THE $*$-AUTONOMOUS CATEGORY OF UNIFORM SUP SEMI-LATTICES

Dedicated to the memory of Heinrich Kleisli, 1930–2011.

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Abstract. In [Barr & Kleisli 2001] we described $\star$-autonomous structures on two full subcategories of topological abelian groups. In this paper we do the same for sup semi-lattices except that uniform structures play the role that topology did in the earlier paper.

1. Introduction

Sup semi-lattices. The main purpose of this paper is to show that certain categories that are based on sup semi-lattices with a uniform structure are $*$-autonomous. The main tool used for this is the chu construction. We begin by describing briefly what these terms mean.

Closed symmetric monoidal categories. It is well-known that if A and B are abelian goups then $\mathrm{Hom}(A, B)$ can, in a natural way, be given the structure of an abelian group. In fact, it can be shown that this structure is unique if we require that for any $A' \longrightarrow A$ and $B \longrightarrow B'$, the induced map $\mathrm{Hom}(A, B) \longrightarrow \mathrm{Hom}(A', B')$ be a homomorphism of abelian groups. It is common to denote this abelian group by $\mathrm{hom}(A, B)$ to distinguish the abelian group from its underlying set. Moreover, there is a tensor product $A \otimes B$ of abelian groups which is also an abelian group and is characterized by natural isomorphisms

$$\mathrm{Hom}(A \otimes B, C) \cong \mathrm{Hom}(A, \mathrm{hom}(B, C))$$

(Actually, it is also true that $\mathrm{hom}(A \otimes B, C) \cong \mathrm{hom}(A, \mathrm{hom}(B, C))$.)

There are a great many categories that have this structure, including modules over a commutative ring, certain well-behaved categories of topological spaces ([Barr 1978]) and, what is relevant for this paper, the category of sup semi-lattices. By a sup semi-lattice (SSL) we mean a partially ordered set in which every finite subset has a least upper bound or sup. This includes the empty set, so that an SSL has a bottom element, which we usually call 0. A morphism of SSLs is a function that preserves all finite sups

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(including 0). The tensor products can be shown to exist by the general adjoint functor theorem, but we can give a more or less explicit description in the case of SSLs. Given A and B form the free SSL generated by the product of the underlying sets and then factor out the least congruence E for which $(0,b)E0$, $(a,0)E0$, $(a \vee a', b)E((a,b) \vee (a',b))$, and $(a, b \vee b')E((a,b) \vee (a,b'))$ for all $a, a' \in A$ and $b, b' \in B$. Note that $A \otimes B \cong B \otimes A$. It can be shown that $(A \otimes B) \otimes C \cong A \otimes (B \otimes C)$ and that $A \otimes \mathsf{I} \cong A$, where the tensor unit I is the two-element Boolean algebra. These isomorphisms are subject to a number of **coherence conditions** which are tabulated in many places, for example, [Eilenberg & Kelly 1966].

Such a category is called a **closed symmetric monoidal category**, although an older name for this is **autonomous**.

$*$-Autonomous categories. A closed symmetric monoidal category is called **$*$-autonomous** if it contains an object K with the property that for every object A, the canonical map $A \longrightarrow \mathrm{hom}(\mathrm{hom}(A, \mathsf{K}), \mathsf{K})$, described below, is an isomorphism. Then K is called the dualizing object and we usually write $A^* = \mathrm{hom}(A, \mathsf{K})$. The canonical map $A \longrightarrow A^{**}$ is given as the image of the identity map under

$$\mathrm{Hom}(\mathrm{hom}(A,\mathsf{K}), \mathrm{hom}(A,\mathsf{K})) \cong \mathrm{Hom}(\mathrm{hom}(A,\mathsf{K}) \otimes A, \mathsf{K}) \cong \mathrm{Hom}(A, \mathrm{hom}(\mathrm{hom}(A,\mathsf{K}),\mathsf{K}))$$

in which we have made implicit use of the symmetry of the tensor product.

Usually, we denote the closed structure in a $*$-autonomous category by $\multimap$. It is easy to see that there is a close connection between the $\multimap$ and $\otimes$, described by a canonical isomorphism $A \otimes B \cong (A \multimap B^*)^*$ or equivalently $A \multimap B \cong (A \otimes B^*)^*$ so that the internal hom and the tensor determine each other.

A few examples of $*$-autonomous categories were described in [Barr 1979]. They included certain categories of topological abelian groups, of topological vector spaces, and Banach spaces equipped with a second topology (weaker than that of the norm). The only one that did not involve an explicit topology was complete sup semi-lattices.

The Chu construction. In addition to the examples of $*$-autonomous categories just described, there was an appendix to [Barr 1979] in which P-H Chu exposed what has become known as the Chu construction, which we describe briefly.

The Chu construction was motivated by George Mackey's approach to topological vector spaces, see [Mackey 1945]. Instead of putting a topology on a vector space X, he specified a vector space L of admissible maps to the ground field field K (R or C in his situation). So he defined a "linear system" as a vector space X, together with a subspace L of its "conjugate", that is, dual space. He denoted this linear system X_L. To get the actual Chu construction, we generalize this to a pair (X, L) where L has a linear map into the conjugate space. To get the chu (in constrast to the Chu) construction we have, instead, to specialize Mackey's construction to require, in addition, that L contain enough linear maps to separate the points of X, although in some places he added that condition. Mackey did not say what a map between pairs is, still less what the category of pairs is, but he did note the explicit duality of exchanging X and L.

In [Schaefer 1971, IV. 1], we find the definition of a "dual pair" $\langle F, G\rangle$ to consist a two vector spaces, equipped with a bilinear pairing $\langle -, -\rangle : F \times G \longrightarrow \mathsf{K}$ (nowadays we describe it as a linear transformation $F \otimes G \longrightarrow K$) in which F contains enough elements to separate the points of G and vice versa. Again, nothing was said about maps between dual pairs, let alone a category, but the definitions seem obvious.

For our purposes, we begin with a closed symmetric monoidal category $\mathcal{C}$ and fixed object K of $\mathcal{C}$. By $\mathrm{Chu}(\mathcal{C}, \mathsf{K})$ we mean the category whose objects are pairs (A, X) of objects of $\mathcal{C}$ equipped with a pairing $A \otimes X \longrightarrow \mathsf{K}$. A morphism $(A, X) \longrightarrow (B, Y)$ is a pair (f, u) of arrows, $f : A \longrightarrow B$ and $u : Y \longrightarrow X$ such that the diagram

$$\begin{tikzcd}
A \otimes Y \arrow[r, "f\otimes Y"] \arrow[d, "A\otimes u"'] & B \otimes Y \arrow[d] \\
A \otimes X \arrow[r] & \mathsf{K}
\end{tikzcd}$$

commutes. The arrow on the right is the pairing on (B, Y) and the one on the bottom is the pairing on (A, X). This definition of morphism can be internalized to produce an object $[(A, X), (B, Y)]$ of $\mathcal{C}$ as the pullback:

$$\begin{tikzcd}
{[(A, X), (B, Y)]} \arrow[r] \arrow[d] & \hom(A, B) \arrow[d] \\
\hom(Y, X) \arrow[r] & \hom(A \otimes Y, \mathsf{K})
\end{tikzcd}$$

The right and lower arrows in this square arise from the maps $B \longrightarrow \hom(Y, \mathsf{K})$ and $X \longrightarrow \hom(A, \mathsf{K})$, respectively. We define an internal hom (denoted $\multimap$) in the Chu category by $(A, X) \multimap (B, Y) = ([(A, X), (B, Y)], A \otimes Y)$. The dualizing object is the pair (I, K) where I is the tensor unit and the pairing is the isomorphism $\mathsf{I} \otimes \mathsf{K} \longrightarrow \mathsf{K}$. It turns out, not surprisingly, that $(A, X)^* = (X, A)$.

