in I} v_i^u$$

and

$$\bigland^u \left( \biglor_{i \in I} v_i \right) = \left( \biglor_{i \in I} (v_i^u)' \right)'$$

Then we want equations of the form $u^u \lor (v^u)' = u^\uparrow$, $v^u \land (v^u)' = v_{\downarrow}$ that force $[u_1, u^\uparrow]$ to be a boolean algebra, evidently complete, with $\lor$ is infinite join and, by duality, $\land$ its meet. Then the complete distributive law will state that for all sets $I$ and $J$ and $I \times J$ indexed families $v_{ij}$

$$\bigland_{i \in I} \biglor_{j \in J} v_{ij} = \biglor_{s: I \to J} \bigland_{i \in I} v_{i(s(j))}$$

This equation is imposed on the whole algebra, but of course is equivalent to the assumption that $[u_1, u^\uparrow]$ is a completely distributive complete boolean algebra, which is equivalent to its being atomic (see, for example, [Johnstone, 1982], VII.1.16, page 285).

We denote by $\textbf{Grid}$ the category of grids and homomorphisms.
3.1 Proposition A grid has the following properties:
1. \[ u_1' = u_1 \] and \[ u_1'' = u_1 \];
2. \[ u_1' = u_1 \] and \[ u_1'' = u_1 \];
3. \[ (u \land v)_1 = u_1 \land v_1 \];
4. \[ u_1 \downarrow_1 = u_1 \downarrow_1 = u_1 \uparrow_1 = u_1 \uparrow_1 \]

Proof.
1. Since \([u_1, u_1']\) is a boolean algebra and \(\cdot\) is the complement operation, the complement of the top element is the bottom and vice versa.
2. This is immediate since, for example, \[ u_1' = u_1' \land u_1'' = u_1' \land u = u_1 \].
3. \[ (u \land v)_1 = u \land v_1 \] and also \[ (u \land v)_1 = u_1 \land v \] so that \[ (u \land v)_1 = u_1 \land v_1 \] since evidently \(u_1 \leq u\) and \(v_1 \leq v\).
4. \[ u_1 \downarrow_1 = (u \land u_1')_1 = u_1 \land u_1' = u_1 \]
\[ u_1 \uparrow_1 = (u \lor u_1')_1 = u_1 \lor u_1' = u_1 \]
and similarly for the other two.

The following is true because \([u_1, u_1']\) is a boolean algebra.

3.2 Proposition Let \(G\) be a grid, \(u \in G\) and \(v, w \in [u_1, u_1']\).
Then
1. \( (v \land w)' = v' \lor w' \) and \( (v \lor w)' = v' \land w' \);
2. \( v_1 = u_1 \) and \( v_1 = u_1 \);

3.3 Corollary A grid \(G\) is partitioned by sets of the form \([u_1, u_1']\).

4 The main theorem

4.1 Theorem The category of topological spaces is dual to the full subcategory of grids defined by the Horn clause
\[ (u_1 \lor 1_1 = v_1 \lor 1_1) \Rightarrow (u_1 = v_1) \] (*)

Proof. Define \( \Phi : \text{Top}^{\text{op}} \rightarrow \text{Grid} \) by letting \( \Phi(X) \) be the set of all pairs \((U, A)\) where \(U\) is an open subset of \(X\) and \(A\) is an arbitrary subset of \(U\). The order relation is the restriction of the product order and both \(\lor\) and \(\land\) are coordinatewise. \((U, A)' = (U, U - A)\). The derived operations are \((U, A)_1 = (U, U)\) and \((U, A)_1 = (U, \emptyset)\). It is clear that
this is a grid and also satisfies the Horn clause. We define a functor 
$\Psi : \text{Grid}^{op} \to \text{Top}$. Suppose $G$ is a grid. Then $[1, 1]$ is a complete 
atomic boolean algebra whose set of atoms we denote by $X$. Then the 
interval $[1, 1]$ can be thought of as the set of subsets of $X$. We will 
use capitals to denote elements of $[1, 1]$. Say that $U \in [1, 1]$ is open 
if there is a $u \in G$ such that $u \uparrow \lor 1 = U$. It follows from the fact that 
$\uparrow$ commutes with $\lor$ that the union of open sets is open and from the 
fact that $\uparrow$ commutes with $\land$ and the distributivity that an intersection 
of two open sets is open. Thus we have a topology on $X$. The set $X$ 
with this topology is $\Psi(G)$. If $f : G \to G'$ is a grid homomorphism, 
then $f(1) = f(1) \downarrow 1 = 1 \downarrow 1$ so that $f$ takes the interval $[1, 1]$ of $G$ to the 
corresponding interval of $G'$. Moreover, since $f$ preserves $\lor$ and $\land$, 
it is a morphism of CABAs, which is induced by a function we denote 
$\Psi(f) : X' \to X$, the set of atoms of $[1, 1]$ in $G'$ and $G$, resp. Moreover, 
the duality of CABAs and sets is such that the inverse image function 
of $\Psi(f)$ is $f$ itself, so that showing that $\Psi(f)$ is continuous is equivalent 
to showing that $f$ takes open sets to open sets. But if $U = u \uparrow \lor 1$ is 
open in $X$, then $f(U) = f(u) \uparrow \lor 1$ is open in $X'$.

