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MOLECULAR TOPOSES*

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0. Introduction

In [2], Barr and Diaconescu characterized those Grothendieck toposes $\mathscr{E}$ for which the inverse image, Δ, of the geometric morphism $\Gamma: \mathscr{E} \to \mathscr{S}et$, is logical. It was shown (among other things) that this happens precisely when the lattice of subobjects of every object of $\mathscr{E}$ is a complete atomic boolean algebra. Toposes satisfying this property are called *atomic*. These results were relativised to the case where $\Gamma: \mathscr{E} \to \mathscr{S}$ is an arbitrary morphism of elementary toposes. Their proofs used Mikkelsen's theorem [4] which says that a logical functor between toposes has a left adjoint if and only if it has a right adjoint, in order to obtain a left adjoint Λ to Δ. (E.g. in the $\mathscr{S}et$ based case, ΛA is the set of atomic subobjects of A.)

The purpose of this paper is to obtain analogous theorems characterizing those Grothendieck toposes $\mathscr{E}$ for which Δ has a left adjoint. For reasons which will become clear later, these toposes are called *molecular*. It is an exercise in [7, p. 414, Ex 7.6] that Sh(X) is molecular if and only if X is locally connected.

We also treat the relative case, where $\mathscr{S}et$ is replaced by an arbitrary elementary topos $\mathscr{S}$. These results may be taken as a definition and characterizations of what it means for an elementary topos to be locally connected over another topos. It is presumably because of topological considerations such as these that Joyal raised the question that resulted in this paper.

Tierney has also shown that our conditions are closely related to the problem of determining when a pullback of elementary toposes satisfies the Beck condition.

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1. The strong case

Let $\mathscr{S}$ be an elementary topos and $\mathscr{E}$ an $\mathscr{S}$-topos, i.e., $\mathscr{E}$ is an elementary topos and comes equipped with a geometric morphism into $\mathscr{S}$,

$$\mathscr{E} \underset{\Delta}{\overset{\Gamma}{\rightleftarrows}} \mathscr{S}.$$

We want to find conditions under which Δ has a left adjoint (more precisely, an indexed left adjoint).

We shall use the following theorem due to W. Butler [1].

Theorem 1. *Assume that in the following diagram of functors*

$$\begin{tikzcd} \mathscr{A} \arrow[rr, "\Phi"] \arrow[dr, shift left, "U"] & & \mathscr{B} \arrow[dl, shift left, "V"] \\ & \mathscr{C} \arrow[ul, shift left, "F"] \arrow[ur, shift left, "G"] & \end{tikzcd}$$

(1) *F is left adjoint to U and G is left adjoint to V,*
(2) *$\Phi F \cong G$,*
(3) *U is tripleable,*
(4) *Φ preserves coequalizers of U split pairs.*
Then Φ has a right adjoint.

Sketch of proof. Since U is tripleable, every object of $\mathscr{A}$ appears in a coequalizer

$$FUFUA \underset{FU\varepsilon A}{\overset{\varepsilon FUA}{\rightrightarrows}} FUA \xrightarrow{\varepsilon A} A$$

where ε is the counit of the adjunction $F \dashv U$. When Φ has a right adjoint Ψ there is a coequalizer

$$FUFU\Psi \underset{FU\varepsilon\Psi}{\overset{\varepsilon FU\Psi}{\rightrightarrows}} FU\Psi \xrightarrow{\varepsilon\Psi} \Psi.$$

There is also an isomorphism $U\Psi \cong V$ induced by (2). This leads us to define Ψ to be the coequalizer of

$$FUFV \underset{F\alpha}{\overset{\varepsilon FV}{\rightrightarrows}} FV$$

where α is defined by

$$\begin{tikzcd}
UFV \arrow[rr, "\alpha"] \arrow[d, "\eta' UFV"'] & & V \\
VGUFV & & VGV \arrow[u, "V\varepsilon'"'] \\
V\Phi FUFV \arrow[u, phantom, "\cong" description] \arrow[rr, "V\Phi\varepsilon FV"'] & & V\Phi FV \arrow[u, phantom, "\cong" description]
\end{tikzcd}$$

and η' and ε' are the unit and counit of the adjunction $G \dashv V$. That the coequalizer exists follows from tripleableness and the fact that

$$\begin{tikzcd}
UFUFV \arrow[r, shift left, "U\varepsilon FV"] \arrow[r, shift right, "UF\alpha"'] & UFV \arrow[l, bend left=50, "\eta UFV"] \arrow[r, "\alpha"] & V \arrow[l, bend left=50, "\eta V"]
\end{tikzcd}$$

is a split coequalizer diagram. □

$\mathscr{S}$ is a cartesian closed category and $\mathscr{E}$ may be considered as an $\mathscr{S}$-category if we define $\mathrm{Hom}(E, E')$ to be $\Gamma(E'^E)$. Then both Γ and Δ are strong functors and the adjointness $\Delta \dashv \Gamma$ is also strong.

Theorem 2. *Δ has a strong left adjoint if and only if Δ preserves exponentiation.*

Proof. Suppose that Δ has a strong left adjoint Λ. This means that

$$\Gamma(\Delta X^E) \cong X^{\Lambda E}$$

with the isomorphism natural in X and E. Then, for any X, Y in $\mathscr{S}$ and E in $\mathscr{E}$ we have the following sequence of natural bijections

$$\begin{array}{c}
E \to \Delta(X^Y) \\ \hline
\Lambda E \to X^Y \\ \hline
Y \to X^{\Lambda E} \\ \hline
Y \to \Gamma(\Delta X^E) \\ \hline
\Delta Y \to \Delta X^E \\ \hline
E \to \Delta X^{\Delta Y}
\end{array}$$

so, by the Yoneda lemma, $\Delta(X^Y) \cong \Delta X^{\Delta Y}$.

Conversely, assume that Δ preserves exponentiation and consider the following diagram

$$\begin{tikzcd}
\mathscr{S}^{\mathrm{op}} \arrow[rr, "\Delta^{\mathrm{op}}"] \arrow[dr, shift left, "\Omega^{(-)}"] & & \mathscr{E}^{\mathrm{op}} \arrow[dl, shift left, "\Gamma((\Delta\Omega)^{(-)})"] \\
& \mathscr{S} \arrow[ul, shift left, "\Omega^{(-)}"] \arrow[ur, shift left, "(\Delta\Omega)^{(\Delta -)}"] &
\end{tikzcd}$$

It satisfies the conditions of Butler's theorem: the bijections

$$[X, \Gamma(\Delta\Omega)^E]_{\mathscr{S}} \cong [\Delta X, \Delta\Omega^E]_{\mathscr{E}} \cong [E, \Delta\Omega^{\Delta X}]_{\mathscr{E}} \cong [\Delta\Omega^{\Delta X}, E]_{\mathscr{E}^{\mathrm{op}}}$$

show that $(\Delta\Omega)^{(\Delta -)}$ is left adjoint to $\Gamma((\Delta\Omega)^{(-)})$, that $\Omega^{(-)}$ is tripleable is proved in [5], and the other conditions are obvious. Thus Δ^{op} has a right adjoint, i.e. Δ has a left adjoint Λ.

Now, for E in $\mathscr{E}$ and X, Y in $\mathscr{S}$ we have the following natural bijections

$$\begin{array}{c}
Y \to \Gamma(\Delta X^E) \\ \hline
\Delta Y \to \Delta X^E \\ \hline
E \to \Delta X^{\Delta Y} \\ \hline
E \to \Delta(X^Y) \\ \hline
\Lambda E \to X^Y \\ \hline
Y \to X^{\Lambda E}
\end{array}$$

so $\Gamma(\Delta X^E) \cong X^{\Lambda E}$. This shows that the adjunction is strong. □

Remark. Another proof of the existence of Λ can be found in [6, p. 123] once we observe that since Δ is left exact, it will preserve exponentiation if and only if it preserves $\mathscr{S}$-$\varinjlim$.

Recall from [3, Proposition 8.2] that $\mathscr{E}$ is bounded (or has generators) over $\mathscr{S}$ if and only if there exists an object (of generators) G in $\mathscr{E}$ such that for any E in $\mathscr{E}$ there exist an I in $\mathscr{S}$, a subobject $G_0 \rightarrowtail \Delta I \times G$, and an epimorphism $G_0 \twoheadrightarrow E$. The following proposition is especially useful in applying theorems about triples, such as Butler's, to bounded toposes.

Theorem 3. *If $\mathscr{E}$ is bounded over $\mathscr{S}$ with object of generators G, then the functor*

$$\Gamma(\Omega^{G \times (\,)}) : \mathscr{E}^{\mathrm{op}} \to \mathscr{S}$$

is tripleable.

Proof. The natural isomorphisms

$$\begin{aligned}
[X, \Gamma(\Omega^{G \times E})]_{\mathscr{S}} &\cong [\Delta X, \Omega^{G \times E}]_{\mathscr{E}} \cong [G \times E \times \Delta X, \Omega]_{\mathscr{E}} \\
&\cong [E, \Omega^{G \times \Delta X}]_{\mathscr{E}} \cong [\Omega^{G \times \Delta X}, E]_{\mathscr{E}^{\mathrm{op}}}
\end{aligned}$$

show that $\Omega^{G \times \Delta(\,)}$ is left adjoint to $\Gamma(\Omega^{G \times (\,)})$.

