$*$-autonomous categories, revisited

Michael Barr

1999-06-11

1 Introduction

This is to show that the category $\mathrm{Chu}_{se}(\mathscr{V},K)$ is equivalent to the category of reflexive objects in the category of linearly topologized vector spaces for the two notions of duality from [Barr, 1979]. To see this, we begin by defining, for an object A of $\mathscr{A}$, two extreme topologies on the same point set, one the least topology that has the same functionals as A and one the greatest such topology.

We assume without further mention that in any extensional object (V,V') of Chu_{se}, V' can be assumed to be a set of functionals. Also in any map $(f,f')\colon(V,V')\longrightarrow(W,W')$, $f' = -\circ f$.

Let $A^\sharp$ denote the space with the same underlying vector space as A and in which a linear subspace $U\subseteq A$ is open when every functional on $|A|$ that vanishes on U is continuous. If U_1 and U_2 are two such subspaces, it is easily seen that every functional on $|A|$ that vanishes on $U_1\cap U_2$ is the intersection of a functional that vanishes on U_1 and one that vanishes on U_2 and if these are individually continuous, so is their sum and thus we have a topology. When U is cofinite dimensional, then there are functionals ϕ_1, ϕ_2, $\ldots$, ϕ_n such that $U=\ker(\phi_1)\cap\ker(\phi_2)\cap\cdots\cap\ker(\phi_n)$ and if each of these functionals, which evidently vanish on U, is continuous, then U is open in A. Clearly the identity $A^\sharp\longrightarrow A$ is continuous.

Let $A^\flat$ denote the space with the same underlying vector space as A and in which a linear subspace $U\subseteq A$ is open if it is open in A and of finite codimension. This is clearly a topology and it is clear that it has the same continuous functionals as A and that the identity $A\longrightarrow A^\flat$ is continuous. In fact, any topology on the point set of A that has the same continuous functionals has a topology between those of $A^\flat$ and $A^\sharp$. We will sometimes refer to them as the weak and strong topologies, respectively.

Define a functor $F\colon\mathscr{A}\longrightarrow\mathrm{Chu}_{se}(\mathscr{V},K)$ by $FA=(|A|,\mathscr{V}(A,K))$. Define functors $R,L\colon\mathrm{Chu}_{se}(\mathscr{V},K)\longrightarrow\mathscr{A}$ as follows. $R(V,V')$ is the vector space V equipped with the weak topology for the functionals in V'. $L(V,V')=R(V,V')^\sharp$. The topology is such that U is open in $L(V,V')$ iff every linear functional on V that vanishes on U belongs to V'.

1.1 Proposition. *$L \dashv F \dashv R$ and the adjunction morphisms $LF \longrightarrow \mathrm{id} \longrightarrow RF$ are isomorphisms.*

Proof. Let (V,V') be an object of $\mathrm{Chu}_{se}(\mathscr{V},K)$. Suppose that A is an object of $\mathscr{A}$ and that $(f,f')\colon(|A|,\mathscr{V}(A,K)) \longrightarrow (V,V')$. Then $f\colon|A| \longrightarrow V$ and whenever $\phi \in V'$, $\phi \circ f$ is a continuous linear functional on A. Now let $U \subseteq R(V,V')$ be an open subspace. Then there are a finite number of functionals ϕ_1, ϕ_2, ..., ϕ_n such that $U = \ker(\phi_1) \cap \ker(\phi_2) \cap \cdots \cap \ker(\phi_n)$. Since all of $\phi_1 \circ f$, $\phi_2 \circ f$, ..., $\phi_n \circ f$ are continuous, it follows that

$$f^{-1}(U) = f^{-1}(\ker(\phi_1)) \cap f^{-1}(\ker(\phi_2)) \cap \cdots \cap f^{-1}(\ker(\phi_n))$$

is open in A. Thus f is continuous. Now suppose that $f\colon A \longrightarrow R(V,V')$ is a continuous linear map. Suppose that $\phi \in V'$. Then $\phi \circ f$ is a continuous functional on A so that $(f, - \circ f)\colon(|A|,\mathscr{V}(A,K)) \longrightarrow (V,V')$ is a map of $\mathrm{Chu}_{se}(\mathscr{V},K)$. This shows that $F \dashv R$.