Chu and chu. For our purposes, we require a full subcategory of the Chu category. This is determined by a factorization system but here we will use only the regular epic/monic system that exists in any equational category. We say that the object (A, X) is **separated** if the map $A \longrightarrow \hom(X, \mathsf{K})$ induced by the pairing is monic, and **extensional** if the induced map $X \longrightarrow \hom(A, \mathsf{K})$ is monic. The name comes from thinking of X as representing functions on A; the extensionality condition on functions is that two are equal if they have the same value on every argument. The full subcategory of separated extensional Chu objects is denoted $\mathrm{chu}(\mathcal{C}, \mathsf{K})$. It is also $*$-autonomous. The original $(A, X) \multimap (B, Y)$ is always separated, but the formula has to be adjusted somewhat to make it also extensional. See [Barr 1998] for details.

Topologies and Uniformities. In earlier works, we and others have described a number of $*$-autonomous categories constructed by topological algebras based on some well-known closed symmetric monoidal categories, see [Barr & Kleisli 1999, Barr 2000, Barr & Kleisli 2001, Barr 2006, Barr et. al. 2010]. The underlying categories were in every case categories whose homsets had canonical abelian group structures. Among other things, such categories have the property that finite sums are canonically isomorphic to finite products. Abelian group structures are not necessary as the isomorphisms follow from commutative monoid structures (see 2.2). However there was a second, less obvious use of the abelian group structure. A topological abelian group has a canonical uniform structure and continuous homomorphisms are automatically uniform. In monoids, this fails. However, when the earlier proofs are analyzed, it becomes apparent that it was the uniform structure we used rather than the topology. To apply the same ideas to the category of sup semi-lattices we found it necessary to use uniform structures rather than topological ones. This shows up most clearly in Proposition 2.4. In a forthcoming paper we hope to show how at least some of the same ideas work for a category of topological sup semi-lattices.

The previous papers, mentioned above, were based on categories that were closed monoidal, enriched over abelian groups, and had enough injectives. The abelian group structure meant that quotient objects could be formed by factoring out a subgroup and a continuous homomorphism was continuous if and only if it was continuous at 0. These advantages are lost when replacing the abelian group structure by a commutative monoid structure. Similarly, in the previous papers, there was an object $\mathbf{K}$ that was an injective cogenerator and whose internal object of endomorphisms was the tensor unit. Unfortunately, the category of commutative monoids does not have any non-zero injectives. However the full subcategory of sup semi-lattices (SSLs) does have an injective cogenerator: the two-element Boolean algebra. We therefore deal here with the category of SSLs and the category of uniform SSLs, that is, those equipped with a uniform structure in which the lattice sup is a uniform function.

Notation and conventions. We will be using the following notation and conventions throughout this paper.

SSL means sup semi-lattice and $\mathcal{S}s\ell$ denotes the category of SSLs and functions that preserve finite (including empty) sups.

If A is an SSL, then a subset $T \subseteq A$ will be called **$\vee$-closed** if whenever $a, a' \in T$, so is $a \vee a'$. It misses being a sub-SSL only by not necessarily containing 0.

If A is an SSL, then for $a \in A$, $a{\downarrow}$ denotes $\{a' \in A \mid a' \le a\}$; $a{\uparrow}$ denotes $\{a' \in A \mid a \le a'\}$.

If A is an SSL and $T \subseteq A$ is a subset, we let $T{\uparrow} = \bigcup_{t \in T} t{\uparrow}$ and $T{\downarrow} = \bigcup_{t \in T} t{\downarrow}$. They are called the **up-closure** and **down-closure**, respectively, of T.

If A is an SSL and $T \subseteq A$ is a subset, we let $\overline{T} = \bigcap_{t \in T} t{\uparrow}$ (the set of upper bounds of T) and let $\underline{T} = \bigcap_{t \in T} t{\downarrow}$ (the set of lower bounds of T). Note that if $\bigvee T$ exists, then $\overline{T} = (\bigvee T){\uparrow}$ and similarly if $\bigwedge T$ exists, then $\underline{T} = (\bigwedge T){\downarrow}$.

USSL means sup semi-lattice with a unform structure in which the sup operation is uniform and $\mathcal{U}ss\ell$ denotes the category of USSLs and uniform morphisms; uniform morphisms will be called **unimorphisms**.

All spaces are Hausdorff.

Discrete means uniformly discrete, that is the diagonal is an entourage.

We identify the category of SSLs as the full subcategory of discrete USSLs.

$\mathbf{2} = \{0, 1\}$ with $0 < 1$. If A is a USSL, then a unimorphism $A \longrightarrow \mathbf{2}$ will be called a $\mathbf{2}$-valued unimorphism.

If A is a USSL, then $|A|$ is its underlying (discrete) SSL and $\|A\|$ is its underlying set.

If A and B are USSLs, then $A \multimap B$ denotes the set of unimorphisms from A to B with the uniformity inherited from the product uniformity on $B^{\|A\|}$ and $\hom(A, B) = |A \multimap B|$ (of course, $\mathrm{Hom}(A, B) = \|A \multimap B\|$).

If A is a USSL, then $A^{\#}$ denotes $A \multimap \mathbf{2}$.

If A and B are SSLs, a morphism $A \longrightarrow B$ will be called a uniform embedding if it is an isomorphism, both algebraic and uniform to a sub-SSL of B.

We denote by $\mathcal{C}$ the category of USSLs that can be uniformly embedded into a product of discrete USSLs. Following a useful suggestion of a referee, we point out that this is not the same as having a uniform embedding into a power of $\mathbf{2}$. For example, the SSL of discrete integers $\mathbf{Z}$ cannot be uniformly embedded into a power of $\mathbf{2}$ since a compact set cannot have an infinite uniformly discrete subset (although it could have a topologically discrete one).

If A is an object of $\mathcal{C}$, then A^* denotes $A^{\#}$, reuniformized with a generally finer uniformity that is characterized as the finest uniformity among objects of $\mathcal{C}$ with the same underlying SSL structure and the same set of $\mathbf{2}$-valued unimorphims as $A^{\#}$ (Theorem 4.2 shows that this exists).

If A is any set, $\Delta(A)$ denotes the diagonal of $A \times A$.

A USSL A **has enough 2-valued maps** if there are enough unimorphisms to $\mathbf{2}$ to separate the points of A.

If A is a USSL whose canonical map $A \longrightarrow A^{\#\#}$ is bijective, we will say that A is **prereflexive**. If it is an isomorphism, we will say that A is **weakly reflexive**. If the canonical map $A \longrightarrow A^{**}$ $(= A^{\#*})$ is an isomorphism, we will say that A is **strongly reflexive**. Note that "weak" and "strong" refer only to the strength of the uniformities.