$\mathscr{E}^{\mathrm{op}}$ has all reflexive coequalizers and by the results of [5], $\Omega^{(\ )}$ transforms reflexive coequalizers to split ones, so that $\Gamma(\Omega^{G\times(\ )})$, which is isomorphic to $\Gamma((\Omega^{(\ )})^G)$, must preserve reflexive coequalizers. We still have to show that $\Gamma(\Omega^{G\times(\ )})$ is faithful (and therefore, since $\mathscr{E}$ is balanced, that it reflects isomorphisms) in order that it satisfy the conditions of the RTT (see [5]).

Let $f, g : A \rightrightarrows B$ be different morphisms of $\mathscr{E}$. Since $\Omega^{(\ )}$ is faithful, $\Omega^f, \Omega^B : \Omega^B \rightrightarrows \Omega^A$ are different. By the generating property of G there exist I in $\mathscr{S}$, a subobject $G_0 \rightarrowtail \Delta I \times G$, and an epimorphism $G_0 \rightrightarrows \Omega^B$. Since Ω^B is injective this epimorphism lifts to an epimorphism $\Delta I \times G \to \Omega^B$. Thus we have the correspondences

$$\begin{array}{lc}
\Delta I \times G \twoheadrightarrow \Omega^B \underset{\Omega^g}{\overset{\Omega^f}{\rightrightarrows}} \Omega^A & \text{different} \\
\hline
\Delta I \to \Omega^{G\times B} \rightrightarrows \Omega^{G\times A} & \text{different} \\
\hline
I \to \Gamma(\Omega^{G\times B}) \rightrightarrows \Gamma(\Omega^{G\times A}) & \text{different}
\end{array}$$

and so $\Gamma(\Omega^{G\times f})$ and $\Gamma(\Omega^{G\times g})$ must be different, i.e. $\Gamma(\Omega^{G\times(\ )})$ is faithful. □

We are now able to prove the following generalization of Theorem 2.

Theorem 4. *Let $\mathscr{E}$ and $\mathscr{F}$ be $\mathscr{S}$-toposes with $\mathscr{E}$ bounded over $\mathscr{S}$, and $f : \mathscr{F} \to \mathscr{E}$ a geometric morphism over $\mathscr{S}$. Then $f^* : \mathscr{E} \to \mathscr{F}$ has an $\mathscr{S}$-strong left adjoint if and only if f^* preserves exponentials of the form $E^{\Delta X}$.*

Proof. We denote the structural geometric morphisms from $\mathscr{E}$ and $\mathscr{F}$ to $\mathscr{S}$ by the same symbol Γ (with inverse images Δ). To say that f^* has an $\mathscr{S}$-strong left adjoint $f_!$ means that there are isomorphisms

$$\Gamma(E^{f_!F}) \cong \Gamma((f^*E)^F) \tag{1}$$

natural in E and F. For any X in $\mathscr{S}$, E in $\mathscr{E}$, and F in $\mathscr{F}$, we have the following natural bijections

$$\begin{aligned}
[F, f^*(E^{\Delta X})]_{\mathscr{F}} &\cong [f_!F, E^{\Delta X}]_{\mathscr{E}} \cong [\Delta X, E^{f_!F}]_{\mathscr{E}} \\
&\cong [X, \Gamma(E^{f_!F})]_{\mathscr{S}}
\end{aligned} \tag{2}$$

and

$$\begin{aligned}
[F, (f^*E)^{(f^*\Delta X)}]_{\mathscr{F}} &\cong [F, (f^*E)^{\Delta X}]_{\mathscr{F}} \cong [\Delta X, (f^*E)^F]_{\mathscr{F}} \\
&\cong [X, \Gamma((f^*E)^F)]_{\mathscr{S}}.
\end{aligned} \tag{3}$$

Now (1) tells us that $[X, \Gamma(E^{f_!F})]_{\mathscr{S}} \cong [X, \Gamma((f^*E)^F)]_{\mathscr{S}}$, so

$$[F, f^*(E^{\Delta X})]_{\mathscr{F}} \cong [F, (f^*E)^{(f^*\Delta X)}]_{\mathscr{F}}$$

and by the Yoneda lemma, $f^*(E^{\Delta X}) \cong (f^*E)^{(f^*\Delta X)}$.

Conversely, assume that f^* preserves exponentials of the form $E^{\Delta X}$. The diagram

$$\begin{tikzcd}
\mathscr{E}^{\mathrm{op}} \arrow[rr, "f^{*\mathrm{op}}"] \arrow[dr, shift left, "\Gamma(\Omega^{G\times(\ )})"] & & \mathscr{F}^{\mathrm{op}} \arrow[dl, shift left, "\Gamma(F_0^{(\ )})"] \\
& \mathscr{S} \arrow[ul, shift left, "\Omega^{G\times\Delta(\ )}"] \arrow[ur, shift left, "F_0^{\Delta(\ )}"] &
\end{tikzcd}$$

satisfies the conditions of Butler's theorem if we take F_0 to be $f^*(\Omega^G)$. Indeed, $\Gamma(\Omega^{G\times(\ )})$ is tripleable by theorem 3, $F_0^{\Delta(\ )}$ is always left adjoint to $\Gamma(F_0^{(\ )})$, and

$$f^*(\Omega^{G\times\Delta X}) \cong f^*((\Omega^G)^{\Delta X}) \cong f^*(\Omega^G)^{f^*\Delta X} \cong f^*(\Omega^G)^{\Delta X}.$$

Thus $f^{*\mathrm{op}}$ has a right adjoint, i.e. f^* has a left adjoint $f_!$. Since $f^*(E^{\Delta X}) \cong (f^*E)^{(f^*\Delta X)}$, (2) and (3) give that

$$[X, \Gamma(E^{f_!F})]_{\mathscr{S}} \cong [X, \Gamma((f^*E)^F)]_S$$

so $\Gamma(E^{f_!F}) \cong \Gamma((f^*E)^F)$, i.e. the adjunction $f_! \dashv f^*$ is $\mathscr{S}$-strong. $\square$

Remark. As $\mathscr{S}$-categories $\mathscr{E}$ and $\mathscr{F}$ are cotensored, the cotensor of E with X given by $E^{\Delta X}$. Thus the condition of the previous theorem can be stated as: f^* preserves cotensors.

2. The indexed case

An $\mathscr{S}$-topos $\mathscr{E}$ is not merely a category enriched over $\mathscr{S}$ but has the much richer structure of an $\mathscr{S}$-indexed category with small homs. This means that $\mathscr{E}$ comes equipped with a notion of families of objects of $\mathscr{E}$ indexed by an object of $\mathscr{S}$ and satisfies some conditions. For I in $\mathscr{S}$, the category $\mathscr{E}^I$ of I-indexed families of objects of $\mathscr{E}$ is defined to be the slice category $\mathscr{E}/\Delta I$. For $\alpha : J \to I$ in $\mathscr{S}$, the substitution functor $\alpha^* : \mathscr{E}^I \to \mathscr{E}^J$ is defined by pulling back along $\Delta\alpha$. When we specialize this to the case $\mathscr{E} = \mathscr{S}$, we get the canonical indexing of $\mathscr{S}$, namely $\mathscr{S}^I = \mathscr{S}/I$.

To be indexed, a functor Φ must come equipped with functors Φ^I defined on the I-families, compatible with the substitution functors in the sense that $\alpha^*\Phi^I \cong \Phi J\alpha^*$ for all α. For example, if $f : \mathscr{F} \to \mathscr{E}$ is a geometric morphism of $\mathscr{S}$-toposes, then f^* can be made into an indexed functor by defining f^{*I} by

$$(F \to \Delta I) \mapsto (f^*F \to f^*\Delta I \cong \Delta I).$$

We can also give an indexing to f_* by defining f_*^I at $E \to \Delta I$ as the top line in the pullback square

$$\begin{tikzcd} P \arrow[r] \arrow[d] & \Delta I \arrow[d] \\ f_*E \arrow[r] & f^*\Delta I \cong f_* f^* \Delta I\,. \end{tikzcd}$$

It turns out that f^{*I} is a left exact left adjoint to f_*^I, making f into an indexed geometric morphism. In particular, Γ and Δ are part of an indexed geometric morphism. The reader is referred to [6] for more details on indexed categories.

The enrichment of $\mathscr{E}$ over $\mathscr{S}$ is completely determined by the indexing of $\mathscr{E}$ and the requirement that for E_1 and E_2 in $\mathscr{E}$, $\mathrm{Hom}(E_1, E_2)$ is an object of $\mathscr{S}$ with the property

$$[I, \mathrm{Hom}(E_1, E_2)]_{\mathscr{S}} \cong [I^*E_1, I^*E_2]_{\mathscr{E}^I}$$

natural in I (I^* is the substitution functor corresponding to the unique morphism $I : I \to 1$). Because of this, indexed functors are automatically strong, and because of *this*, indexedness is harder to achieve but more useful.

