ON MACKEY TOPOLOGIES IN TOPOLOGICAL ABELIAN GROUPS

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ABSTRACT. Let $\mathcal{C}$ be a class of topological abelian groups and SP$\mathcal{C}$ denote the full subcategory of subobjects of products of objects of $\mathcal{C}$. We say that SP$\mathcal{C}$ has Mackey coreflections if there is a functor that assigns to each object A of SP$\mathcal{C}$ an object τA that has the same group of characters as A and is the finest topology with that property. We show that the existence of Mackey coreflections in SP$\mathcal{C}$ is equivalent to the injectivity of the circle with respect to topological subgroups of groups in $\mathcal{C}$.

1. Introduction

Consider a class $\mathcal{C}$ of topological abelian groups that includes the circle group $K = \mathsf{R}/\mathsf{Z}$. If A is a topological group, then a **character** on A is a continuous homomorphism $\chi: A \longrightarrow K$. We let $A\hat{}$ denote the discrete group of all characters of A. Let SP$\mathcal{C}$ denote the closure of $\mathcal{C}$ with respect to arbitrary products and subobjects.

If A is an object of SP$\mathcal{C}$, we say that A is a **Mackey group** in SP$\mathcal{C}$ if whenever $\widetilde{A}$ in SP$\mathcal{C}$ is the same underlying group as A with a topology finer than that of A such that the identity $\widetilde{A} \longrightarrow A$ induces an isomorphism $A\hat{} \longrightarrow \widetilde{A}\hat{}$, then $\widetilde{A} = A$. In other words, if A has the finest possible topology with the same group of characters. We denote by SP$\mathcal{C}_\tau$ the full subcategory of Mackey groups.

We say that SP$\mathcal{C}$ **admits Mackey coreflections** if there is a functor τ: SP$\mathcal{C} \longrightarrow$ SP$\mathcal{C}$ that assigns to each group A of SP$\mathcal{C}$ a Mackey group τA that has the same underlying group group as A as well as the same set of functionals and additionally has the property that for $f: A \longrightarrow B$, then $\tau f: \tau A \longrightarrow \tau B$ is the same function as f.

Since τA has the finest topology on the group A that has the same characters as A it follows that the identity function $\tau A \longrightarrow A$ is continuous. Since for $f: A \longrightarrow B$, $\tau f = f$, it follows that

$$\begin{tikzcd}
\tau A \arrow[r, "\tau f"] \arrow[d, "="'] & \tau B \arrow[d, "="] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

commutes so that $\tau A \longrightarrow A$ is the component at A of a natural transformation $\iota: \tau \longrightarrow \mathrm{Id}$, the latter denoting the identity functor. Conversely, if we supposed that this square commutes, then evidently $\tau f = f$ so that naturality of ι would give another definition of Mackey coreflections.

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If SP$\mathcal{C}$ admits Mackey coreflections, the group A is a Mackey group if and only if $\tau A = A$. In particular, $\tau\tau A = \tau A$. If A is a Mackey group and $A \longrightarrow B$ is a morphism in SP$\mathcal{C}$, then we have $\tau A = A \longrightarrow \tau B$, while the continuity of $\tau B \longrightarrow B$ gives the other half of the equality $\mathrm{Hom}(A, B) = \mathrm{Hom}(A, \tau B)$ and thus we see that τ determines a coreflection of the inclusion SP$\mathcal{C}_\tau \longrightarrow$ SP$\mathcal{C}$.

There has been some effort into finding general conditions on a class of groups that ensure the existence of Mackey coreflections. See, for example, [Chasco *et. al.*, 1999].

On the other hand, there has been some effort into finding a class of topological abelian groups on which the circle group is injective. It is known for locally compact groups, for example. In [Banaszczyk, 1991], especially 8.3 on page 82, it is shown that there is a large class of topological abelian groups, called **nuclear groups**, on which the circle is injective. We refer the reader to the source for the somewhat obscure definition. The class described there as the largest class of groups that is closed under the operations of taking products, subgroups, and Hausdorff quotients.

What has not seemed to be noticed before (at least, as far as we can determine) is that these two questions are essentially equivalent. This is the main result of this paper, Theorem 4.1. The main tool in this construction is the so-called Chu construction, described below.