Suppose that $(f,f')\colon(V,V') \longrightarrow (|A|,\mathscr{V}(A,K))$ is a map of $\mathrm{Chu}_{se}(\mathscr{V},K)$. Let U be an open linear subspace of A. Then A/U is discrete which means that every linear functional on $|A|$ that vanishes on U is in $\mathscr{V}(A,K)$. But then $f' = - \circ f$ takes every such functional into V' which means that every functional on V that vanishes on $f^{-1}(U)$ is in V' and hence that $f^{-1}(U)$ is open in $L(V,V')$. Now suppose that $f\colon L(V,V') \longrightarrow A$ is continuous. Let ϕ be a functional on A. Then $\phi \circ f$ is a continuous functional on V. Thus $\ker(\phi)$ is open in $L(V,V')$. Thus every linear functional on V that vanishes on $\ker(\phi)$, of which $\phi \circ f$ is one, belongs to V'. Thus $(f, - \circ f)\colon(V,V') \longrightarrow (|A|,\mathscr{V}(V,K))$ is a morphism. This shows that $L \dashv F$.

$LF \longrightarrow \mathrm{id}$ is an isomorphism iff F is full and faithful iff $\mathrm{id} \longrightarrow RF$ is an isomorphism, so it suffices to show that one of these holds. So let (V,V') be an object of $\mathrm{Chu}_{se}(\mathscr{V},K)$. Then every continuous linear functional ϕ on $L(V,V')$ has a kernel that is open, which means that every functional on V that vanishes on $\ker(\phi)$, of which ϕ is one, belongs to V' and so $V' = \mathscr{V}(L(V,V'),K)$. □

As a result of this, the full subcategories of the linearly topologized vector spaces consisting of the weakly topologized spaces as well as the full subcategory of strongly topologized spaces are equivalent to $\mathrm{Chu}_{se}(\mathscr{V},K)$ and hence to each other. Moreover, since $\mathrm{Chu}_{se}(\mathscr{V},K)$ is $*$-autonomous, so are these categories. In the case of the weakly topologized spaces, the topology on the dual spaces, as well as on the general homsets, is that of pointwise convergence. For the strongly topologized spaces, there seems to be no better description than to form the homsets with pointwise convergence and retopologize using the $()^\sharp$ operator.

2 Abelian topological groups

All groups considered are abelian, without further notice. We make the same assumptions as before. That in any object (V, V') of an extensional Chu space, we can suppose that V' is a set of characters on V and in any map (f, f') of such, $f' = - \circ f$.

Say a topological group is subcompact if it is isomorphic to a subgroup of a compact group. Say it is locally compactly represented (LCR) if it is isomprphic to a subgroup of a product of locally compact groups.

Call a topology weak if it is the weak topology for all the characters. Since a compact topological group is embedded in a product of circles (from Pontrjagin duality between compact and discrete groups) weak is actually equivalent to subcompact. Call an LCR topology strong if there is no stronger LCR topology that has the same set of characters.

2.1 Proposition. *The topology of every LCR group A can be strengthened to a strong topology that has the same set of characters.*

Proof. Let A be an LCR group. Let L be a locally compact group. Say that a group homomorphism $f\colon |A| \longrightarrow L$ is strongly continuous if for every character $\chi\colon L \longrightarrow K$, the composite $\chi \circ f$ is a (continuous) character on A. It is clear that every continuous homomorphism is strongly continuous. Thus if we let $A^\sharp$ denote the same underlying group topologized with the weak topology with respect to all strongly continuous homomorphisms into locally compact spaces, it is clear that $A^\sharp \longrightarrow A$ is continuous. It is also clear that $A^\sharp$ is LCR and that it has the same set of characters as A. Suppose, if possible, that A' is the same group topologized with an LCR topology that has the same set of characters as A and hence as $A^\sharp$. In order to show that $A^\sharp \longrightarrow A'$ is continuous, it is sufficient to show that for all locally compact L every $f\colon |A| \longrightarrow L$ that is continuous on A' is also continuous on $A^\sharp$. But the definition of $A^\sharp$ is such that in order for f to be continuous on $A^\sharp$ it is sufficient to show that whenever $\chi\colon L \longrightarrow K$ is a character on L, $\chi \circ f$ is a character on $A^\sharp$. But is f continuous on A', $\chi \circ f$ is a character on A' and $A^\sharp$ have the same characters, so the conclusion follows. □

For any LCR group A, let $A^\flat$ denote the same group with the weak topology for its characters. It is clear that this is a weak topology and is the weakest topology with the same characters as A. Thus we have $A^\sharp \longrightarrow A \longrightarrow A^f lat$. These two represent the range of LCR topologies that have the same characters as A.