We believe that the correct way to relativise the results of [7, p. 414] to an arbitrary base topos $\mathscr{S}$ is to require that the Λ and $f_!$ of Propositions 2 and 4 be *indexed* left adjoints.

Definitions. A geometric morphism $f : \mathscr{F} \to \mathscr{E}$ over $\mathscr{S}$ will be called *$\mathscr{S}$-essential* iff f^* has an $\mathscr{S}$-indexed left adjoint $f_!$. $\mathscr{E}$ will be called *S-molecular* if Γ is $\mathscr{S}$-essential.

Example. If $\mathbf{C}$ is a category object in $\mathscr{S}$, then $\mathscr{S}^{\mathbf{C}}$ is $\mathscr{S}$-molecular. Indeed, the left adjoint to Δ is $\varinjlim_{\mathbf{C}}$, which is indexed since α^* preserves everything in its construction.

If Δ has an ordinary left adjoint Λ, we can define $\Lambda^I : E^I \to \mathscr{S}^I$ by

$$\Lambda^I(E \xrightarrow{p} \Delta I) = (\Lambda E \xrightarrow{\bar{p}} I)$$

where $\bar{p}$ is the morphism corresponding to p under the adjointness $\Lambda \dashv \Delta$. It is easily seen that $\Lambda^I \dashv \Delta^I$, *but* Λ is not necessarily indexed as the following example shows. Let $\mathscr{S} = \mathscr{S}et^2$, $\mathscr{E} = \mathscr{S}et \times \mathscr{S}et$, $\Gamma(A, B) = (A \times B \to B)$, $\Delta(A \to B) = (A, B)$, and $\Lambda(A, B) = (A \to A + B)$. Taking α to be the unique morphism $(0 \to 1) \to (1 \to 1)$ shows that Λ is not indexed.

In concrete terms, for Λ to be indexed means that if (1) is a pull back, then so is (2):

$$\begin{tikzcd} E' \arrow[r] \arrow[d, "q"'] \arrow[dr, phantom, "(1)"] & E \arrow[d, "p"] \\ \Delta J \arrow[r, "\Delta\alpha"'] & \Delta I \end{tikzcd}, \qquad \begin{tikzcd} \Lambda E' \arrow[r] \arrow[d, "\bar{q}"'] \arrow[dr, phantom, "(2)"] & \Lambda E \arrow[d, "\bar{p}"] \\ J \arrow[r, "\alpha"'] & I\,. \end{tikzcd}$$

Intuitively, take $J=1$, $\alpha \in I$ then the condition means that $(\Lambda E)_\alpha = \Lambda(E_\alpha)$, or thinking of E as $\sum E_\alpha$, $\Lambda(\sum E_\alpha) = \sum(\Lambda E_\alpha)$, i.e. Λ preserves internal sums, which is a reasonable condition.

Theorem 5. *$\mathscr{E}$ is $\mathscr{S}$-molecular if and only if Δ preserves Π_α for all α in $\mathscr{S}$.*

Proof. For any $\alpha : J \to I$ in $\mathscr{S}$, any object $p : E \to \Delta I$ of $\mathscr{E}^I$, and any object $x : X \to J$ of $\mathscr{S}^J$, if Δ has a left adjoint Λ we have the following natural isomorphisms

$$[\alpha^* \Lambda^I(p), x]_{\mathscr{S}^J} \cong [\Lambda^I(p), \Pi_\alpha(x)]_{\mathscr{S}^I} \cong [p, \Delta^I \Pi_\alpha(x)]_{\mathscr{E}^I}$$

and

$$[\Lambda^J \alpha^*(p), x]_{\mathscr{S}^J} \cong [\alpha^*(p), \Delta^J(x)]_{\mathscr{E}^J} \cong [p, \Pi_{\Delta\alpha} \Delta^J(x)]_{\mathscr{E}^I}.$$

Thus, by the Yoneda lemma,

$$\alpha^* \Lambda^I \cong \Lambda^J \alpha^* \quad \text{if and only if} \quad \Delta^I \Pi_\alpha \cong \Pi_{\Delta\alpha} \Delta^J,$$

i.e. Λ is indexed if and only if Δ preserves Π. To complete the proof, notice that if Δ preserves Π_I (the I here denotes the unique $I \to 1$) then it preserves exponentiation since $X^I = \Pi_I(I^* X)$. So by Theorem 2, Δ has a left adjoint. □

Similarly, for a geometric morphism $f : \mathscr{F} \to \mathscr{E}$ over $\mathscr{S}$, if f^* has an ordinary left adjoint $f_!$, we can define $f_!^I$ by

$$f_!^I(F \xrightarrow{p} \Delta I) = (f_! F \xrightarrow{\bar{p}} \Delta I)$$

where $\bar{p}$ corresponds by adjointness to $F \xrightarrow{p} \Delta I \cong f^* \Delta I$. Then $f_!^I$ is left adjoint to f^{*I} and makes $f_!$ into an indexed left adjoint iff (2) below is a pullback whenever (1) is:

$$\begin{tikzcd} F' \arrow[r] \arrow[d, "p'"'] \arrow[dr, phantom, "(1)"] & F \arrow[d, "p"] \\ \Delta \arrow[r, "\Delta\alpha"'] & \Delta I \end{tikzcd}, \qquad \begin{tikzcd} f_! F' \arrow[r] \arrow[d, "\bar{p}'"'] \arrow[dr, phantom, "(2)"] & f_! F \arrow[d, "\bar{p}"] \\ \Delta J \arrow[r, "\Delta\alpha"'] & \Delta I. \end{tikzcd}$$

The proof of the following proposition is similar to that of Theorem 5, and will be omitted.

Theorem 6. *A morphism of $\mathscr{S}$-toposes, $f : \mathscr{F} \to \mathscr{E}$, is $\mathscr{S}$-essential iff f^* preserves $\Pi_{\Delta\alpha}$ for all α in $\mathscr{S}$.* □

It is an open question whether f^* preserving exponentials of the form $E^{\Delta X}$ implies that f^* also preserves $\Pi_{\Delta\alpha}$ (even when $f^* = \Delta$).

3. $\mathcal{S}$-definable subobjects

For $\mathscr{E}$ an $\mathcal{S}$-topos, the canonical morphism $d:\Delta\Omega\to\Omega$ is defined to be the characteristic morphism of $\Delta t:\Delta 1\rightarrowtail\Delta\Omega$, i.e. such that

$$\begin{tikzcd}
\Delta\Omega \arrow[r, "d"] & \Omega \\
\Delta 1 \arrow[r] \arrow[u, tail, "\Delta t"] & 1 \arrow[u, tail, "t"']
\end{tikzcd}$$

is a pullback. If $e:E\to\Delta\Omega$ is any morphism, then the subobject classified by de is given by the pull back

$$\begin{tikzcd}
E \arrow[r, "e"] & \Delta\Omega \\
E_0 \arrow[r] \arrow[u, tail] & \Delta 1 \arrow[u, tail, "\Delta t"']
\end{tikzcd}$$

Part of the following proposition appears in [4, p. 75] where d and related morphisms are studied in detail.

Proposition 7. *$\Delta\Omega$ is a lattice object in $\mathscr{E}$ and d is a lattice homomorphism.*

Proof. The structure of a lattice is that of a finitary algebraic theory which will be preserved by Δ because it is left exact.

The squares in the following diagrams are pullbacks and the triangles are image factorizations.

$$\begin{tikzcd}
 & \Delta\Omega\times\Delta\Omega \arrow[rr, "d\times d"] & & \Omega\times\Omega \arrow[r, "v"] & \Omega \\
 & \cdot \arrow[rr] \arrow[u, tail] & & \cdot \arrow[r] \arrow[u, tail] & 1 \arrow[u, tail] \\
\Delta\Omega\times\Delta 1+\Delta 1\times\Delta\Omega \arrow[rr] \arrow[uur, "\binom{\Delta\Omega\times\Delta t}{\Delta t\times\Delta\Omega}"] \arrow[ur] & & \Omega\times 1+1\times\Omega \arrow[uur, "\binom{\Omega\times t}{t\times\Omega}"] \arrow[ur] & &
\end{tikzcd}$$

$$\begin{tikzcd}
 & \Delta(\Omega \times \Omega) \arrow[r, "\Delta(v)"] & \Delta\Omega \arrow[r, "d"] & \Omega \\
 & \Delta(\cdot) \arrow[r] \arrow[u, tail] & \Delta 1 \arrow[r] \arrow[u, tail, "\Delta t"'] & 1 \arrow[u, tail] \\
\Delta(\Omega \times 1 + 1 \times \Omega) \arrow[uur, "\Delta\binom{\Omega \times t}{t \times \Omega}"] \arrow[ur] & & &
\end{tikzcd}$$

Since Δ preserves binary products and coproducts, $v \cdot d \times d$ and $d \cdot \Delta(v)$ classify the same subobject and so $v \cdot d \times d = d \cdot \Delta(v)$, i.e. d preserves v.

The other parts are similar but easier. □

When $\mathscr{S}$ is boolean, $\Delta\Omega = 2$ and $d = \binom{t}{f}$ which is always monic. However not all d are monic if $\mathscr{S}$ is not boolean as the following proposition shows.