If A is a USSL and $\varphi : A \longrightarrow \mathbf{2}$ is a $\mathbf{2}$-valued unimorphism, we write $\ker \varphi = \varphi^{-1}(0)$.

In connection with the last item, it is clear that $\ker \varphi$ is sup-closed, down-closed, and clopen, but those conditions are not sufficient to be the kernel of a $\mathbf{2}$-valued unimorphism. It must also be the case that $\{\ker \varphi, A - \ker \varphi\}$ is a uniform cover of A.

EXAMPLE. Let $\mathbf{N}$ denote the non-negative integers with the usual order and the discrete uniformity. The kernel of an SSL homomorphism $\varphi : \mathbf{N} \longrightarrow \mathbf{2}$ can either be all of $\mathbf{N}$ or $n\!\downarrow$ for some $n \in \mathbf{N}$. The first is the kernel of the 0 homomorphism and we call it 0. We denote by φ_n the homomorphism whose kernel is $n\!\downarrow$. Clearly $\varphi_n \le \varphi_m$ if and only if $m \le n$. Thus $\mathbf{N}^\#$ has elements $0 \le \cdots \le \varphi_n \le \varphi_{n-1} \le \cdots \le \varphi_1 \le \varphi_0$. The usual argument shows that this uniform space is closed in $\mathbf{2}^{\|\mathbf{N}\|}$ and is therefore compact, so it suffices to see what its topology is. If $n \in \mathbf{N}$ and $p_n : \mathbf{2}^{\|\mathbf{N}\|} \longrightarrow \mathbf{2}$ is the product projection, then the subbasic open sets are $p_n^{-1}(0) = \{0, \ldots, \varphi_n\}$ and $p_n^{-1}(1) = \{\varphi_{n-1}, \ldots, \varphi_0\}$. Thus a basic open neighbourhood of φ_n is $p_n^{-1}(0) \cap p_{n+1}^{-1}(1) = \{\varphi_n\}$ while the basic open sets at 0 are simply the complements of finite sets. In other words, $\mathbf{N}^\#$ can be identified as the one-point compactification of the discrete set $\{\varphi_n \mid n \in \mathbf{N}\}$. Since $\mathbf{N}^\#$ is compact, we can compute $\mathbf{N}^{\#\#}$ as the continuous maps $\mathbf{N}^\# \longrightarrow \mathbf{2}$. They can be identified with $\mathbf{N}$ because the SSL homomorphism that vanishes everwhere except at 0 is not continuous. Thus $\mathbf{N} \longrightarrow \mathbf{N}^{\#\#}$ is bijective and $\mathbf{N}$ is pre-reflexive. Note that a compact space cannot contain an infinite uniformly discrete subspace (see the paragraph preceding Theorem 4.2), so $\mathbf{N}$ cannot be weakly reflexive. We will show in 4.4 that it is strongly reflexive.

2. Basic properties

2.1. SEMI-ADDITIVE CATEGORIES. A category is called **semi-additive** if its homsets have the structure of commutative monoids in such a way that composition of morphisms distributes over the monoid operation (that is generally denoted "+", although in a sup semi-lattice we will denote it $\vee$). This means that for every pair of objects A, B there is a zero morphism, usually denoted $0 : A \longrightarrow B$ and for any two morphisms $f, g : A \longrightarrow B$, there is a sum $f + g : A \longrightarrow B$. Moreover, for any $h : A' \longrightarrow A$ and $k : B \longrightarrow B'$, we have $k0h = 0$ and $k(f + g)h = kfh + kgh$, both from A' to B'. If these monoids are actually groups and the category has finite products, then the category is called **additive**.

A category with finite products is said to have **finite biproducts** if every finite product is also a finite sum in a canonical way. This means two things. First, the empty sum and the empty product are the same, that is the category is pointed. We will denote this object by 0. Second, for each pair of objects A and B, there is an object $A \oplus B$, equipped with arrows $u : A \longrightarrow A \oplus B$, $v : B \longrightarrow A \oplus B$, $p : A \oplus B \longrightarrow A$ and $q : A \oplus B \longrightarrow B$ such that $A \oplus B$, together with u and v constitute a categorical sum of A and B and $A \oplus B$, together with p and q, constitute their product. These are subject to the requirements that u, v, p, q be natural in A and B, that pu and qv be the respective identity maps and that pv and qu be the respective zero maps.

The following is well known (see, for example, [Freyd 1964, Section 2.4]) and actually characterizes semi-additive categories with finite products. Note that although Freyd states Theorems 2.41 and 2.42 for abelian categories, he makes no actual use of any properties of abelian categories save for semi-additivity. There are no exactness arguments and no subtraction.

2.2. Proposition. *A semi-additive category with finite products has biproducts.*

Proof. Let us (temporarily) denote the terminal object by 1. For any object A, there is at least one map $0 : 1 \longrightarrow A$. The identity $1 \longrightarrow 1$ must also be the 0 map since 1 is terminal and is the target of exactly one map from any object. Thus if $f : 1 \longrightarrow A$ is any map, we have $f = f.\mathrm{id} = f0 = 0$, which shows that 1 is also initial.

To show that $A \times B$ is the sum in $\mathcal{C}$ of A and B, we begin with the product projections $p : A \times B \longrightarrow A$ and $q : A \times B \longrightarrow B$. A map $C \longrightarrow A \times B$ is given by a pair (k, ℓ), where $k : C \longrightarrow A$ and $\ell : C \longrightarrow B$ are uniquely determined by the equations $p(k, \ell) = k$ and $q(k, \ell) = \ell$. In order to show that $A \times B$ is the sum of A and B, define $u = (\mathrm{id}, 0) : A \longrightarrow A \times B$ and $v = (0, \mathrm{id}) : B \longrightarrow A \times B$. Now suppose that $f : A \longrightarrow C$ and $g : B \longrightarrow C$. We claim that $h = fp + gq : A \times B \longrightarrow C$ is the unique map for which $hu = f$ and $hv = g$. We have $hu = (fp + gq)u = fpu + gqu = f + 0 = f$ and similarly $hv = g$. Now suppose $h' : A \times B \longrightarrow C$ is another map with the same properties. We claim that $up + vq : A \times B \longrightarrow A \times B$ is the identity. In fact $p(up+vq) = pup+pvq = p(\mathrm{id}, 0)p+p(0, \mathrm{id})q = p+0 = p$ and similarly $q(up + vq) = q$ and we know the identity of $A \times B$ is the unique map with those two properties. Thus $h' = h'(up + vq) = h'up + h'vq = fp + gq = h$. ■

Remark. In the case of SSLs, the sum is denoted $\vee$ rather than $+$, but the proposition remains valid.