Now let $\mathscr{V}$ denote the category of abelian groups with its usual closed monoidal structure and $\mathscr{A}$ the category of LCR topological abelian groups. Let K denote the circle group. Define a functor $F\colon \mathscr{A} \longrightarrow \mathrm{Chu}_{se}(\mathscr{V}, K)$ by $FA = (|A|, \mathscr{V}(A, K))$. Define functors $R, L\colon \mathrm{Chu}_{se}(\mathscr{V}, K) \longrightarrow \mathscr{A}$ as follows. $R(V, V')$ is the topological group V equipped with the weak topology for the characters in V'. $L(V, V') = R(V, V')^\sharp$.

2.2 Proposition. *The circle is injective in the category of LCR groups with respect to the class of topological embeddings.*

Proof. It is sufficient to show that homomorphisms can be extended across an embedding of the form $A \subseteq \prod_{i\in I} L_i$, where the L_i are locally compact. So suppose that $\chi\colon A \longrightarrow K$ is a character. let U be a open neighborhood of 0 in K that includes no non-zero subgroup. Then $\chi^{-1}(U)$ is open in A which means it is of the form $A \cap V$, where V is a open neighborhood of 0 in $\prod L_i$. Every neighborhood of 0 in the product includes a product $\prod_{i\in J} L_i$ where J is some cofinite subset of I. Let $B \subseteq A$ denote $A \cap \prod_{i\in J} L_i$. Then $\chi(B) \subseteq K$ is a subgroup and is thus 0. Then χ induces a character on B/A and that is a subgroup of the locally compact group $\prod_{i\in I-J} L_i$, which is a locally compact group. This reduces the question to the case of a subgroup A of a locally compact group L. Any character on A, like any continuous group homomorphism, is uniformly continuous and thus extends to the closure of A in L. This we can restrict to the case that A is a closed subgroup. But then A is locally compact and classical Pontrjagin duality provides the required extension. □

2.3 Theorem. *$L \dashv F \dashv R$ and the adjunction morphisms $LF \longrightarrow \mathrm{id} \longrightarrow RF$ are isomorphisms.*

Proof. Let $(f, f')\colon (|A|, \mathscr{V}(A,K)) \longrightarrow (V, V')$ be given. Then $f\colon |A| \longrightarrow V$ is a group homomorphism and whenever $\chi \in V'$, the composite $\chi \circ f$ is a character on A. When V is given the weak topology for all its characters, f becomes continuous, so that $f\colon A \longrightarrow R(V,V')$. To go the other way, we begin by noting that if χ is a character on $R(V,V')$, then If $f\colon A \longrightarrow R(V,V')$ is given, then any $\chi \in V'$ is a character on $R(V,V')$ so that $\chi \circ f$ is a character on A. Thus $(f, - \circ f)\colon (|A|, \mathscr{V}(A,K)) \longrightarrow (V,V')$ is a morphism in Chu_{se}. Thus $F \dashv R$. To see that $RF \cong \mathrm{id}$, we must show that the only characters on $R(V,V')$ are the elements of V'. Any character χ on V extends by injectivity of K to a character on $K^{V'}$ and, as seen in the proof of the injectivity of K, this character is defined mod a cofinite subset of V', which means that there is a finite set $\chi_1,\ \chi_2,\ \ldots,\ \chi_k$ of characters and a character ν on $K^{\chi_1} \times K^{\chi_2} \times \cdots \times K^{\chi_k}$ such that for any $a \in A$, $\chi(a) = \nu(\chi_1(a), \chi_2(a), \ldots, \chi_k(a))$. But ν is just a sequence $(n_1, n_2, \ldots, n_k)$ of integers and

$$\chi(a) = \nu(\chi_1(a), \chi_2(a), \ldots, \chi_k(a)) = n_1\chi_1(a) + n_2\chi_2(a) + \cdots + n_k\chi_k(a)$$

which shows that $\chi = n_1\chi_1 + n_2\chi_2 + \cdots + n_k\chi_k \in V'$.