Proposition 8. *Let $f : X \to Y$ be a continuous function and (f_*, f^*) the corresponding geometric morphism $\mathrm{Sh}(X) \to \mathrm{Sh}(Y)$. Then $d : f^*\Omega \to \Omega$ is monic if and only if f^{-1} preserves interiors of sets of the form $V \cup G$ where V is open and G is closed in Y. In particular, if f is open, then d is monic.*

Proof. $d : f^*\Omega \to \Omega$ is monic if and only if for every $x \in X$, $d_x : (f^*\Omega)_x \to \Omega_x$ is monic. To describe Ω_x explicitly, write $U_1 \subseteq_x U_2$ (resp. $U_1 =_x U_2$) if there exists an open neighbourhood V of x such that $U_1 \cap V \subseteq U_2 \cap V$ (resp. $U_1 \cap V = U_2 \cap V$). It is easily seen that $\subseteq_x$ is a preorder and $=_x$ is an equivalence relation on the opens of X. Δ_x is the set of equivalence classes, $[U]_x$ for U open in X, and $\subseteq_x$ induces an order relation on Ω_x, making it into a lattice. We have a similar description for $(f^*\Omega)_x$ which is Ω_{fx}. Then $d_x([V]_{fx}) = [f^{-1}V]_x$. Since d_x is a lattice homomorphism, it is monic if and only if

$$[V_1]_{fx} \leq [V_2]_{fx} \iff [f^{-1}V_1]_x \leq [f^{-1}V_2]$$

i.e. if and only if

$$V_1 \subseteq_{fx} V_2 \iff f^{-1}V_1 \subseteq_x f^{-1}V_2.$$

It is easily seen that $V_1 \subseteq_{fx} V_2$ is equivalent to $fx \in (V_2 \cup V_1')^0$, and $f^{-1}V_1 \subseteq_x f^{-1}V_2$ is equivalent to $x \in (f^{-1}V_2 \cup (f^{-1}V_1)')^0$, where $(\ )^0$ denotes interior and $(\ )'$ complement. Thus d_x is monic for every x if and only if

$$f^{-1}((V_2 \cup V_1')^0) = (f^{-1}V_2 \cup (f^{-1}V_1)')^0,$$

i.e.

$$f^{-1}((V_2 \cup V_1')^0) = (f^{-1}(V_2 \cup V_1'))^0.$$

Finally, it is well known that f is open if and only if f^{-1} preserves all interiors. □

For example, if $f:|\mathbf{R}|\to\mathbf{R}$ is the identity function from the reals with the discrete topology to the reals with the usual topology, the corresponding d is not monic since f^{-1} does not even preserve interiors of closed sets.

In the same vein we have the following proposition also due to Mikkelsen [4, p. 81].

Proposition 9. *The morphism $d:\Delta\Omega\to\Omega$ is monic if and only if Δ preserves implication (i.e. for any subobjects X_1, $X_2\rightarrowtail X$, $\Delta(X_1\Rightarrow X_2)=(\Delta X_1\Rightarrow\Delta X_2)$ as subobjects of ΔX).*

Definition. Let $\mathscr{E}$ be an $\mathscr{S}$-topos for which d is monic. A subobject $E_0\rightarrowtail E$ in $\mathscr{E}$ is called *$\mathscr{S}$-definable* (or d-subobject, for short) if its characteristic morphism factors through d.

Clearly, $\Delta\Omega$ classifies $\mathscr{S}$-definable subobjects in the sense that the association

$$(\varphi:E\to\Delta\Omega)\mapsto(\varphi^*\Delta 1\xrightarrow{\varphi^*\Delta t}E)$$

is a bijection between morphisms into $\Delta\Omega$ and d-subobjects of E.

Examples. If $\mathscr{S}$ is boolean, then $\mathscr{S}$-definable means complemented.

If $\mathscr{S}$ is a Grothendieck topos and $\mathbf{C}$ an ordinary small category, then $\mathscr{S}^{\mathbf{C}}$ is a topos over $\mathscr{S}$ and $\Phi_0\rightarrowtail\Phi$ is $\mathscr{S}$-definable if and only if for every $c:C\to C'$ in $\mathbf{C}$,

$$\begin{tikzcd}
\Phi_0 C' \arrow[r, tail] & \Phi C' \\
\Phi_0 C \arrow[r, tail] \arrow[u, "\Phi_0 c"] & \Phi C \arrow[u, "\Phi c"']
\end{tikzcd}$$

is a pullback.

Definition. An *I-family of $\mathscr{S}$-definable* subobjects is an $\mathscr{S}/I$-definable subobject of $\mathscr{E}/\Delta I$ (which is an $\mathscr{S}/I$-topos via Γ^I and Δ^I, and whose d is $\Delta I\times d$ which is still monic).

The following proposition summarizes some of the properties of $\mathscr{S}$-definable subobjects.

Proposition 10. (i) *If $X_0\rightarrowtail X$ is a subobject in $\mathscr{S}$, then $\Delta X_0\rightarrowtail\Delta X$ is a d-subobject.*

(i) *The inverse image (i.e. pullback) of a d-subobject is a d-subobject.*

(ii) *The d-subobjects of an object E are closed under finite intersections and unions, and implication.*

(iv) *The graph of any morphism $E \to \Delta X$ is a d-subobject.*

(v) *If $p_0 : E_0 \to \Delta I$ is a subobject of $p : E \to \Delta I$ in $\mathscr{E}/\Delta I$, then p_0 is an I-family of d-subobjects if and only if E_0 is a d-subobject of E (i.e. $\sum_I p_0$ is a d-subobject of $\sum_I p$).*

Proof. (i) and (ii) are trivial.

The first part of (iii) is a restatement of Proposition 7. If $E_1, E_2 \rightarrowtail E$ are d-subobjects with classifying morphisms k_1, $k_2 : E \to \Delta\Omega$ then we have the following pullbacks

$$\begin{tikzcd}
E \arrow[r, "{(k_1,k_2)}"] & \Delta\Omega \times \Delta\Omega \arrow[r, phantom, "\cong"] & \Delta(\Omega \times \Omega) \\
E_1 \arrow[u, tail] \arrow[r] & \Delta 1 \times \Delta\Omega \arrow[u, tail, "\Delta t \times \Delta\Omega"'] \arrow[r, phantom, "\cong"] & \Delta(1 \times \Omega) \arrow[u, tail, "\Delta(t \times \Omega)"']
\end{tikzcd}$$

and

$$\begin{tikzcd}
E \arrow[r, "{(k_1,k_2)}"] & \Delta\Omega \times \Delta\Omega \arrow[r, phantom, "\cong"] & \Delta(\Omega \times \Omega) \\
E_2 \arrow[u, tail] \arrow[r] & \Delta\Omega \times \Delta 1 \arrow[u, tail, "\Delta\Omega \times \Delta t"'] \arrow[r, phantom, "\cong"] & \Delta(\Omega \times 1) \arrow[u, tail, "\Delta(\Omega \times t)"']
\end{tikzcd}$$

and so $E_1 \Rightarrow E_2$ is the pullback along (k_1, k_2) of $\Delta(t \times \Omega) \Rightarrow \Delta(\Omega \times t)$ which by Proposition 9 is equal to $\Delta(t \times \Omega \Rightarrow \Omega \times t)$ so by (i) and (ii), $E_1 \Rightarrow E_2$ is a d-subobject.

Applying (i) and (ii) to the following pullback will give (iv):

$$\begin{tikzcd}
E \times \Delta X \arrow[r, "\varphi \times \Delta X"] & \Delta X \times \Delta X \arrow[r, phantom, "\cong"] & \Delta(X \times Y) \\
E \arrow[u, tail, "{(E,\varphi)}"] \arrow[r] & \Delta X \arrow[u, tail] \arrow[r, phantom, "\cong"] & \Delta X \arrow[u, tail, "\Delta(\mathrm{diag})"']
\end{tikzcd}$$

Finally, (v) follows from the fact that the d for $\mathscr{E}/\Delta I$ as an $\mathscr{S}/I$-topos is

$$\begin{tikzcd}
\Delta I \times \Delta\Omega \arrow[rr, tail, "\Delta I \times d"] \arrow[dr, "p_1"'] & & \Delta I \times \Omega \arrow[dl, "p_1"] \\
& \Delta I &
\end{tikzcd}$$

and the characteristic morphism of $p_0 \rightarrowtail p$ is

$$\begin{tikzcd} E \arrow[rr, "(p{,}\chi_{E_0})"] \arrow[dr, "p"'] & & \Delta I \times \Omega \arrow[dl, "p_1"] \\ & \Delta I & \end{tikzcd}$$

Arbitrary $\mathcal{S}$-unions or $\mathcal{S}$-intersections of d-subobjects need not be d-subobjects, even when $\mathcal{S} = \mathcal{S}et$, as the following example shows. Let X be the topological space $\{\pm 1/n \mid n \in \mathbf{N}\} \cup \{0\}$ with the usual topology, and take $\mathscr{E} = \mathrm{Sh}(X)$. The d-subobjects of 1 in $\mathscr{E}$ are the complemented subobjects, i.e. the clopen sets. Each of the singletons $\{1/n\}$ is clopen but $\bigcup\{1/n\}$ is not. Taking complements, we see that the d-subobjects are not closed under $\cap$ either.