2. Preliminaries

The results of this section are known. See, for example, [Hewitt and Ross, 1963]. We include proofs of the simpler facts for the convenience of the reader.

We denote by K the circle group, which for our purposes it is convenient to think of as R/Z. For any $n \geq 2$, we let U_n denote the image of the open interval $(-2^{-n}, 2^{-n}) \subseteq \mathsf{R}$. These evidently form a neighborhood base at 0 for the topology.

2.1. Theorem. *For any locally compact abelian group L, subgroup $M \subseteq L$, and character $\chi: M \longrightarrow K$, there is a character $\psi: L \longrightarrow K$ such that $\psi|M = \chi$.*

Proof. It is shown in [Banaszczyk, 1991] that locally compact groups are nuclear (7.10) and that nuclear groups have this extension property (8.3). ∎

We should mention that the proof that locally compact abelian groups are nuclear depends on deep theorems from [Hewitt and Ross, 1963] that are used to embed any locally compact abelian group in a product $C \times D \times E$ where C is compact, D is discrete and E is a finite dimensional Euclidean space.

2.2. Lemma. *Let $x \in U_2$ such that $2x \in U_n$ for $n > 2$. Then $x \in U_{n+1}$.*

Proof. If $2x \in U_n$ then either $x \in U_{n+1}$ or $x - \frac{1}{2} \in U_{n+1}$, but in the latter case, $x \notin U_2$. ∎

2.3. Lemma. *Suppose A is a topological abelian group and $\chi: A \longrightarrow K$ is a homomorphism such that $\chi^{-1}U_2$ is open in A. Then χ is continuous.*

Proof. Let $V_2 = \chi^{-1}U_2$. For each $n > 2$, let V_n be a neighborhood of 0 in A such that $V_n + V_n \subseteq V_{n-1}$. We claim that $\chi^{-1}U_n \supseteq V_n$. This is true by definition for $n = 2$. Assuming it is true for n, let $x \in V_{n+1}$. Then $2x \in V_n$ and so $\chi(2x) = 2\chi(x) \in U_n$ together with $\chi(x) \in \chi(V_{n+1}) \subseteq \chi(V_2) \subseteq U_2$, gives, by Lemma 2.2, that $\chi(x) \in U_{n+1}$. ∎

2.4. Lemma. *Suppose $\{A_i \mid i \in I\}$ is a family of topological groups and A is a subgroup of $\prod_{i\in I} A_i$. Then for any character $\chi\colon A \longrightarrow K$, there is a finite subset $J \subseteq I$ such that χ factors through the image of A in $\prod_{i\in J} A_i$.*

Proof. Suppose $\chi\colon A \longrightarrow K$ is a continuous character. Then $\chi^{-1}(U_2)$ is an open neighborhood of 0 in A. Thus there is a finite subset $\{i_1, \ldots, i_n\} \subseteq I$ and open neighborhoods $V_{i_1}, \ldots, V_{i_n}$ of $A_{i_1}, \ldots, A_{i_n}$, respectively, such that

$$\chi^{-1}(U_2) \supseteq A \cap \left(V_{i_1} \times \cdots \times V_{i_n} \times \prod_{i\neq i_1,\ldots,i_n} A_i\right)$$

which implies that

$$\chi\left(A \cap \left(0 \times \cdots \times 0 \times \prod_{i\neq i_1,\ldots,i_n} A_i\right)\right) \subseteq U_2$$

and since U_2 contains no non-zero subgroup of K,

$$\chi\left(A \cap \left(0 \times \cdots \times 0 \times \prod_{i\neq i_1,\ldots,i_n} A_i\right)\right) = 0$$

It follows that, algebraically at least, χ factors through the quotient

$$\widetilde{A} = \frac{A}{A \cap \left(0 \times \cdots \times 0 \times \prod_{i\neq i_1,\ldots,i_n} A_i\right)}$$

topologized as a subobject of $A_{i_1} \times \cdots A_{i_n}$ by a character we will denote $\widetilde{\chi}$. To show that $\widetilde{\chi}$ is continuous, it is sufficient to observe that

$$\widetilde{\chi}^{-1}(U_2) \supseteq \widetilde{A} \cap (V_{i_1} \times \cdots \times V_{i_n})$$

and invoke Lemma 2.3. ∎

By taking A to be the whole product, we derive the following.