Suppose that $f\colon L(V,V') \longrightarrow A$. Then $f\colon V \longrightarrow |A|$ is a group homomorphism and for any character $\chi\colon A \longrightarrow K$, $\chi \circ f$ is a character on $L(V,V')$. But $L(V,V')$ has the same characters as $R(V,V')$ and we have just seen that any character on $R(V,V')$ is in V'. Thus $(f, - \circ f)\colon (V,V') \longrightarrow FA$ is a morphism. If $(f,f')\colon (V,V') \longrightarrow FA = (|A|, \mathscr{V}(A,K))$ is given, then $f\colon V \longrightarrow A$ is continuous when V has the discrete topology. To see it remains continuous in the strong topology, we must show that for any locally compact L and $g\colon A \longrightarrow L$, the composite $g \circ f$ is continuous on $L(V,V')$ which is equivalent to its being strongly continuous on $R(V,V')$. So we must show that for any such L and g and any character χ on L, $\chi \circ g \circ f \in V'$. But extensionality implies that $f' = - \circ f$ and so the fact that $\chi \circ g$ is a character on A implies that $\chi \circ g \circ f = f'(\chi \circ g) \in V'$, as required. The fact that $RF \cong \mathrm{id}$ implies that $LF \cong \mathrm{id}$. □

It follows that the categories of strongly and weakly topologized linear spaces are equivalent to Chu_{se} and hence to each other. Since the latter is $*$-autonomous, so are the other two. The dualities are always the space of functionals, in one case with the weak topology and in the other with the strong topology as described above. The second dual of A for the strong duality is $A^\sharp$ and for the weak duality is $A^\flat$.

3 Topological abelian groups

We let **LC** denote the category of locally compact abelian groups. If L is a such a group, we let L^* denote the usual dual group of continuous homomorphisms into the circle K topologized by uniform convergence on compact subsets (the compact/open topology). Let **SPLC** denote the full subcategory of topological abelian groups that can be embedded topologically into a product of locally compact groups. (**SPLC** stands for subobject of product of locally compact.) We are going to introduce a pre-duality on **SPLC**. For A in **SPLC**, let A^* denote the set of continuous linear homomorphisms into K with the weak topology for all homomorphisms $A^* \longrightarrow L^*$ for all continuous homomorphisms $L \longrightarrow A$ with L locally compact. If A should be locally compact, then there is an initial element of the set, namely the identity of A^* and this topology is then just as defined above.

3.1 Proposition. *Every element of A^{**} is evaluation at a unique element of A and the resultant map $A^{**} \longrightarrow A$ is a continuous bijection, although not generally an isomorphism.*

Proof. Although A^* has the weak topology for a proper class of continuous homomorphisms, it is clear that there is a set $\{f_i\colon L_i \longrightarrow A\}$ of maps with locally compact domains such that A^* is embedded in $\prod L_i{}^*$. Let $\alpha\colon A^* \longrightarrow K$ be a continuous homomorphism. Let U be a neighborhood of 0 in K that includes no non-zero subgroup. Then $\alpha^{-1}(U)$ is open in A^* which means that there is a neighborhood $V \in \prod L_i{}^*$ such that $\alpha^{-1}(U) \subseteq V \cap A^*$; equivalently $\alpha(V \cap A^*) \subseteq U$. Since U contains no non-zero subgroup, it follows that if $B \subseteq V$ is any subgoup, then $\alpha(A^* \cap B) = 0$. Any neighborhood of 0 in $\prod L_i{}^*$ contains a subgroup of the from $B = \prod_{j\in J} L_j{}^*$ for some cofinite subset $J \subseteq I$. Thus $A^* \cap B \subseteq \ker \alpha$. Let $I_0 = I - J$. Then $A' = A^*/(A^* \cap B)$ is embedded in the locally compact $L = \prod_{i\in I_0} L_i{}^*$ and α factors as $\alpha' \circ \pi$ where $\pi\colon A^* \longrightarrow A'$ is the projection. Since K is complete, α' extends to a homomorphism α'' on the closure A'' of A' which is locally compact and, by the usual duality theory, extends to a homomophism $\beta\colon L \longrightarrow K$. Now $L = \prod_{i\in I_0} L_i{}^*$ so that $L^* = \sum_{i\in I_0} L_i$, which means that there are elements $x_i \in L_i$ such that $\beta(\chi_i) = \chi_i(x_i)$ for $i \in I_0$ and $\chi_i \in L_i{}^*$. For