The following example shows that the composite of two d-subobjects need not be a d-subobject (although, if $\mathcal{S}$ is boolean, it must). Take

$$\mathcal{S} = \mathcal{S}et^{\cdot \to \cdot} \quad \text{and} \quad \mathscr{E} = \mathcal{S}et^{\cdot \rightrightarrows \cdot}$$

with

$$\Gamma(A \underset{g}{\overset{f}{\rightrightarrows}} B) = (Eq(f, g) \to B) \quad \text{and} \quad \Delta(A \xrightarrow{f} B) = (A \underset{f}{\overset{f}{\rightrightarrows}} B).$$

Then it is easily checked that d is monic and

$$\begin{tikzcd} A_0 \arrow[r, tail] \arrow[d, shift right, "f_0"'] \arrow[d, shift left, "g_0"] & A \arrow[d, shift right, "f"'] \arrow[d, shift left, "g"] \\ B_0 \arrow[r, tail] & B \end{tikzcd}$$

is a d-subobject if and only if $\forall a\ (fa \in B_0 \equiv ga \in B_0)$. The two monos below satisfy this property but not their composite:

$$\begin{tikzcd} \phi \arrow[r, equal] \arrow[d, shift right] \arrow[d, shift left] & \phi \arrow[r, tail] & \{0\} \arrow[d, shift right, "0"'] \arrow[d, shift left, "1"] \\ \{0\} \arrow[r, tail] & \{0, 1\} \arrow[r, equal] & \{0, 1\} \end{tikzcd}$$

Definition. We shall say that a morphism $f: E \to E'$ is *$\mathcal{S}$-defined* if it fits in a pullback diagram of the form

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d] & E' \arrow[d] \\ \Delta I \arrow[r, "\Delta\alpha"'] & \Delta J' \end{tikzcd}$$

Intuitively this means that both E and E' can be partitioned into direct sums in such a way that f restricted to each summand of E is an isomorphism into some summand of E'.

Since pulling back preserves image factorizations it follows that the image of an $\mathcal{S}$-defined morphism is an $\mathcal{S}$-defined subobject. Also, since $\mathcal{S}$-defined subobjects are $\mathcal{S}$-defined morphisms we cannot expect the $\mathcal{S}$-defined morphisms to be closed under composition in general.

Before proving the main theorem of this section, we need the following lemma.

Lemma 11. *In a regular category, $R \rightarrowtail X \times Y$ is the graph of a morphism $X \to Y$ if and only if*

(i) $\exists_Y R = 1_X$ *where* $\exists_Y : \mathrm{Sub}(X \times Y) \to \mathrm{Sub}(X)$

(ii) $p_1^* R \cap p_2^* R = \exists_\delta R$ *where*

$$p_1^*, p_2^*, \exists_\delta : \mathrm{Sub}(X \times Y) \to \mathrm{Sub}(X \times Y \times Y)$$

i.e. the following is a limit diagram in $\mathcal{S}et$

$$\begin{tikzcd} {[X, Y]} \arrow[r, tail] & \mathrm{Sub}(X \times Y) \arrow[r, shift left, "\exists_\delta"] \arrow[r, shift right, "p_1^* \wedge p_2^*"'] \arrow[d, shift right, "\exists_Y"'] \arrow[d, shift left, "\ulcorner 1_X \urcorner"] & \mathrm{Sub}(X \times Y \times Y) \\ & \mathrm{Sub}(X) & \end{tikzcd}$$

Proof. If $(f, g) : R \rightarrowtail X \times Y$, then (i) says that f is a regular epi. By noting that the kernel pair of f can be calculated in stages as in the following diagram

$$\begin{tikzcd} R \arrow[r, tail, "{(f,g)}"] & X \times Y \arrow[r, "p"] & X \\ p_1^* R \arrow[r] \arrow[u] & X \times Y \times Y \arrow[r, "X \times p_2"'] \arrow[u, "X \times p_1"] & X \times Y \arrow[u, "p"] \\ p_1^* R \cap p_2^* R \arrow[r, tail] \arrow[u, tail] & p_2^* R \arrow[r] \arrow[u, tail] & R \arrow[u, tail, "{(f,g)}"] \end{tikzcd}$$

a simple diagram chase will tell us that (ii) is equivalent to f being monic. □

Theorem 12. *$\mathscr{E}$ is $\mathscr{S}$-molecular if and only if*

(1) *d is monic,*

(2) *for every E in $\mathscr{E}$ there exists an object ΛE in $\mathscr{S}$ and a lattice isomorphism $\Gamma(\Delta\Omega^E) \cong \Omega^{\Lambda E}$, and*

(3) *the composite of $\mathscr{S}$-definable morphisms is $\mathscr{S}$-definable.*

Proof. Assume $\mathscr{E}$ molecular. Let $f, g : E \to \Delta\Omega$ be such that $df = dg$. Then we get a diagram

$$\begin{tikzcd}
E \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & \Delta\Omega \arrow[r, "d"] & \Omega \\
E_0 \arrow[u, tail] \arrow[r] & \Delta 1 \arrow[u, tail, "\Delta t"'] \arrow[r] & 1 \arrow[u, tail, "t"']
\end{tikzcd}$$

in which the right hand square and the composite rectangle and hence the left hand square are pullbacks whether f or g is taken as the top map. Since Λ is an indexed left adjoint

$$\begin{tikzcd}
\Lambda E \arrow[r, shift left, "\bar{f}"] \arrow[r, shift right, "\bar{g}"'] & \Omega \\
\Lambda E_0 \arrow[u] \arrow[r] & 1 \arrow[u]
\end{tikzcd}$$

are pullbacks as well, where $\bar{f}$ and $\bar{g}$ correspond to f and g respectively under the adjunction. Since a subobject has a unique classifying map, $\bar{f} = \bar{g}$ whence $f = g$. Thus d is monic.

Indexed adjointness implies strong adjointness, of which (2) is an instance.

If $f : E \to E'$ is an $\mathscr{S}$-definable morphism, it appears in a pullback diagram

$$\begin{tikzcd}
E' \arrow[r] & \Delta I' \\
E \arrow[u, "f"] \arrow[r] & \Delta I \arrow[u, "\Delta\alpha"']
\end{tikzcd}$$

which by indexed adjointness, gives a pullback diagram

$$\begin{tikzcd}
\Lambda E' \arrow[r] & I' \\
\Lambda E \arrow[u, "\Lambda f"] \arrow[r] & I \arrow[u, "\alpha"']
\end{tikzcd}$$

In the following diagram

$$\begin{tikzcd} E' \arrow[r, "\eta E'"] & \Delta\Lambda E' \arrow[r] & \Delta I' \\ E \arrow[u, "f"] \arrow[r, "\eta E"'] & \Delta\Lambda E \arrow[u, "\Delta\Lambda f"'] \arrow[r] & \Delta I \arrow[u, "\Delta\alpha"'] \end{tikzcd}$$

the right hand square is a pullback since Δ is left exact, and the rectangle is a pullback by hypothesis, so the left hand square is also a pullback. If $f': E' \to E''$ is also $\mathscr{S}$-definable then the rectangle

$$\begin{tikzcd} E'' \arrow[r] & \Delta\Lambda E'' \\ E' \arrow[u, "f'"] \arrow[r] & \Delta\Lambda E' \arrow[u] \\ E \arrow[u, "f"] \arrow[r] & \Delta\Lambda E \arrow[u] \end{tikzcd}$$

is a pullback, thus showing that $f'f$ is $\mathscr{S}$-definable.

Conversely, assume conditions (1), (2), and (3). For any E in $\mathscr{E}$ and X in $\mathscr{S}$ we have the following lattice isomorphisms:

$$\begin{aligned} \mathrm{Sub}(\Lambda E \times X) &\cong [\Lambda E \times X, \Omega] \cong [X, \Omega^{\Lambda E}] \cong [X, \Gamma(\Delta\Omega^E)] \\ &\cong [\Delta X, \Delta\Omega^E] \cong [E \times \Delta X, \Delta\Omega] \cong d\text{-}\mathrm{Sub}(E \times \Delta X), \end{aligned}$$

which are natural in X and where d-Sub$(E \times \Delta X)$ represents the lattice of $\mathscr{S}$-definable subobjects of $E \times \Delta X$. The monomorphism $d: \Delta\Omega \to \Omega$ induces an inclusion of lattices

$$d\text{-}\mathrm{Sub}(E \times \Delta X) \rightarrowtail \mathrm{Sub}(E \times \Delta X)$$

which is also natural in X. Therefore we get a monomorphism of lattices

$$\mathrm{Sub}(\Lambda E \times X) \rightarrowtail \mathrm{Sub}(E \times \Delta X)$$

which is natural in X. Furthermore, since images of $\mathscr{S}$-definable morphisms are $\mathscr{S}$-definable subobjects, (3) implies that d-Sub$(E \times \Delta X)$ is closed under $\exists_f$ for $\mathscr{S}$-definable f.