2.5. Corollary. *The natural map $\sum(A_i\hat{\ }) \longrightarrow (\prod A_i)\hat{\ }$ is an isomorphism.*

3. The category chu

We give a brief description here of the category $\mathsf{chu} = \mathsf{chu}(\mathsf{Ab}, K)$ on which our development depends. The objects of chu are pairs (G, G') of abelian groups, equipped with a non-singular pairing $\langle -,-\rangle\colon G\otimes G' \longrightarrow K$. Non-singular means that for all $x\in G$, if $x\neq 0$ there is a $\phi\in G'$ with $\langle x,\phi\rangle\neq 0$ and for all $\phi\in G'$, if $\phi\neq 0$ there is an $x\in G$ with $\langle x,\phi\rangle\neq 0$. It is an immediate consequence that G' can be thought of as a subset of $G\hat{}$ and G can be thought of as a subset of $G'\hat{}$. Using that identification, we define a **morphism** $f\colon (G,G')\longrightarrow (H,H')$ in chu to be a group homomorphism $f\colon G\longrightarrow H$ such that for all $\phi\in H'$, $\phi\circ f\in G'$. Consequently, there is an induced morphism $f'\colon H'\longrightarrow G'$ given by $f'\phi=\phi\circ f$. It has the property that for $x\in G$, fx is the unique element of H such that for all $\phi\in H'$, $\phi(fx)=f'(\phi)(x)$ so that f' also determines f. Thus we can define a duality by $(G,G')^*=(G',G)$ with the pairing given by $\langle\phi,x\rangle=\langle x,\phi\rangle$ for $\phi\in G'$ and $x\in G$.

Let us denote by $[(G,G'),(H,H')]$ the subspace of $\mathrm{Hom}(G,H)$ consisting of the morphisms described above. There is a tensor product in the category, given by

$$(G,G')\otimes(H,H')=(G\widetilde{\otimes}H,[(G,G'),(H',H)])$$

This definition requires a bit of explanation. First, there is a pairing between $G\otimes H$ and $[(G,G'),(H',H)]$ given by $\langle x\otimes y,f\rangle=\langle y,fx\rangle$, for $x\in G$, $y\in H$ and a morphism $f\colon(G,G')\longrightarrow(H',H)$. This is non-singular in the second variable, but not in the first and we let $G\widetilde{\otimes}H$ be $G\otimes H$ modulo the elements of the tensor product that are annihilated by all f. Similarly, there is an internal hom given by

$$(G,G')\multimap(H,H')=([(G,G'),(H,H')],G\widetilde{\otimes}H)$$

The resultant category is what is called a $*$-autonomous category. See [Barr, 1998] for the definition of that term and further details.

For any topological abelian group A, we denote by $|A|$ the underlying discrete group. There is a functor $F\colon \mathrm{SP}\mathcal{C}\longrightarrow\mathsf{chu}$ defined by $FA=(|A|,A\hat{})$ with evaluation as the pairing. There is a functor $R\colon\mathsf{chu}\longrightarrow\mathrm{SP}\mathcal{C}$ defined by letting $R(G,G')$ be the abelian group G topologized as a subgroup of $K^{G'}$, topologized by the product topology. This can also be described as the weak topology for the characters in G'.

3.1. Theorem. *The functor R is full and faithful and right adjoint to F.*

Proof. To show that R is full and faithful, it is sufficient to show that $FR\cong\mathrm{Id}$ for which it is sufficient to show that the natural map $G'\longrightarrow R(G,G')\hat{}$ is an isomorphism. But a character on $R(G,G')$ extends to the closure of $R(G,G')$ in $K^{G'}$, which is compact and hence, by the injectivity of K on compact groups, to $K^{G'}$. By Corollary 2.5, a character on $K^{G'}$ takes the form $n_1\chi_1+n_2\chi_2+\cdots+n_k\chi_k$, where $\chi_1,\chi_2,\ldots,\chi_k\in G'$ and $n_1,n_2,\ldots,n_k$ are characters on K, that is integers. But then $\chi=n_1\chi_1+n_2\chi_2+\cdots+n_k\chi_k\in G'$. If $f\colon FA=(|A|,A\hat{})\longrightarrow(G,G')$ is given, then $f\colon|A|\longrightarrow G$ has the property that for $\phi\in G'$, $\phi\circ f\in A\hat{}$. But this means that the composite $A\xrightarrow{f}G\longrightarrow K^{G'}$ is continuous, so that $f\colon A\longrightarrow R(G,G')$ is continuous. ∎