$\chi \in A^*$,

$$\alpha(\chi) = \beta(\langle {f_i}^*(\chi)\rangle) = \sum f_i(\chi)(x_i) = \sum \chi(f_i(x_i)) = \chi(\sum f_i(x_i))$$

so that α is evaluation at $\sum f_i(x_i)$. This shows the surjectivity. For injectivity, it is sufficient to show that for any $x \in A$, there is some $\chi \in A^*$ with $\chi(x) \neq 0$. Let A be embedded in $\prod L_i$. Then there is some i such that the image of x in L_i is non-zero and some functional on L_i that does not vanish on that image.

In order that a map $B \longrightarrow A$ be continuous, it is necessary that for each $A \longrightarrow L$ with L locally compact, the composite $B \longrightarrow A \longrightarrow L$ be continuous. Applied to $A^{**} \longrightarrow A$, we see that for any $A \longrightarrow L$, we have $L^* \longrightarrow A^*$ is continuous and hence, so is $A^{**} \longrightarrow L^{**} = L$, which is the composite. Thus $A^{**} \longrightarrow A$ is continuous. $\square$

3.2 Proposition. *The circle K is injective in* **SPLC** *with repsect to topological embeddings.*

Proof. Let $A \subseteq B$ and suppose $B \subseteq \prod L_i$, with the L_i locally compact. Then also $A \subseteq \prod L_i$ and the argument of the preceding proposition shows that $\sum {L_i}^* = (\prod L_i)^* \longrightarrow A^*$ is surjective from which it is immediate that $B \twoheadrightarrow A$ as well. $\square$

We call a group A *reflexive* if $A^{**} \longrightarrow A$ is an isomorphism and denote by **RSPLC** the full subcategory of reflexive groups. It is clear that A^* is reflexive for any A in **SPLC** and that $A \mapsto A^{**}$ is a coreflector from **SPLC** to **RSPLC**.

We want to define an internal hom on **RSPLC**. We begin with one on **SPLC**. We define $A \multimap B$ to be the set of continuous homomorphisms topologized as a subspace of $B^{|A|} \times A^{*|B^*|}$. This means that $A \multimap B$ has the weak topology for all the evaluations at elements of A and for all valuations of the duals at elements of B^*.

3.3 Proposition. *Let* **Z** *be the discrete group of integers. Then for any A,* $\mathbf{Z} \multimap A \cong A$ *and* $A \multimap K \cong A^*$.

Proof. The set of homomorphisms $\mathbf{Z} \longrightarrow A$ can be identified with A; the only question is the topology. Since evaluation at $1 \in \mathbf{Z}$ is continuous, the topology is at least that of A. On the other hand, all evaluations are continuous on A and for all $\chi \in A^*$, the composite $A \longrightarrow \mathbf{Z}^{*|A^*|} \longrightarrow K$ is just χ, which is continuous, so that the given topology on A is no finer than the given one. Thus $\mathbf{Z} \multimap A \cong A$.

$A \multimap K$ is topologized with the weak topology for all the functionals as well as for all the maps $A^* \longrightarrow A^*$ of the form $A \xrightarrow{\chi} K \xrightarrow{n} K$, that is $\chi \mapsto n\chi$. In particular, $\chi \mapsto \chi$ must be continuous, so the topology is at least the given one and that one suffices to make all those maps continuous. For the first factor, that is a consequence of the fact that all $\mathbf{Z} \longrightarrow A$ are continuous and for the second it follows from the continuity of multiplication by n. $\square$

3.4 Proposition. *Suppose A and B are objects of* **RSPLC**. *Then $A \multimap B \cong B^* \multimap A^*$.*

Proof. They are clearly isomorphic groups. The first is topologized as a subspace of $B^{|A|} \otimes A^{*|B^*|}$ and the second as a subobject of the isomorphic $A^{*|B^*|} \times B^{**|A^{**}|}$. □

3.5 Proposition. *Suppose that A and B are reflexive. Then the canonical map $\mathrm{Hom}(A, B \multimap C^{**}) \longrightarrow \mathrm{Hom}(A, B \multimap C)$ is an isomorphism. Moreover, $(B \multimap C^{**})^{**} \longrightarrow (B \multimap C)^{**}$ is an isomorphism.*