2.3. Proposition. *The object* $\mathbf{2}$ *is an injective cogenerator in* $\mathcal{Ssl}$.

Proof. We begin by showing that $\mathbf{2}$ is injective. Suppose $A \subseteq B$ and $\varphi : A \longrightarrow \mathbf{2}$ is a morphism. Let I be the kernel of φ. One easily sees that $I{\downarrow}$, the down-closure of I in B, is an ideal and that $A \cap I{\downarrow} = I$ and the $\mathbf{2}$-valued morphism whose kernel is I obviously extends φ. Next suppose that $a \neq a'$ in A. Then either $a' \not\leq a$ or $a \not\leq a'$. In the former case, $a{\downarrow}$ is the kernel of a $\mathbf{2}$-valued morphism φ for which $\varphi(a') = 1$ and $\varphi(a) = 0$. ■

2.4. Proposition. *Let the uniform space* X *be embedded in a product* $\prod_{s \in S} X_s$ *in which each* X_s *is discrete and let* D *be a discrete uniform space. Then for any uniform function* $f : X \longrightarrow D$*, there is a finite subset* $T \subseteq S$ *and a map* $h : \prod_{t \in T} X_t \longrightarrow D$ *such that* f *factors as* $X \hookrightarrow \prod_{s \in S} X_s \xrightarrow{p} \prod_{t \in T} X_t \xrightarrow{h} D$*, with* $p : \prod_{s \in S} X_s \longrightarrow \prod_{t \in T} X_t$ *the product projection.*

Proof. Since D is discrete, $\Delta(D)$ is an entourage and hence $(f \times f)^{-1}(\Delta(D))$ must be an entourage in X. There must be an entourage $U \subseteq \prod_{s \in S}(X_s \times X_s)$ such that $(f \times f)^{-1}(\Delta(D)) = (X \times X) \cap U$. Basic entourages have the form $\prod_{s \in S-T}(X_s \times X_s) \times \prod_{t \in T} \Delta(X_t)$ for finite subsets $T \subseteq S$. Thus there must be a finite $T \subseteq S$ such that the equivalence relation E defined by

$$E = (X \times X) \cap \left(\prod_{s \in S-T} (X_s \times X_s) \times \prod_{t \in T} \Delta(X_t) \right)$$

is included in $(f \times f)^{-1}(\Delta(D))$. If $x, x' \in X$ are such that $(x, x') \in (f \times f)^{-1}(\Delta(D))$, then clearly $f(x_1) = f(x_2)$ so that f is well defined mod $(f \times f)^{-1}(\Delta(D))$. In particular, if

$Y = X/E$ then f induces a map $g : Y \longrightarrow D$ such that f is the composite $X \longrightarrow Y \xrightarrow{g} D$. Clearly Y is a subspace of the discrete space $\prod_{t \in T} X_t$ from which it is immediate that g can be extended to a uniform map $h : \prod_{t \in T} X_t \longrightarrow D$ and our conclusion follows. ■

3. The category $\mathcal{C}$

Recall that $\mathcal{C}$ denotes the category of all USSLs that are uniformly embedded in a product of discrete SSLs.

3.1. Theorem. *The object* $\mathbf{2}$ *is an injective cogenerator in* $\mathcal{C}$ *with respect to uniform embeddings.*

Proof. Let $A \subseteq B$ be a uniform embedding. Since B can be embedded in a product, say $\prod B_s$, of discrete objects, to prove injectivity, it is sufficient that any unimorphism $A \longrightarrow \mathbf{2}$ can be extended to the product. To do this, we apply the construction used in Proposition 2.4. The only thing to be noted is that the extension from g to h exists because $\mathbf{2}$ is injective in the discrete spaces by Proposition 2.3. ■

The following result is crucial. It replaces the arguments based on continuity in abelian groups by those based on uniformity in SSLs.

3.2. Theorem. *Suppose* $A \hookrightarrow \prod_{s \in S} A_s$ *(the latter with the product uniformity) is an inclusion in* $\mathcal{C}$ *and* $\varphi : A \longrightarrow \mathbf{2}$ *is a unimorphism. Then there is a finite subset* $T \subseteq S$ *and for each* $t \in T$*, there is a unimorphism* $\psi_t : A_t \longrightarrow \mathbf{2}$ *such that* ψ *is the composite* $A \longrightarrow \prod_{s \in S} A_s \xrightarrow{\bigvee_{t \in T} \psi_t p_t} \mathbf{2}$ *where* $p_t : \prod_{s \in S} A_s \longrightarrow A_t$ *is the product projection.*

Proof. Apply once more the construction of Proposition 2.4, using the fact that the finite product $\prod X_t$ is a biproduct. ■

3.3. Proposition. *Every object of* $\mathcal{C}$ *is pre-reflexive.*

Proof. Let A be an object of $\mathcal{C}$. The definition of $A^\#$ embeds it into $\mathbf{2}^{\|A\|}$. Suppose that $\varphi : A^\# \longrightarrow \mathbf{2}$ is a unimorphism. Then from Theorem 3.2, there is a finite subset $T \subseteq \|A\|$ and there are morphisms $\{\varphi_t : \mathbf{2} \longrightarrow \mathbf{2} \mid t \in T\}$ such that

$$\begin{tikzcd}
A^\# \arrow[rr, hook] \arrow[dd, "\varphi"'] & & \mathbf{2}^{\|A\|} \arrow[dl] \\
& \mathbf{2}^T \arrow[dl, "\bigvee_{t \in T} \varphi_t p_t"] & \\
\mathbf{2} & &
\end{tikzcd}$$

commutes. But this is nothing but evaluation at the element $\bigvee\{t \in T \mid \varphi_t = \mathrm{id}\}$ which belongs to A. ■

3.4. Corollary. *Every compact object of $\mathcal{C}$ is weakly reflexive.*

Proof. The bijection $C \longrightarrow C^{\#\#}$ is continuous, the domain is compact, and the codomain is Hausdorff. ■

4. Weak and strong uniformities

Every object A of $\mathcal{C}$ maps injectively into a power of $\mathbf{2}$ (specifically $\mathbf{2}^{\|A^{\#}\|}$). If the uniformity on A is such that this injection is a uniform embedding, we will say that A has the **weak uniformity**.

4.1. Proposition. *There is an idempotent endofunctor σ on $\mathcal{C}$ such that for any object A of $\mathcal{C}$, $|\sigma A| = |A|$, $|(\sigma A)^{\#}| = |A^{\#}|$, and the uniformity on σA is the coarsest possible with these two properties. It follows that A is weak if and only if the bijection $A \longrightarrow \sigma A$ is an isomorphism.*

Proof. Since A has enough $\mathbf{2}$-valued unimorphisms to separate points, there is an injection $A \longrightarrow \mathbf{2}^{\|A^{\#}\|}$. Let σA be the induced uniformity on A. To see that σ is a functor, observe that $A \longrightarrow B$ induces $\|B^{\#}\| \longrightarrow \|A^{\#}\|$ and now look at the diagram

$$\begin{tikzcd} \sigma A \arrow[r, hook] \arrow[d] & \mathbf{2}^{\|A^{\#}\|} \arrow[d] \\ \sigma B \arrow[r, hook] & \mathbf{2}^{\|B^{\#}\|} \end{tikzcd}$$

in which the left hand map is uniform because the top and right hand maps are uniform and the bottom arrow is an embedding. ■

We will say that a uniformity on A is **strong** if whenever B is such that $|A| = |B|$ and $|A^{\#}| = |B^{\#}|$, then the identity $A \longrightarrow B$ is uniform. This means that the uniformity on A is as strong as it can be without allowing more unimorphisms to $\mathbf{2}$. It is not obvious that strong uniformities exist (unless A is discrete), but we will show they always do. Incidentally, it is worth pointing out that an infinite discrete object (such as $\mathbf{N}$) cannot have a weak uniformity since a compact space cannot contain an infinite (uniformly) discrete subspace. For if it is discrete, then there must be some entourage on the compact space for which the each set in the corresponding uniform cover contains at most one element of the discrete space. Clearly such a cover cannot have a finite refinement.