Consider the diagram (of solid arrows)

$$\begin{tikzcd}
 & {[\Lambda E, X]} \arrow[rr, dashed, shift left, "\varphi"] \arrow[d, tail, "j"'] & & {[E, \Delta X]} \arrow[ll, dashed, shift left, "\psi'"] \arrow[d, tail, "i"] \arrow[dll, dashed, "\psi"'] \\
 & \mathrm{Sub}(\Lambda E \times X) \arrow[rr, tail, "\theta"] \arrow[dl, shift left] \arrow[dl, shift right] \arrow[dd, shift left] \arrow[dd, shift right] & & \mathrm{Sub}(E \times \Delta X) \arrow[dl, shift left] \arrow[dl, shift right] \arrow[dd, shift left] \arrow[dd, shift right] \\
\mathrm{Sub}(\Lambda E) \arrow[rr, tail] & & \mathrm{Sub}(E) & \\
 & \mathrm{Sub}(\Lambda E \times X \times X) \arrow[rr, tail] & & \mathrm{Sub}(E \times \Delta X \times \Delta X)
\end{tikzcd}$$

where the left and right parts are diagrams of the type appearing in Lemma 11, and the horizontal arrows come from the above mentioned injections of lattices and the fact that Δ preserves finite products.

All corresponding squares commute: the ones involving $\ulcorner 1_{\Lambda E} \urcorner$ and $p_1^* \wedge p_2^*$ since the inclusion is a lattice homomorphism and is natural in X; the ones involving $\exists_X$ and $\exists_\delta$ by the preceding remarks and the fact that if an adjoint pair restricts (as functors) to full subcategories, the restrictions are still adjoint.

Therefore, by Lemma 11, there exists a function φ such that $i\varphi = \theta j$ and φ is necessarily monic. Since the graph of any morphism $E \to \Delta X$ is $\mathcal{S}$-definable (Proposition 10(iv)) there exists a function ψ such that $\theta\psi = i$. Since the horizontal maps are monic, ψ equalizes the morphisms in the left hand limit diagram and so factors through j by ψ'. So $i\varphi\psi' = \theta j \psi' = \theta\psi = i$ and since i is monic, $\varphi\psi' = 1$. Then $\varphi\psi'\varphi = \varphi$ which implies $\psi'\varphi = 1$. Therefore $[\Lambda E, X] \cong [E, \Delta X]$ and Λ extends uniquely to a left adjoint for Δ.

Next we show that the conditions (1), (2), (3) are stable in the sense that if they hold for $\Gamma : \mathscr{E} \to \mathcal{S}$ then they also hold for $\Gamma^I : \mathscr{E}/\Delta I \to \mathcal{S}/I$.

This is clear for condition (1) since the morphism corresponding to d in $\mathscr{E}/\Delta I$ is $\Delta I \times d$.

Let $p : E \to \Delta I$ be an object of $\mathscr{E}/\Delta I$ and $\alpha : J \to I$ an object of $\mathcal{S}/I$, then we have the following bijections of lattices

$$\begin{array}{ll}
 & \mathcal{S}/I\text{-definable} \quad (p_0) \rightarrowtail (p) \times (\Delta\alpha) \\
(1) & \overline{} \\
 & \mathcal{S}\text{-definable} \quad E_0 \rightarrowtail E_{\Delta I} \times \Delta J \\
(2) & \overline{} \\
 & \mathcal{S}\text{-definable} \quad E_0 \rightarrowtail E \times \Delta J \quad \text{such that} \quad E_0 \subseteq \alpha^*(E, p) \\
(3) & \overline{} \\
 & X_0 \rightarrowtail \Lambda E \times J \quad \text{such that} \quad X_0 \subseteq \alpha^*(\Lambda E, \bar{p}) \\
(4) & \overline{} \\
 & X_0 \rightarrowtail \Lambda E_I \times J \quad \text{in} \quad \mathcal{S} \\
(5) & \overline{} \\
 & (x_0) \rightarrowtail (\bar{p}) \times (\alpha) \quad \text{in} \quad \mathcal{S}/I.
\end{array}$$

Bijection (1) follows from Proposition 10(v). By $\alpha^*(E, p)$ we mean the subobject of $E \times \Delta J$ defined by the pullback

$$\begin{tikzcd}
\cdot \arrow[r, tail, "\alpha^*(E,p)"] \arrow[d] & E \times \Delta J \arrow[d, "E\times\Delta\alpha"] \\
E \arrow[r, tail, "(E,p)"'] & E \times \Delta J
\end{tikzcd}$$

which is easily seen to be $E_{\Delta I} \times \Delta J \rightarrowtail E \times \Delta J$. By Proposition 10 (ii, iv) it follows that $\alpha^*(E, p)$ is an $\mathcal{S}$-definable subobject. Then condition (3) gives bijection (2) in the downward direction, whereas the upward direction comes from pulling back along $\alpha^*(E, p)$ (Proposition 10(ii)). The correspondence (3) comes from the isomorphism d-Sub$(E \times \Delta J) \cong$ Sub$(\Lambda E \times J)$ mentioned above, and the fact that under this lattice isomorphism (E, p) corresponds to $(\Lambda E, \bar{p})$ by construction, and by naturality, $\alpha^*(E, p)$ will correspond to $\alpha^*(\Lambda E, \bar{p})$. Finally, bijections (4) and (5) are similar to (2) and (1), but simpler since we are dealing with arbitrary subobjects. This shows that condition (2) is stable.

That condition (3) is stable follows immediately from the fact that

$$\begin{tikzcd}
E \arrow[rr, "f"] \arrow[dr] & & E' \arrow[dl] \\
& \Delta I &
\end{tikzcd}$$

is $\mathcal{S}/I$-defined if and only if $f: E \to E'$ is $\mathcal{S}$-defined. The "only if" part is trivial. To see the "if" part, take a pullback

$$\begin{tikzcd}
E \arrow[r, "f"] \arrow[d] & E' \arrow[d] \\
\Delta J \arrow[r, "\Delta\alpha"'] & \Delta K
\end{tikzcd}$$

in $\mathscr{E}$, then the following is a pullback in $\mathscr{E}/\Delta I$

$$\begin{tikzcd}
E \arrow[r, "f"] \arrow[d] & E' \arrow[d] \arrow[ddr, bend left] & \\
\Delta(J\times I) \arrow[r, "\Delta(\alpha\times I)"'] \arrow[drr, bend right, "\Delta_{p_2}"'] & \Delta(K\times I) \arrow[dr, "\Delta_{p_2}"] & \\
& & \Delta I
\end{tikzcd}$$

If $q : E' \to \Delta I'$ and $\varphi : I \to I'$, the above bijection of lattices becomes

$$\frac{\mathcal{S}/I\text{-definable } (p_0) \rightarrowtail (\varphi^* q) \times (\Delta\alpha)}{(x_0) \rightarrowtail (\overline{\varphi^* q}) \times (\alpha) \quad \text{in} \quad \mathcal{S}/I.}$$

On the other hand we have the following bijections of lattices

$$\begin{array}{r|c}
 & \mathcal{S}/I\text{-definable} \quad (p_0) \rightarrowtail (\varphi^* q) \times (\Delta\alpha) \\
\hline
(1) & \mathcal{S}\text{-definable} \quad E_0 \rightarrowtail E' \times_{\Delta I'} \Delta J \\
\hline
(1) & \mathcal{S}/I'\text{-definable} \quad (q_0) \rightarrowtail (q) \times \Delta(\varphi\alpha) \\
\hline
\text{(condition (2))} & (y_0) \rightarrowtail (\bar{q}) \times (\varphi\alpha) \quad \text{in} \quad \mathcal{S}/I' \\
\hline
(5) & X_0 \rightarrowtail \Lambda E_{I'} \times J \quad \text{in} \quad \mathcal{S} \\
\hline
(5) & (x_0) \rightarrowtail (\varphi^* \bar{q}) \times (\alpha) \quad \text{in} \quad \mathcal{S}/I.
\end{array}$$

In the presence of conditions (1) and (3) condition (2) uniquely determines the value of the left adjoint to Δ so we conclude from the above bijections that

$$\Lambda^I(\varphi^* q) = \varphi^* \bar{q} = \varphi^* \Lambda^{I'}(q),$$

i.e. Λ is indexed. □

Remark. $\Gamma(\Delta\Omega^E)$ is the $\mathcal{S}$-object of $\mathcal{S}$-definable subobjects of $\mathcal{E}$, and taking into account Mikkelsen's theorem [4, p. 70] that the complete atomic heyting algebra objects of $\mathcal{S}$ are precisely those objects of the form Ω^X for some X, we see that condition 2 could be stated: for every E in $\mathcal{E}$, the lattice object (in $\mathcal{S}$) of $\mathcal{S}$-definable subobjects of E is a complete atomic heyting algebra object.

Proposition 13. *Condition* (3) *of Theorem* 12 *can be replaced by either of the two following conditions*:

(3') *for any $\mathcal{S}$-definable morphism $f : E \to E'$, $\exists_f : \mathrm{Sub}(E) \to \mathrm{Sub}(E')$ preserves $\mathcal{S}$-definable subobjects,*

(3'') (a) *the composite of $\mathcal{S}$-definable subobjects is an $\mathcal{S}$-definable subobject.*

(b) *the $\mathcal{S}$-union of a family of $\mathcal{S}$-definable subobjects is again one.*

Proof. (3') $\Rightarrow$ (3'') since (a) is an instance of (3') for f monic and (b) for f a projection $\Delta I \times E' \to E'$.