Proof. First we observe that $B \multimap C^{**} \longrightarrow B \multimap C$ is a bijection since B is reflexive. A map $f\colon A \longrightarrow B \multimap C$ assigns to each element of $a \in A$ a continuous homomorphism $f(a)\colon B \longrightarrow C$ and does so continuously on A. Since B is reflexive, $f(a)\colon B \longrightarrow C^{**}$ is also continuous. In order to be continuous on A it must be that for all $b \in B$, the composite

$$A \xrightarrow{f} B \multimap C^{**} \xrightarrow{\mathrm{eval}_b} C^{**} \tag{a^{**}}$$

is continuous and for each $\gamma \in C^{***}$, the composite

$$A \xrightarrow{f} B \multimap C^{**} \xrightarrow{*} C^{***} \multimap B^* \xrightarrow{\mathrm{eval}_\gamma} B^* \tag{b^{**}}$$

be continuous. But A is reflexive so that a map $A \longrightarrow B^{**}$ is continuous if and only if the corresponding $A \longrightarrow B$ is. Also $B^{***} \longrightarrow B^*$ is an isomorphism for any B so that (a^{**}) and (b^{**}) are equivalent to

$$A \xrightarrow{f} B \multimap C \xrightarrow{\mathrm{eval}_b} C \tag{a}$$

and

$$A \xrightarrow{f} B \multimap C \xrightarrow{*} C^* \multimap B^* \xrightarrow{\mathrm{eval}_\gamma} B^* \tag{b}$$

Thus $\mathrm{Hom}(A, B \multimap C^{**}) \cong \mathrm{Hom}(A, B \multimap C)$ and then $(B \multimap C^{**})^{**} \cong (B \multimap C)^{**}$. □

3.6 Proposition. *Let A and B be objects of* **SPLC**. *Then there is a natural map $|B^*| \otimes |A| \longrightarrow |(A \multimap B)^*|$ and that map is surjective.*

Proof. If $\beta \in |B^*|$ and $a \in |A|$, then we let $\beta \otimes a$ be a character on $A \multimap B$ by the formula $(\beta \otimes a)f = \beta(f(a))$. By definition, $A \multimap B$ is embedded in $B^|A| \times A^{*|B^*|}$. The injectivity of K implies that there is a surjection

$$\sum_{|A|} B^* + \sum_{|B^*|} A^{**} \twoheadrightarrow (A \multimap B)^*$$

At the level of abelian groups this implies, taking into account that $|A^{**}| = |A|$, that

$$\sum_{|A|} |B^*| + \sum_{|B^*|} |A| \twoheadrightarrow |(A \multimap B)^*|$$

This means that for any $\chi\colon A \multimap B \longrightarrow K$, there are elements $a_1, \ldots, a_n \in A$, $\beta_1, \ldots, \beta_n \in B^*$, $\beta'_1, \ldots, \beta'_m \in B^*$ and $a'_1, \ldots, a'_m \in A$ such that for all $f\colon A \longrightarrow B$,

$$\chi(f) = \sum_{i=1}^{n} \beta_i(f(a_i)) + \sum_{j=1}^{m} \beta'_j(f(a'_j))$$

This means that χ is the image of the element

$$\sum_{i=1}^{n} \beta_i \otimes a_i + \sum_{j=1}^{m} \beta'_j \otimes a'_j$$

which completes the proof. $\square$

By the way, it is worth remarking that this map is not always injective. Here is an example. Let $\mathbf{Z}$ denote the group of integers with the usual topology and let $\mathbf{Z}$ denote the group of integers with the weak topology for all its characters. This is not discrete because $\widehat{\mathbf{Z}}$ is embedded into a compact group and no compact group can include an infinite discrete subgroup. Since the same is true of any infinite subgroup of $\widehat{\mathbf{Z}}$, it follows that any map of $\widehat{\mathbf{Z}} \longrightarrow \mathbf{Z}$ has finite range. But no non-zero subgroup of $\mathbf{Z}$ is finite, so that $\widehat{\mathbf{Z}} \multimap \mathbf{Z} = 0$ and also $(\widehat{\mathbf{Z}} \multimap \mathbf{Z})^* = 0$. On the other hand, $|\widehat{\mathbf{Z}}| \otimes |\mathbf{Z}| \cong \mathbf{Z}$ is non-zero.