4.2. Theorem. *There is an idempotent endofunctor τ on $\mathcal{C}$ such that for any object A of $\mathcal{C}$, $|\tau A| = |A|$, $|(\tau A)^{\#}| = |A^{\#}|$ and the uniformity on τA is the finest possible with these properties. It follows that A is strong if and only if the bijection $\tau A \longrightarrow A$ is an isomorphism.*

Proof. Let $\{A_s \longrightarrow A \mid s \in S\}$ range over the set of all bijective unimorphisms that induce bijections $A^{\#} \longrightarrow A_s^{\#}$ and for which $A_s \in \mathcal{C}$. Define τA so that the diagram

$$\begin{tikzcd} \tau A \arrow[r] \arrow[d] & \prod_{s\in S} A_s \arrow[d] \\ A \arrow[r] & A^S \end{tikzcd}$$

is a pullback. The bottom arrow is the diagonal, which is a uniform embedding, from which it follows that the top arrow is also a uniform embedding. Since the right hand arrow is an isomorphism of the underlying SSLs, so is the left hand arrow. Now suppose that $\varphi : \tau A \longrightarrow \mathbf{2}$ is a unimorphism. From Theorem 3.1 we see that φ can be extended to a unimorphism $\psi : \prod_{s\in S} A_s \longrightarrow \mathbf{2}$. From Theorem 3.2, we see that there is a finite subset $T \subseteq S$ and a family of unimorphisms $\{\psi_t : A_t \longrightarrow \mathbf{2} | t \in T\}$ such that ψ factors as $\prod_{s\in S} A_s \longrightarrow \prod_{t\in T} A_t \xrightarrow{\bigvee \psi_t p_t} \mathbf{2}$. Since each A_t has the same set of $\mathbf{2}$-valued unimorphisms as A, it follows that each ψ_t is uniform on A. The commutativity of the diagram

$$\begin{tikzcd} \tau A \arrow[r] \arrow[rr, bend left] \arrow[dr, "\varphi"'] & A^T \arrow[d, "\bigvee \psi_t p_t"] & A \arrow[l] \arrow[dl, "\bigvee \psi_t"] \\ & \mathbf{2} & \end{tikzcd}$$

combined with the fact that the top arrow is a bijection, shows that $\varphi = \bigvee \psi_t$ is uniform on A. Thus A and τA have the same set of $\mathbf{2}$-valued unimorphisms.

Next we show that τ is a functor. Suppose we have a unimorphism $f : B \longrightarrow A$. Let C be the USSL defined so that

$$\begin{tikzcd} C \arrow[r, "h"] \arrow[d, "k"'] & \tau A \arrow[d, "g"] \\ B \arrow[r, "f"'] & A \end{tikzcd}$$

is a pullback. The right hand vertical arrow and therefore the left hand vertical arrow are bijections. Suppose $\varphi \in C^{\#}$. We will show that there is a $\nu \in B^{\#}$ such that $\varphi = \nu k$, which will show that $B^{\#} \longrightarrow C^{\#}$ is a bijection and hence that the uniformity on C lies between those of τB and B, which suffices, since then we have $\tau B \longrightarrow C \longrightarrow \tau A$. The definition of pullback implies that there is a uniform embedding $C \hookrightarrow B \times \tau A$. Injectivity of $\mathbf{2}$, in conjunction with the fact that finite products in SSL are also sums, implies that there is a $(\psi, \rho) \in B^{\#} \times (\tau A)^{\#}$ such that $\varphi = \psi k \vee \rho h$. Since $(\tau A)^{\#} = A^{\#}$, there is a $\mu \in A^{\#}$ such that $\rho = \mu g$. Then we have $\varphi = \psi k \vee \mu g h = \psi k \vee \mu f k = (\psi \vee \mu f)k$. Thus $\nu = \psi \vee \mu f$ is the required map. ∎

Remark. The fact that τ is a functor was never even mentioned in [Barr 2006, Theorem 4.1, $2 \Longrightarrow 3$]. But the above argument can be repeated verbatim, just substituting "+" for "$\vee$" and "continuous" for "uniform" to fill that gap.

Terminology. Recall that A has a weak uniformity when $A = \sigma A$ and that A has a strong uniformity when $A = \tau A$. We denote by A^* the USSL $\tau(A^\#)$. Then $A^\#$ has a weak uniformity and A^* has a strong uniformity.

As an obvious application of the above results, we have,

4.3. Corollary. *The bijections* $\tau A \longrightarrow A \longrightarrow \sigma A$ *induce isomorphisms*

$$(\sigma A)^\# \longrightarrow A^\# \longrightarrow (\tau A)^\# \qquad \textit{and} \qquad (\sigma A)^* \longrightarrow A^* \longrightarrow (\tau A)^* \qquad \blacksquare$$

As another application, we have:

4.4. Theorem. *A discrete SSL is strongly reflexive. An infinite discrete SSL is not weakly reflexive.*

Proof. To take the last point first, we note that a discrete space cannot have the weak uniformity since a compact space cannot contain an infinite (uniformly) discrete subspace. If A is discrete then A and $A^{\#\#}$ have the same $\mathbf{2}$-valued unimorphisms, namely the elements of $A^\# \cong A^{\#\#\#}$ and hence $A^{\#*} = \tau(A^{\#\#})$ has a uniformity at least as fine as that of A. But A is discrete and there is no finer uniformity. $\blacksquare$

5. The category chu($\mathcal{S}s\ell$,$\mathbf{2}$)

By Chu($\mathcal{S}s\ell$, $\mathbf{2}$) we mean the category whose objects are pairs (A, X) of SSLs together with a pairing $A \otimes X \longrightarrow \mathbf{2}$. A morphism $(f, g) : (A, X) \longrightarrow (B, Y)$ consists of SSL morphisms $f : A \longrightarrow B$ and $g : Y \longrightarrow X$ (note the direction of the second arrow) such that the square

$$\begin{tikzcd}
A \otimes Y \arrow[r, "f \otimes Y"] \arrow[d, "A \otimes u"'] & B \otimes Y \arrow[d] \\
A \otimes X \arrow[r] & \mathbf{2}
\end{tikzcd}$$

commutes. The unspecified arrows are the pairings. This becomes a $*$-autonomous category when you define $(A, X)^* = (X, A)$, $(A, X) \multimap (B, Y)$ as

$$([A, B] \times_{[A \otimes Y]} [Y, X], A \otimes Y)$$

(which is just the internalization of the preceding diagram) and $(A, X) \otimes (B, Y) = ((A, X) \multimap (Y, B))^*$.

The full subcategory of Chu($\mathcal{S}s\ell$, $\mathbf{2}$) consisting of the pairs (A, X) for which both induced maps $A \longrightarrow \hom(X, \mathbf{2})$ and $X \longrightarrow \hom(A, \mathbf{2})$ are monic, is denoted chu($\mathcal{S}s\ell$, $\mathbf{2}$). This chu category is also $*$-autonomous, see [Barr 1998], using the surjection/injection factorization system.

6. The main theorem

As above, for a USSL A we denote by σA and τA the weak and strong uniformities, respectively, on A.