(3'') $\Rightarrow$ (3') since, if f appears in a pullback diagram

$$\begin{tikzcd}
E \arrow[r, "f"] \arrow[d, "g"'] & E' \arrow[d] \\
\Delta I \arrow[r, "\Delta\alpha"'] & \Delta J
\end{tikzcd}$$

then f can be written as

$$E \overset{(g,f)}{\rightarrowtail} \Delta I \times E' \xrightarrow{p_2} E'$$

and (g,f) is $\mathscr{S}$-definable by

$$\begin{tikzcd} E \arrow[r, tail, "(g,f)"] \arrow[d] & \Delta I \times E' \arrow[d] \\ \Delta J \arrow[r, tail, "\delta"'] & \Delta J \times \Delta J \end{tikzcd}$$

Since images of $\mathscr{S}$-definable morphisms are $\mathscr{S}$-definable subobjects, (3) clearly implies (3′).

Finally, a careful look at the proof of Theorem 12 shows that all that is used is condition (3′) and the fact that the hypotheses are stable. But (3′) is stable for the same reasons that (3) is. □

4. Molecules

Assume that d is monic for $\Gamma : \mathscr{E} \to \mathscr{S}$.

Definition. An object M in $\mathscr{E}$ is called a *molecule* (or $\mathscr{S}$-indecomposable) if the canonical $\Omega \to \Gamma(\Delta\Omega^M)$ (corresponding to diag$: \Delta\Omega \to \Delta\Omega^M$) is an isomorphism. An *I-family of molecules* is a molecule in $\mathscr{E}/\Delta I$ relative to $\mathscr{S}/I$.

Equivalently, M is a molecule if and only if for every I in $\mathscr{S}$ and every $\mathscr{S}$-definable subobject $M_0 \rightarrowtail \Delta I \times M$ there exists a unique $I_0 \rightarrowtail I$ such that

$$\begin{tikzcd} M_0 \arrow[dr, tail] \arrow[dd, phantom, "\cong" rotate=90] & \\ & \Delta I \times M \\ \Delta I_0 \times M \arrow[ur, tail] & \end{tikzcd}$$

Intuitively, of an I-family of $\mathscr{S}$-definable subobjects of M, I_0 of them are equal to M and the others are 0.

An I-family of molecules is an object $M \to \Delta I$ such that for every $\alpha : J \to I$ and every $\mathscr{S}$-definable subobject $M_0 \rightarrowtail \Delta J_{\Delta I} \times M$, there exists a unique $\beta : K \rightarrowtail J$ such that

$$\begin{tikzcd} M_0 \arrow[r, tail] \arrow[d] & \Delta J_{\Delta I} \times M \arrow[d] \\ \Delta K \arrow[r, tail, "\Delta\beta"'] & \Delta J \end{tikzcd}$$

is a pullback.

It is clear from this formulation that molecules are stable under substitution, i.e. if $M \to \Delta I$ is a family of molecules then so is $\Delta J_{\Delta I} \times M \to \Delta J$ for every $J \to I$.

Theorem 14. *$\mathscr{E}$ is molecular over $\mathscr{S}$ if and only if*

(a) *d is monic and*

(b) *every object of $\mathscr{E}$ is a sum of molecules.*

Proof. Assume $\mathscr{E}$ molecular. By Theorem 12, d is monic. Now for any E, let $h: E \to \Delta\Lambda E$ be the unit of the adjunction $\Lambda \dashv \Delta$. For any $\alpha: J \to \Lambda E$, if

$$\begin{tikzcd} E' \arrow[r] \arrow[d] & E \arrow[d] \\ \Delta J \arrow[r, "\Delta\alpha"'] & \Delta\Lambda E \end{tikzcd}$$

is a pullback, then by indexedness of the adjunction

$$\begin{tikzcd} \Lambda E' \arrow[r] \arrow[d] & \Lambda E \arrow[d, equal] \\ J \arrow[r, "\alpha"'] & \Lambda E \end{tikzcd}$$

is also a pullback, so $J \cong \Lambda E'$ and the morphism $E' \to \Lambda J$ is essentially the unit $h': E' \to \Delta\Lambda E'$. For an $\mathscr{S}$-definable $E_0 \rightarrowtail E'$ we have a pullback

$$\begin{tikzcd} E' \arrow[r] & \Delta\Omega \\ E_0 \arrow[u, tail] \arrow[r] & \Delta 1 \arrow[u, tail, "\Delta t"'] \end{tikzcd}$$

so by adjointness there exists a unique $\Lambda E' \to \Omega$ such that

$$\begin{tikzcd} & \Delta\Lambda E' \arrow[d] & \Lambda E' \arrow[d] \\ E' \arrow[ur] \arrow[r] & \Delta\Omega & \Omega \end{tikzcd} .$$

Pulling back in stages we get pullbacks

$$\begin{tikzcd} E' \arrow[r, "h"] & \Delta\Lambda E' \arrow[r] & \Delta\Omega \\ E_0 \arrow[u, tail] \arrow[r] & \Delta K \arrow[u, tail] \arrow[r] & \Delta 1 \arrow[u, "\Delta t"'] \end{tikzcd}$$

and the following the left square by the isomorphism $\Lambda E' \cong J$, we get a pullback

$$\begin{tikzcd} E' \arrow[r] & \Delta J \\ E_0 \arrow[u, tail] \arrow[r] & \Delta K \arrow[u, tail] \end{tikzcd}$$

which shows that $h : E \to \Delta\Lambda E$ is a family of molecules.

Since $\sum_{\Lambda E}(h) = E$, this proves (b).

Conversely, assume (a) and (b) are satisfied. We shall show that the conditions of Theorem 12 are satisfied. (1) is obvious. Given any E in $\mathscr{E}$, express it as a sum of molecules indexed by some object which we will call ΛE:

$$h : E \to \Delta\Lambda E.$$

Then h is a molecule in $\mathscr{E}/\Delta\Lambda E$ and for any I is $\mathscr{S}$

$$\begin{tikzcd} E \times \Delta I \arrow[rr] \arrow[d] & & E \arrow[dd] \\ \Delta\Lambda E \times \Delta I \arrow[d, phantom, "\cong" rotate=-90] & & \\ \Delta(\Lambda E \times I) \arrow[rr] & & \Delta\Lambda E \end{tikzcd}$$

is a pullback, so for any $\mathscr{S}$-definable subobject $E_0 \rightarrowtail E \times \Delta I$ there exists a unique $I_0 \rightarrowtail \Lambda E \times I$ such that

$$\begin{tikzcd} E_0 \arrow[r, tail] \arrow[d] & E \times \Delta I \arrow[d] \\ \Delta I_0 \arrow[r, tail] & \Delta(\Lambda E \times I) \end{tikzcd}$$

is a pullback. This gives us the isomorphism of lattices required to establish (2).

We prove (3″) of Proposition 13 rather than (3). To see 3″(a), let $E_0 \rightarrowtail E_1$ and $E_1 \rightarrowtail E$ be two $\mathscr{S}$-definable subobjects and write E as a sum of molecules

$$h : E \to \Delta\Lambda E.$$

Since (h) is a ΛE family of molecules there exists a unique $K_1 \rightarrowtail \Lambda E$ making

$$\begin{tikzcd} E_1 \arrow[r, tail] \arrow[d] & E \arrow[d] \\ \Delta K_1 \arrow[r, tail] & \Delta \Lambda E \end{tikzcd}$$

into a pullback. Since $E_0 \rightarrowtail E_1$ is $\mathscr{S}$-definable, the defining property of a family of molecules says that there exists a unique $K_0 \rightarrowtail K_1$ such that

$$\begin{tikzcd} E_0 \arrow[r, tail] \arrow[d] & E_1 \arrow[d] \\ \Delta K_0 \arrow[r, tail] & \Delta K_1 \end{tikzcd}$$

is a pullback. Pasting these two pullbacks exhibits $E_0 \rightarrowtail E$ as a pullback of $\Delta(K_0 \rightarrowtail \Lambda E)$ and so $E_0 \rightarrowtail E$ is $\mathscr{S}$-definable.