3.7 Theorem. *For any objects A, B and C of* $\mathbf{SPLC}$ *with A and B reflexive, $A \multimap (B \multimap C)$ is isomorphic to $B \multimap (A \multimap C)$ by the map that interchanges variables and is evidently natural in all arguments.*

Proof. We will first show that they have the same elements and then that they have the same topology. An element $f \in A \multimap (B \multimap C)$ can be described as a function $f\colon |A| \times |B| \longrightarrow C$ that is bilinear, such that for each $a \in A$, the function $f(a,-)\colon B \longrightarrow C$ is continuous, such that for each $b \in B$ the function $f(-,b)\colon A \longrightarrow C$ is continuous and such that for each $\gamma \in C^*$ the function $A \longrightarrow B^*$ that takes $a \in A$ to $\gamma(f(a,-))$ is continuous. The first three conditions are symmetric between A and B and since they are reflexive, there is a one-one correspondence between continuous homomorphisms $A \longrightarrow B^*$ and continuous homomorphisms $B \longrightarrow A^*$. Thus the two groups have the same elements.

The topology on $A \multimap (B \multimap C)$ is as a subspace of

$$(B \multimap C)^{|A|} \times A^{*|(B \multimap C)^*|}$$

Since $B \multimap C \subseteq C^{|B|} \times B^{*|C^*|}$ and $A^{*|(B \multimap C)^*|} \subseteq A^{*|C^*|\otimes|B|}$ we see that $A \multimap (B \multimap C)$ is a subspace of

$$(C^{|B|} \times B^{*|C^*|})^{|A|} \times A^{*|C^*|\otimes|B|}$$

The condition imposed by the second term is that for any element $\sum \gamma_i \otimes b_i \in |C^*| \otimes |B|$, the map $A \mathrel{-\!\!\bullet} (B \mathrel{-\!\!\bullet} C) \longrightarrow A^*$ defined by $f \mapsto \sum \gamma_i(f(-, b_i))$ is continuous. Since these are topological groups, addition is continuous and it is equivalent to suppose that for all pairs $\langle \gamma, b \rangle \in |C^*| \times |B|$, the map $A \mathrel{-\!\!\bullet} (B \mathrel{-\!\!\bullet} C) \longrightarrow A^*$ given by $f \mapsto \gamma(f(-, b))$ be continuous. The upshot of all this is that $A \mathrel{-\!\!\bullet} (B \mathrel{-\!\!\bullet} C)$ is topologized as a subspace of

$$C^{|B| \times |A|} \times B^{*|C^*| \times |A|}) \times A^{*|C^*| \times |B|}$$

which is clearly symmetric between A and B. $\square$

When A and B are in **RSPLC**, there is no reason to think that $A \mathrel{-\!\!\bullet} B$ is reflexive, although it is clearly in **SPLC**. Accordingly we define $A \multimap B = (A \mathrel{-\!\!\bullet} B)^{**}$.

3.8 Proposition. *For any objects A, B and C of* **RSPLC**, $A \multimap B \cong B^* \multimap A^*$ *and* $A \multimap (B \multimap C) \cong B \multimap (A \multimap C)$.

Proof. The first assertion is immediate from $A \mathrel{-\!\!\bullet} B \cong B^* \mathrel{-\!\!\bullet} A^*$. For the second, we apply Proposition 3.5 to A and $B \mathrel{-\!\!\bullet} C$ and to B and $A \mathrel{-\!\!\bullet} C$. Then

$$\begin{aligned} A \multimap (B \multimap C) &= (A \mathrel{-\!\!\bullet} (B \mathrel{-\!\!\bullet} C)^{**})^{**} = (A \mathrel{-\!\!\bullet} (B \mathrel{-\!\!\bullet} C))^{**} \\ &\cong (B \mathrel{-\!\!\bullet} (A \mathrel{-\!\!\bullet} C))^{**} = (B \mathrel{-\!\!\bullet} (A \mathrel{-\!\!\bullet} C)^{**})^{**} \\ &= B \multimap (A \multimap C) \end{aligned}$$

$\square$

References

M. Barr (1979), *-Autonomous categories. Lecture Notes in Mathematics **752**, Springer-Verlag, Berlin, Heidelberg, New York.