6.1. THEOREM. *The categories of weak USSLs and strong USSLs are equivalent to each other and to chu($\mathcal{S}s\ell$, $\mathbf{2}$) and are thus $*$-autonomous.*

PROOF. Let us write chu for chu($\mathcal{S}s\ell, \mathbf{2}$). Define a functor $F : \mathcal{U}ss\ell \longrightarrow \mathrm{chu}$ by letting $F(A) = (|A|, |A^*|)$ with evaluation as pairing. If $f : A \longrightarrow B$, define $Ff = (|f|, |f^*|) : FA \longrightarrow FB$. We first define the right adjoint R of F. If (A, X) is an object of chu, let $R(A, X)$ be the object of $\mathcal{U}ss\ell$ for which $|R(A, X)| = A$ and whose uniformity is inherited from the embedding $R(A, X) \hookrightarrow \mathbf{2}^X$. If B is any object of $\mathcal{U}ss\ell$ and $(f, g) : (|B|, |B^*|) \longrightarrow (A, X)$ is given, the compatibility condition in the chu category says that for $b \in |B|$ and $x \in X$, we have $g(x)(b) = x(f(b))$, which says that $g(x)$ is the composite $|B| \xrightarrow{f} A \xrightarrow{\mathrm{ev}_x} \mathbf{2}$ and thus an element of B^*. This is the same as saying that the composite $B \longrightarrow R(A, X) \longrightarrow \mathbf{2}^X \xrightarrow{p_x} \mathbf{2}$ is uniform. But $R(A, X)$ has the uniformity inherited from $\mathbf{2}^X$, so this means that $B \longrightarrow R(A, X)$ is uniform. The uniqueness is clear so that the object function R defines a functor that is right adjoint to F.

A morphism $\varphi : R(A, X) \longrightarrow \mathbf{2}$ extends to some $\psi : \mathbf{2}^X \longrightarrow \mathbf{2}$. It follows from Theorem 3.2 that ψ factors through a finite power, which means that there is a finite subset, say $\{x_1, \ldots, x_n\}$ of X such that $\psi = \mathrm{ev}(x_1) \vee \cdots \vee \mathrm{ev}(x_n)$. But the fact that the original pairing is bilinear implies that the restrictions to A of $\mathrm{ev}(x_1) \vee \cdots \vee \mathrm{ev}(x_n)$ and $\mathrm{ev}(x_1 \vee \cdots \vee x_n)$ coincide. Thus every element of $R(A, X)^*$ belongs to X and hence $FR(A, X) = (A, X)$ so that R is a full embedding. Clearly $R(A, X)$ is always weakly uniformized. Now suppose that A is weakly uniformized and that $A \subseteq \mathbf{2}^X$ is an embedding that determines that uniformity. Every $\varphi \in A^*$ is, as above, represented by an element of the free SSL $\langle X \rangle$ determined by X so that $|A^*|$ is a quotient of $\langle X \rangle$. This gives a canonical function $X \longrightarrow |A^*|$ from which we have the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[dr] & \mathbf{2}^{|A^*|} \arrow[d] \\ & \mathbf{2}^X \end{tikzcd}$$

and if the diagonal arrow is a uniform embedding, so is the top arrow. Thus $A = RF(A)$ if and only if A is weakly topologized.

Next let $L(A, X) = \tau R(A, X)$. Suppose we have $(f, g) : (A, X) \longrightarrow (|B|, |B^*|)$. The definition of a chu morphism implies that for any $\varphi \in B^{\#}$, the composite $A \longrightarrow |B| \xrightarrow{\varphi} \mathbf{2}$

is $g(\varphi)$ and hence that the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & \mathbf{2}^X \arrow[d, "\mathbf{2}^g"] \\ |B| \arrow[r] & \mathbf{2}^{\|B^*\|} \end{tikzcd}$$

commutes. But this means that in $\mathcal{Ussl}$ the square

$$\begin{tikzcd} R(A,X) \arrow[r, hook] \arrow[d, "f"'] & \mathbf{2}^X \arrow[d, "\mathbf{2}^g"] \\ \sigma B \arrow[r, hook] & \mathbf{2}^{\|B^*\|} \end{tikzcd}$$

commutes so that the function f is uniform. Thus we have $R(A,X) \longrightarrow \sigma B$, which gives $\tau R(A,X) \longrightarrow \tau\sigma B = \tau B \longrightarrow B$ is the required map $L(A,X) \longrightarrow B$. Again uniqueness is clear. It is well known that when the right adjoint of a functor is full and faithful, so is its left adjoint (if any) so that we conclude that FL is equivalent to the identity. Clearly, $L(A,X)$ is strongly uniformized. If A is a strongly uniformized SSL, then we know that the adjunction morphism $LF(A) \longrightarrow A$ is a bijection and to see that it is an isomorphism, we need only see that they have the same dual space, which follows immediately from $|A^{\#}| = |A^*|$. ■

7. A topological interlude

Every uniform space has an associated topological space. If $\mathcal{U}$ is a uniform structure on the set X and $U \in \mathcal{U}$, then for each $x \in X$, let $U[x] = \{y \mid (x,y) \in U\}$. The family of all $U[x]$, for $U \in \mathcal{U}$ is a base for a topology on X, called the **uniform topology**. A unimorphism between spaces is continuous in the associated uniform topologies. In this section we see some of the interactions between uniform and topological notions that will be especially useful when the uniform topology is compact. In that case, the uniformity is unique and consists of all neighbourhoods of the diagonal.

7.1. Proposition. *Suppose A is a USSL. Then for each $a \in A$, both $a{\downarrow}$ and $a{\uparrow}$ are closed.*

Proof. Define $f : A \longrightarrow A$ by $f(b) = a \vee b$. Then $f^{-1}(a) = a{\downarrow}$. Define $g : A \longrightarrow A \times A$ by $g(b) = (b, a \vee b)$. Then $g^{-1}(\Delta(A)) = a{\uparrow}$. ■

7.2. Corollary. *For any subset $T \subseteq A$, both $\underline{T}$ and $\overline{T}$ are closed.*

Proof. These sets are the meets of all the $t{\uparrow}$, respectively $t{\downarrow}$, over all $t \in T$. ■

Let T be a subset of the USSL A. We say that T is **directed** if for $t_1, t_2 \in T$, there is an element $t \in T$ with $t_1 \leq t$ and $t_2 \leq t$. We say that T is **down-directed** if T^{op} is directed. If T is directed, then T can be thought of as a net in A, indexed by itself. If T is down-directed, then T can also be thought of as a net in A, indexed by T^{op}.

7.3. THEOREM. *Let A be a USSL and suppose T is a non-empty directed subset of A that has a cluster point c. Then*

1. *c is an upper bound for T;*
2. *c is the least upper bound for T; and*
3. *$c = \bigvee T = \lim T$.*

PROOF.