Finally, to show 3″(b), let $E_0 \rightarrowtail E \times \Delta I$ be an $\mathscr{S}$-definable subobject. As in the proof of (2) above, there exists a unique $I_0 \rightarrowtail \Lambda E \times I$ such that the left square below is a pullback (the right one is trivially)

$$\begin{tikzcd} E_0 \arrow[r, tail] \arrow[d] & E \times \Delta I \arrow[r, "p_1"] \arrow[d] & E \arrow[d] \\ \Delta I_0 \arrow[r, tail] & \Delta(\Lambda E \times I) \arrow[r, "\Delta(p_1)"'] & \Delta \Lambda E. \end{tikzcd}$$

Therefore $E_0 \to E$ is an $\mathscr{S}$-definable morphism and so its image, which is $\bigcup_I E_0 \rightarrowtail E$, is $\mathscr{S}$-definable. □

5. The $\mathscr{S}et$ case

When the base category $\mathscr{S}$ is the category of sets, the above results become simpler. In the first place, all functors into $\mathscr{S}et$ are indexed and so we need only consider ordinary adjoints $\Lambda \dashv \Delta$. Secondly, $\mathscr{S}et$ being Boolean, d is

$$\binom{t}{f} : 1 + 1 \to \Omega,$$

which is always monic. An $\mathscr{S}$-definable subobject is therefore the same thing as a complemented subobject. The definition of molecule also simplifies to that of an object M whose only complemented subobjects are 0 and M (and $M \neq 0$), i.e. M is indecomposable. We should point out, however, that families of objects of $\mathscr{E}$ as we defined them, namely $E \to \Delta I$, are not quite the usual families. They correspond to ordinary families $(E_i)_{i\in I}$ which are bounded in the sense that there exist an object B in $\mathscr{E}$ and monomorphisms $(E_i \rightarrowtail B)_{i\in I}$. If $\mathscr{E}$ has coproducts (which it does when $\mathscr{E}$ is Grothendieck) then these families are the same as ordinary families.

We now restate our characterizations of molecular toposes, incorporating the simplifications which occur when $\mathscr{S}$ is taken to be $\mathscr{S}et$.

Theorem 15. *Let $\mathscr{E}$ be an elementary topos defined over $\mathscr{S}et$. Then the following conditions are equivalent.*

(1) *$\mathscr{E}$ is molecular (i.e. $\Delta : \mathscr{S}et \to \mathscr{E}$ has a left adjoint).*

(2) *Δ preserves exponentiation.*

(3) (a) *For every object E of $\mathscr{E}$, the lattice of complemented subobjects of E is a complete atomic boolean algebra, i.e. there is a set ΛE such that we have an isomorphism of lattices $[E, 2]_{\mathscr{E}} \cong [\Lambda E, 2]_{\mathscr{S}et}$,*

(b) *arbitrary unions of complemented subobjects are again complemented.*

(4) *Every object of E is a sum of molecules (i.e. indecomposables).*

Proof. Since the base category is $\mathscr{S}et$, indexed adjointness is the same as ordinary adjointness and strong adjointness. It follows that (1)$\Leftrightarrow$(2) by Theorem 2.

Condition (3) is equivalent to the conditions of Theorem 12, with (3″). Indeed condition (1) of Theorem 12 and (3″a) of Proposition 13 are automatic when $\mathscr{S} = \mathscr{S}et$ (or any boolean topos), (3″b) is the same as 3(b) above, and condition (2) Theorem 12, is the same as 3(a) above.

Finally (4)$\Leftrightarrow$(1) by Theorem 14. $\square$

Definition. A site $\mathscr{M}$ is called *molecular* if the covering sieves are nonempty and connected (a sieve on M is considered as a full subcategory of the slice category $\mathscr{M}/M$).

Theorem 16. *Let $\mathscr{E}$ be a Grothendieck topos. Then the following are equivalent.*

(1) *$\mathscr{E}$ is molecular (over $\mathscr{S}et$).*

(2) *Every element of some generating set for $\mathscr{E}$ is a sum of molecules.*

(3) *$\mathscr{E}$ is equivalent to a category of sheaves on some site for which the constant presheaves are sheaves.*

(4) *$\mathscr{E}$ is equivalent to a category of sheaves on a molecular site.*

Proof. It follows immediately from Theorem 14 that (1)$\Leftrightarrow$(2).

Now, assume (2) and write each of the generators as a sum of (a set of) molecules and let $\mathscr{M}$ be the full subcategory of $\mathscr{E}$ determined by those molecules used. $\mathscr{M}$ is a small category and its objects generate $\mathscr{E}$. Give $\mathscr{M}$ the topology induced [7, III 3.1] by the canonical topology on $\mathscr{E}$. By Corollary 1.2.1 of [7, IV], $\mathscr{E}$ is equivalent to $\tilde{\mathscr{M}}$ (sheaves on $\mathscr{M}$), the equivalence being given by the functor $\mathscr{E} \to \tilde{\mathscr{M}}$ which sends E to

$[-, E]_{\mathscr{E}}$ restricted to $\mathscr{M}$. For any M in $\mathscr{M}$ and I in $\mathscr{S}et$,

$$[M, \Delta I]_{\mathscr{E}} \cong [\Lambda M, I]_{\mathscr{S}et} \cong [1, I]_{\mathscr{S}et} \cong I$$

where $\Lambda M = 1$ since M is a molecule. Thus the constant presheaf with value I is $[-, \Delta I]_{\mathscr{E}}$ restricted to $\mathscr{M}$, and therefore is a sheaf. This shows that (2) ⇒ (3).

Denote the constant presheaf with value I by $C(I) : \mathscr{M}^{\mathrm{op}} \to \mathscr{S}et$. $C(I)$ is a sheaf if and only if for every M in $\mathscr{M}$ and every covering sieve R of M, the canonical morphism

$$C(I)(M) \to \varprojlim\, (C(I)(M') | (M' \to M) \in R)$$

is an isomorphism, i.e.

$$\mathrm{diag} : I \to I^{\pi_0(R)}$$

is an isomorphism, where $\pi_0(R)$ denotes the set of connected components of R. If I is not 0 or 1, diag is an isomorphism if and only if $\pi_0(R) = 1$. Therefore the constant presheaves are sheaves if and only if the site is molecular. Thus (3)⇔(4).

If $\mathscr{M}$ is any site, the category of presheaves, $\hat{\mathscr{M}}$, is always molecular over $\mathscr{S}et$, so we have adjoint functors $\Lambda \dashv \Delta \dashv \Gamma$. If the constant presheaves are sheaves, then Δ factors through $i : \tilde{\mathscr{M}} \rightarrowtail \hat{\mathscr{M}}$, say $\Delta = i\Delta'$. Then Δ' is easily seen to be left adjoint to $\Gamma i = \Gamma'$ and this adjoint pair is easily seen to be the geometric morphism $\tilde{\mathscr{M}} \to \mathscr{S}et$. Then

$$[\Lambda iE, X]_{\mathscr{S}et} \cong [iE, \Delta X]_{\hat{\mathscr{M}}} \cong [iE, i\Delta' X]_{\hat{\mathscr{M}}} \cong [E, \Delta' M]_{\tilde{\mathscr{M}}}$$

shows that $\Lambda' = \Lambda i$ is left adjoint to Δ'. Thus (3) ⇒ (1).

Remark. As seen from the above proof, it is sufficient that any constant presheaf, whose value is not 0 or 1, be a sheaf in order that the site be molecular.

Example. If X is a locally connected space, then every open set is the disjoint union of its connected components (with the disjoint union topology) and since the opens, considered as subobjects of 1, are a generating family for $\mathrm{Sh}(X)$, and connected opens are molecules, it follows by Theorem 16 that $\mathrm{Sh}(X)$ is molecular. Conversely, if $\mathrm{Sh}(X)$ is molecular, then X must be locally connected since every open set must be a disjoint union of molecules. Geometrically, Λ associates to a local homeomorphism $E \to X$, the set of connected components of E, $\pi_0(E)$.

6. Local character of molecular toposes

The following theorem says that being molecular is a local property (see [7, IV.8]).

Theorem 17. *Let $\Gamma : \mathscr{E} \to \mathscr{S}$ be a morphism of elementary toposes.*

(1) *If $\mathscr{E}$ is $\mathscr{S}$-molecular then $\mathscr{E}/\Delta I$ is $\mathscr{S}/I$-molecular for any I in $\mathscr{S}$.*

(2) *If I has full support* (*i.e.* $I \to\!\!> 1$) *and* $\mathscr{E}/\Delta I$ *is* $\mathscr{S}/I$*-molecular, then* $\mathscr{E}$ *is* $\mathscr{S}$*-molecular.*

Proof. (1) is immediate since to say that Λ is an indexed left adjoint means that for every J, $\Delta^J : \mathscr{S}/J \to \mathscr{E}/\Delta J$ has a left adjoint Λ^J and the Λ^J are compatible with pulling back. Then use the isomorphism $(\mathscr{S}/I)/(\alpha : J \to I) \cong \mathscr{S}/J$.

Now assume that I has full support and that $\mathscr{E}/\Delta I$ is $\mathscr{S}/I$-molecular and consider the commutative diagram of inverse images of geometric morphisms

$$\begin{tikzcd} \mathscr{E}/\Delta J & \mathscr{E} \arrow[l, "(\Delta I)^*"'] \\ \mathscr{S}/I \arrow[u, "\Delta^I"] & \mathscr{S}. \arrow[l, "I^*"] \arrow[u, "\Delta"'] \end{tikzcd}$$

I^* is logical and so preserves Π_α for all α in $\mathscr{S}$ and by Theorem 5, Δ^I also preserves Π. Since $(\Delta I)^*$ preserves Π also and reflects isos, the canonical morphism $\Delta \cdot \Pi_\alpha \to \Pi_{\Delta\alpha} \cdot \Delta$ is an iso in $\mathscr{E}$ and so by Theorem 5, $\mathscr{E}$ is molecular over $\mathscr{S}$.

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