4. The main theorems

4.1. Theorem. *Let $\mathcal{C}$ and $\mathrm{SP}\mathcal{C}$ be as above. Then the first four of the following conditions are equivalent and imply the fifth:*

1. *K is injective with respect to $\mathcal{C}$;*
2. *K is injective with respect to $\mathrm{SP}\mathcal{C}$.*
3. *$\mathrm{SP}\mathcal{C}$ has Mackey coreflections;*
4. *F has a left adjoint L whose counit $LFA \longrightarrow A$ is a bijection;*
5. *the restriction of F to $\mathrm{SP}\mathcal{C}_\tau$ is a natural equivalence.*

Proof. We will show that $1 \iff 2 \implies 3 \implies 4 \implies 2$ and that $4 \implies 5$.

$1 \iff 2$: The property is inherited by subobjects so it is sufficient to show it for products of objects of $\mathcal{C}$. So suppose $A \subseteq \prod_{i\in I} C_i$ with each $C_i \in \mathcal{C}$. There is, by Lemma 2.4, a finite subset $J \subseteq I$ such that χ factors by a character $\widetilde{\chi}$ through the image $\widetilde{A}$ of A in $\prod_{i\in J} C_i$. Since $\mathcal{C}$ is closed under finite finite products, the injectivity of K with respect to $\mathcal{C}$ provides the required extension of $\widetilde{\chi}$ and hence of χ. Thus $1 \implies 2$ while the reverse inclusion is obvious.

$2 \implies 3$: Given an object A of $\mathrm{SP}\mathcal{C}$, let $\{A_i \mid i \in I\}$ range over the set of all abelian groups whose topology is finer than that of A and for which the identity $A_i \longrightarrow A$ induces an isomorphism $A\hat{} \longrightarrow A_i\hat{}$. Form the pullback

$$\begin{tikzcd} \tau A \arrow[r] \arrow[d] & \prod A_i \arrow[d] \\ A \arrow[r] & A^I \end{tikzcd}$$

in which the bottom arrow is the diagonal and the right hand one is the identity on each factor. Since the bottom arrow is an inclusion, so is the upper one up to isomorphism. Taking duals and using the injectivity of K and Corollary 2.5, we have

$$\begin{tikzcd} (\tau A)\hat{} & (\prod A_i)\hat{} \cong \sum A_i\hat{} \arrow[l, two heads] \\ A\hat{} \arrow[u] & \left(A^I\right)\hat{} \cong \sum_{i\in I} A\hat{} \arrow[l] \arrow[u, "\cong"'] \end{tikzcd}$$

so that $A\hat{} \longrightarrow (\tau A)\hat{}$ is surjective, while it is obviously injective. Since the topology on τA is finer than that of any A_i, it has the finest topology that has the same character group.

$3 \implies 4$: Define $L = \tau \circ R$: $\mathsf{chu} \longrightarrow \mathrm{SP}\mathcal{C}$. If $f: (G, G') \longrightarrow FA$, we have $Rf: R(G, G') \longrightarrow RFA$ and then $\tau R(G, G') \longrightarrow \tau RFA$. Now RFA has the same elements and same character group as A and so does τA. Since $\tau A \longrightarrow A \longrightarrow RFA$ is continuous, it follows that τA has exactly the properties that characterize τRFA. Thus we have $L(G, G') \longrightarrow \tau RFA = \tau A \longrightarrow A$. This gives an injection $\mathrm{Hom}((G, G'), FA) \longrightarrow \mathrm{Hom}(L(G, G'), A)$ and the other inclusion is obvious since LF is evidently the identity.