1. Suppose that $t \in T$ with $c \not\geq t$ so that $c \in A - t{\uparrow}$. But then for all $s \geq t$ of T, $s \notin A - t{\uparrow}$ so T is not frequently in the neighbourhood $A - t{\uparrow}$ of c, which contradicts the fact that c is a cluster point of T.
2. Suppose b is another upper bound for T. Then $c \vee b$ is, by continuity of $\vee$, a cluster point of $T \vee b$. But since b is an upper bound for T, $T \vee b$ is constant at b and b is its only cluster point. Thus $c \vee b = b$, whence $c \leq b$ so that c is the least upper bound.
3. Let U be an entourage. The fact that $\vee$ is uniform implies that there is an entourage V such that $V \vee V \subseteq U$. Since c is a cluster point of T, there must be some $t \in T$ such that $t \in V[c]$, meaning $(c, t) \in V$. For any $s \in T$ with $s \geq t$, we also have $(s, s) \in V$. But then $(c, t) \vee (s, s) = (c \vee s, t \vee s) = (c, s) \in U$ so that $s \in U[c]$. This shows that T is eventually in every neighbourhood of c so that $c = \lim T$. ■

7.4. THEOREM. *Suppose T is down-directed and that $c \in A$ is a cluster point of T^{op}. Then*

1. *c is a lower bound for T;*
2. *c is the greatest lower bound for T; and*
3. *If A has the weak uniformity, then $c = \lim T$.*

PROOF.

1. This is the dual of the proof of 7.3.1 and depends only on the fact that down sets are closed.
2. Suppose that b is another lower bound for T. If $b \not\leq c$, then $A - b{\uparrow}$ is a neighbourhood of c and hence must contain some $t \in T$, which contradicts the hypothesis that b is a lower bound for T.

3. The weak uniformity on A has a subbase the sets $U_\varphi = (\varphi \times \varphi)^{-1}(\Delta(\mathbf{2})) = \{(a,b) \mid \varphi(a) = \varphi(b)\}$ for $\varphi \in A^*$. Thus the topology at $a \in A$ has as subbase the sets of the form $U_\varphi[a] = \{b \in A \mid \varphi(b) = \varphi(a)\}$. When $\varphi(a) = 0$, the set $U_\varphi[a] = \ker\varphi$ and these sets are closed under finite intersection. When $\varphi(a) = 1$, the set $U_\varphi[a] = \{b \mid \varphi(b) = 1\} = A - \ker\varphi$ which is up-closed. These sets will not (usually) be closed under finite intersection, but if $\varphi(a) = \psi(a) = 1$, then $U_\varphi[a] \cap U_\psi[a] \subseteq U_{\varphi\vee\psi}$. The result is that the sets of the form $\ker\varphi \cap (A - \ker\psi)$ with $\varphi(a) = 0$ and $\psi(a) = 1$ form a base for the topology at a. For any ψ with $\psi(c) = 1$, we have that $T \subseteq c\!\uparrow \subseteq A - \ker\psi$. But c is a cluster point of T so that no neighbourhood of c can exclude T and so when $\varphi(c) = 0$, there is some $t \in T$ such that for all $s \le t$, we have $s \in T \cap \ker\varphi$ and hence $s \in \ker\varphi \cap (A - \ker\psi)$. Thus T is eventually in every neighbourhood of c. ■

8. Compact USSLs

In this section, we study several properties of compact USSLs. Of course, compactness is a topological property, but, as is well known, compact spaces have a unique uniform structure (all covers are uniform; all neighbourhoods of the diagonal are entourages) and all continuous maps between compact spaces are also uniform. The main tool in this study is the interplay between topological and order properties. We begin with

8.1. Theorem. *Every directed set (respectively, every down-directed set) in a compact USSL has a limit.*

Proof. Every net in a compact space has at least one cluster point. Moreover, a compact USSL must have the weak uniformity since no weaker uniformity can be Hausdorff. Thus Theorems 7.3 and 7.4 apply. ■

8.2. Theorem. *A compact USSL is order complete.*

Proof. Let A be compact and $T \subseteq A$ be a subset. For each finite subset $F \subseteq T$, the set $\overline{F} \neq \emptyset$ since it includes at least $\bigvee F$. It is closed and the set of all $\overline{F}$, for finite subsets $F \subseteq T$ has the finite intersection property and hence their meet $\overline{T}$ is non-empty and closed. For finite $F \subseteq T$, then, since every element of $\overline{T}$ is above every element of F, the set $\overline{F} \cap \underline{(\overline{T})}$ is non-empty and closed. Hence the intersection of all the sets $\overline{F} \cap \underline{(\overline{T})}$ is non-empty and its only possible element is $\bigvee T$. ■

If A is an SSL, then any non-empty $\vee$-closed subset $T \subseteq A$ can be regarded as a net in its inherited order. We will assume this structure whenever we talk of a cluster point or a limit of a $\vee$-closed T.

8.3. Theorem. *Let $f : A \longrightarrow B$ be a USSL morphism. If A is compact, then f preserves arbitrary sups.*

Proof. If $T \subseteq A$, we know that $a = \bigvee T$ exists. We want to show that $f(a) = \bigvee f(T)$. Since f preserves 0, we can assume that $T \neq \emptyset$. Since nothing changes if we replace T by the $\vee$-closed subset it generates and this process is preserved by f, we can suppose that T and hence $f(T)$ are $\vee$-closed. But f is continuous and thereby preserves limits so that $f(a) = \lim f(T)$ and it follows from Theorem 7.3 that $f(a) = \bigvee f(T)$. ∎

8.4. Corollary. *A clopen ideal in a compact USSL is principal.*

Proof. Let A be a compact USSL and $T \subseteq A$ be a clopen ideal. Then T is the kernel of a unimorphism $\varphi : A \longrightarrow \mathbf{2}$. Since φ preserves arbitrary sups, the kernel is principal. ∎

8.5. Lemma. *Let A be a compact USSL and let U be an open subset of A that contains a maximal element a. Then $a\!\downarrow$ is clopen.*

Proof. We know that $a\!\downarrow$ is closed. We want to show that $A - a\!\downarrow$ is closed. So suppose that T is a net in $A - a\!\downarrow$ that converges to an element $b \in a\!\downarrow$. Then $T \vee a$ converges to $b \vee a = a$. Since U is a neighbourhood of a, it follows that $T \vee a$ is eventually in U and so there is a $t \in T$ with $t \vee a \in U$. The maximality of a in U implies that $t \leq a$ which contradicts the assumption that T is a net in $A - a\!\downarrow$. ∎

An immediate consequence of this is that a proper down-closed open set in a compact connected USSL (for example the unit interval) cannot contain a maximal element.

8.6. Theorem. *A compact totally disconnected USSL can be embedded into a power of* $\mathbf{2}$.

Proof. Let A be a compact totally disconnected USSL and let $a \neq b$ be points of A. Replacing, if necessary, b by $a \vee b$, we may suppose that $a < b$. Then $a\!\downarrow$ and $b\!\uparrow$ are disjoint closed subsets, so there is a clopen set U that contains $a\!\downarrow$ and is disjoint from $b\!\uparrow$. Let C be a maximal chain of U such that $a \in C$ and let $c = \bigvee C$. Then $c \in U$ since U is closed. Clearly c is a maximal element of U and thus $c\!\downarrow$ is clopen. Since we supposed that $a \in C$, it follows that $a \in c\!\downarrow$. Thus $c\!\downarrow$ is the kernel of a continuous $\mathbf{2}$-valued morphism φ such that $\varphi(a) = 0$ and $\varphi(b) = 1$. ∎

To describe the dual of a compact USSL, we need the following Definition and Lemmas.