It is clear that $\Psi \circ \Phi \cong \text{Id}$ in any case. We finish the argument by 
letting $G$ be a grid that satisfies $(\ast)$ and showing that $\Phi(\Psi(G)) \cong G$. 
Let $X = \Psi(G)$. Define $\phi : G \to \Phi(X)$ by $\phi(u) = (u \uparrow 1, u \lor 1)$. First 
we show that $\phi$ is a grid morphism.

$$\phi(\bigvee_{i \in I} u_i) = ((\bigvee_{i \in I} u_i) \uparrow 1, (\bigvee_{i \in I} u_i) \lor 1)$$

$$= (\bigvee (u_i \uparrow 1), \bigvee (u_i \lor 1)) = \bigvee (u_i \uparrow 1, u_i \lor 1)$$

for arbitrary index sets $I$, and

$$\phi(u \land v) = ((u \land v) \uparrow 1, (u \land v) \lor 1)$$

$$= ((u \uparrow v) \lor 1, u \land v) \lor 1)$$

$$= ((u \lor 1) \land (v \lor 1), (u \lor 1) \land (v \lor 1))$$

$$= (u \lor 1, u \lor 1) \land (v \lor 1, v \lor 1))$$

$$= \phi(u) \land \phi(v)$$

To see that $\phi$ preserves $'$, we use the fact that $u'$ is the complement of $u$ in the lattice $[u, u']$ and show that $\phi(u')$ is the complement of $\phi(u)$.
in \([\phi(u)], \phi(u)]\). Then

\[
\phi(u) \lor \phi(u') = \phi(u \lor u') = \phi(u^1) \\
= (u^1 \lor 1, u^1 \lor 1) = (u^1 \lor 1, u^1 \lor 1) \\
= (u^1 \lor 1, u \lor 1)^1 = \phi(u)^1
\]
since that is how \(^\dagger\) works in \(\Phi(X)\).

\[
\phi(u) \land \phi(u') = \phi(u \land u') = \phi(u_1) = (u_1^1 \lor 1, u \lor 1) \\
= (u^1 \lor 1, (u \lor 1)) = (u^1 \lor 1, 1) \\
= (u^1 \lor 1, 0) = \phi(u)^1
\]

Thus \(\phi\) is a morphism of grids. I claim that \(\phi\) is an isomorphism. In fact, if \(\phi(u) = \phi(v)\), then \(u^1 \lor 1 = v^1 \lor 1\) which implies that \(u^1 = v^1\). Then

\[
(u \lor 1) \land u^1 = (u \land u^1) \lor (1 \land u^1) = u \lor (1 \land u^1) = u \lor u = u
\]

and similarly \((v \lor 1) \land u^1 = v\) so that \(u = v\) and \(\phi\) is monic. Let \((U, A) \in \Phi(X)\). Then \(U = u^1 \lor 1\) for some \(u \in G\), by definition of the topology on \(X\). Then \(\phi(A \land u^1) = ((A \land u^1) \lor 1, (A \land u^1) \lor 1)\). We have

\[
(A \land u^1) \lor 1 = (A^1 \land u^1) \lor 1 = (1 \land u^1) \lor 1 = u^1 \lor 1 = U
\]

and

\[
(A \land u^1) \lor 1 = (A \lor 1) \land (u^1 \lor 1) = A \land U = A
\]

Thus \(\phi\) is surjective.

**References**


M. C. Pedicchio (to appear), On \(k\)-permutability for categories of \(P\)-algebras.