8.7. Definition. *Let A be a USSL. We say that $a \in A$ is* **regular** *if $a\!\downarrow$ is open (and therefore clopen). If a is regular we let $\varphi_a : A \longrightarrow \mathbf{2}$ be the map whose kernel is $a\!\downarrow$.*

8.8. Lemma. *Assume that A is a compact USSL. Let $R \subseteq A$ be the set of all regular elements of A. Define $f : R \longrightarrow A^{\#}$ by $f(r) = \varphi_r$. Then:*

1. *R is closed in A under finite infs;*
2. *f is order-reversing;*
3. *$f(r \wedge s) = f(r) \vee f(s)$;*
4. *f is a bijection from R to $A^{\#}$.*

Proof.

1. Since A is compact, it is complete and hence has infs. Since $(r \wedge s)\!\downarrow = r\!\downarrow \cap s\!\downarrow$, the conclusion is obvious.

2. Obvious.

3. For any $a \in A$, we have that $\varphi_r(a) = \varphi_s(a) = 0$ if and only if $a \le r$ and $a \le s$ if and only if $a \le r \wedge s$ if and only if $\varphi_{r\wedge s}(a) = 0$.

4. It is obvious that whenever $r \in R$ then $\varphi_r \in A^{\#}$. Conversely, assume that $\varphi \in A^{\#}$. The kernel $K = \varphi^{-1}(0)$ must be clopen. By Corollary 8.4, K is principal and the generator obviously lies in R. ∎

What this means is that $R^{\mathrm{op}} = |A^{\#}|$, the underlying SSL of $A^{\#}$. The next result says that every homomorphism on $|A^{\#}|$ is represented by an element of A and is therefore uniform on $A^{\#}$. Thus $A^{\#}$ has the same **2**-valued morphisms as $|A^{\#}|$. By definition, $A^{\#}$ has the finest topology with the same **2**-valued morphisms as A^*, we conclude that that is $|A^*|$. Thus $A \cong |A^{\#}|^{\#}$ so that $A^{\#} = |A^*|$.

8.9. Theorem. *Let A be as above and let $\gamma : |A^*| \longrightarrow \mathbf{2}$ be an SSL morphism. Then there exists a unique $a \in A$ such that $\gamma(\varphi) = \varphi(a)$ for all $\varphi \in A^*$.*

Proof. Uniqueness is clear since the $\{\varphi_r \mid r \in R\}$ separate the points of A. What we want to find is an $a \in A$ such that $\gamma(\varphi_r) = 0$ if and only if $\varphi_r(a) = 0$ if and only if $a \le r$. Thus a should have the property that $\gamma(\varphi_r) = 0$ when $a \le r$ and $\gamma(\varphi_r) = 1$ when $a \not\le r$. Let $K_\gamma = \{r \in R \mid \gamma(\varphi_r) = 0\}$. We claim that $a = \bigwedge K_\gamma$ is the required element. In fact, for $r \in K_\gamma$, we have $a \le r$ so that $\varphi_r(a) = 0$. We must still show that $r \notin K_\gamma$ implies that $\varphi_r(a) = 1$. Since A is compact it has finite meets. Since γ preserves sups in R^{op}, it follows that K_γ is also closed under finite meet. If $r \in K_\gamma$ and $s \in R - K_\gamma$, it is clear that $r \not\le s$ so that $r \in R - s\!\downarrow$. Thus $r \in r\!\downarrow \cap \bigcap_{s \in R - s\downarrow}(A - s\!\downarrow)$. Compactness implies that

$$\bigcap_{r \in K_\gamma} r\!\downarrow \cap \bigcap_{s \in R - s\downarrow} (A - s\!\downarrow)$$

is non-empty and hence there is an element b in that set. Since $b \in r\!\downarrow$, for every $r \in K_\gamma$, we have that $b \le a$. On the other hand, if $b \notin s\!\downarrow$, then $a \notin s\!\downarrow$ and then $\varphi_a(s) = 1$, as required. ∎

8.10. Corollary. *If A is a compact SSL, then $A^* = R^{\mathrm{op}}$ with the discrete uniformity and the canonical map $A \longrightarrow R^{\mathrm{op}\#}$ is an isomorphism.* ∎

8.11. Two examples. The proofs above actually use compactness rather than just completeness. So it seems reasonable to ask whether every complete SSL has a compact topology in which it is a USSL. Here is an example of a complete SSL that does not admit a compact topology compatible with the sup. We let A consist of an infinite descending sequence $a_0 > a_1 > a_2 > \cdots > a_n > \cdots > 0$ together with an element x such that $a_0 > x > 0$, but x is not comparable to any other element. One easily sees that the sequence $a_0, a_1, \ldots$ can have only one cluster point 0, since any point a_n has a finite neighbourhood $A - a_{n+1}{\downarrow}$, and $A - a_1{\downarrow} = \{a_0, x\}$ is a finite neighbourhood of x. A compact topology has at least one cluster point and here that must be unique so that the sequence converges to 0. But then the sequence $x \vee a_0, x \vee a_1, \ldots$, which is constantly a_0, would have to converge to $x \vee 0 = x$, a contradiction.

The background of the second example is in topological abelian groups. All compact, in fact all locally compact abelian groups, have strong topologies in the sense used here. In particular, if A is compact and $B \longrightarrow A$ is a bijection that induces a bijection $A^* \longrightarrow B^*$, then $B \longrightarrow A$ is an isomorphism. Here we give an example to show that this fails for USSLs.

Let A be the one point compactification of $\mathbf{N}$, but ordered in such a way that $0 < n < \infty$ for any positive integer n, but no two positive integers are comparable. Thus when $n \neq m$ are both positive, then $m \vee n = \infty$. This space is first (even second) countable since it is embeddable into the unit interval (as the points of the form $n/(n+1)$, $n = 0, 1, \ldots, \infty$). Since it is also compact, to show that the $\vee$ operation is uniform, it suffices to show that when $a_1, a_2, \ldots$ converges to a and $b_1, b_2, \ldots$ converges to b, then $a_1 \vee b_1, a_2 \vee b_2, \ldots$ converges to $a \vee b$. But the only way a sequence can converge is if it is eventually constant or it converges to ∞. If both sequences converge to ∞, it is clear that their sup does as well. If, say the first is eventually constant at a, while the second converges to ∞, then for all but finitely many n, we have $a_n \vee b_n = \infty$. Finally if both sequences stabilize at finite a and b, respectively, then depending on whether $a = b$, either all but finitely many $a_n \vee b_n = a$ or all but finitely many $a_n \vee b_n = \infty = a \vee b$. The only ideals are $\{0\}$, the sets $\{0, n\}$ for a positive integer n, and all of A, each of which is open. Let B be the same SSL but with the discrete uniformity. Clearly, it has the same ideals as A so that $A^{\#} \longrightarrow B^{\#}$ is an isomorphism. The topology on $A^{\#}$ is thus the topology of pointwise convergence and hence so is that of $B^{\#}$. But this topology is thus that of the one-point compactification of $\mathbf{N}$. This example illustrates several phenomena.

1. An infinite compact SSL can have its strong uniformity be discrete.
2. An infinite discrete SSL can have its weak uniformity be compact.
3. An infinite compact SSL can be its own weak dual.
4. An infinite discrete SSL can be its own strong dual.

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