$4 \implies 2$: Suppose that $A \subseteq B$ in $\mathrm{SP}\mathcal{C}$. Let $FA = (G, G')$ and $FB = (H, H')$. Then $G \subseteq H$ and we wish to show that the induced $H' \longrightarrow G'$ is surjective. If not, let $\widetilde{G}'$ be the image of $H' \longrightarrow G'$. We claim that there is a pairing on $(G, \widetilde{G}')$ such that $(G, G') \longrightarrow (G, \widetilde{G}') \longrightarrow (H, H')$ are morphisms. In fact, the pairing is given by $G \otimes \widetilde{G}' \longrightarrow G \otimes G' \longrightarrow K$. This is clearly extensional, since $\widetilde{G}'$ is a subgroup of G'. If $x \neq 0$ is in G, let $\phi \in H'$ such that $\langle x, \phi \rangle \neq 0$. If $\widetilde{\phi}$ is the image of ϕ then $\langle x, \widetilde{\phi} \rangle \neq 0$. That $(G, G') \longrightarrow (G, \widetilde{G}') \longrightarrow (H, H')$ are morphisms is clear. We have the diagram

$$\begin{tikzcd}
L(G, G') \arrow[r] \arrow[d] & L(G, \widetilde{G}') \arrow[r] \arrow[d] & L(H, H') \arrow[d] \\
A \arrow[r] \arrow[d] & \widetilde{A} \arrow[r] \arrow[d] & B \arrow[d] \\
R(G, G') \arrow[r] & R(G, \widetilde{G}') \arrow[r] & R(H, H')
\end{tikzcd}$$

in which $\widetilde{A}$ is defined by having the lower right square be a pullback. All the vertical arrows are bijections as are the horizontal arrows in the left hand squares. Moreover, the composite $A \longrightarrow \widetilde{A} \longrightarrow B$ is the inclusion of a subspace and the first arrow is a bijection, which leaves the topology of $\widetilde{A}$ both finer and coarser than that of A. Hence $A = \widetilde{A}$, whence $G' = \widetilde{G}'$ and $H' \longrightarrow G'$ is surjective.

$4 \implies 5$: If F has a left adjoint L, then it follows that K is injective in $\mathrm{SP}\mathcal{C}$ and that L is constructed as above and its image is just $\mathrm{SP}\mathcal{C}_\tau$, which is then equivalent to chu. ∎

Suppose that A and B are topological abelian groups. Say that a homomorphism $f: |A| \longrightarrow |B|$ is **weakly continuous** if for every continuous character $\phi: B \longrightarrow K$, the composite $\phi \circ f: A \longrightarrow K$ is continuous. Clearly if every weakly continuous map out of A is actually continuous, then A is a Mackey space. The converse is also true, provided Mackey coreflections exist. We will say that $\mathcal{C}$ **satisfies Glicksberg's condition** if every weakly continuous homomorphism between objects of $\mathcal{C}$ is continuous. Glicksberg's theorem says that the category of locally compact abelian groups does satisfy Glicksberg's condition [1962, Theorem 1.1].

4.2. Theorem. *Suppose $\mathcal{C}$ satisfies Glicksberg's condition. Then every object of $\mathcal{C}$ is a Mackey group in SP$\mathcal{C}$.*

Proof. It is sufficient to show that if C is an object of $\mathcal{C}$, then for any object A of SP$\mathcal{C}$, any weakly continuous $f\colon C \longrightarrow A$ is continuous. But there is an embedding $A \longrightarrow \prod C_i$ with each C_i in $\mathcal{C}$ and to prove $C \longrightarrow A$ continuous, it is sufficient to show that each composite $C \longrightarrow A \longrightarrow C_i$ is continuous. But for any character $C_i \longrightarrow K$, the composite $A \longrightarrow C_i \longrightarrow K$ is continuous and since $C \longrightarrow A$ is weakly continuous, the composite $C \longrightarrow A \longrightarrow C_i \longrightarrow K$ is also continuous. Thus each composite $C \longrightarrow A \longrightarrow C_i$ is weakly continuous and, by Glicksberg's condition, is continuous. ∎

4.3. Examples. We can identify three examples, though there are certainly more. We list them in order of increasing size.

Weakly topologized groups. If we take for $\mathcal{C}$ the category of compact groups, the resultant SP$\mathcal{C}$ is the category of subcompact groups—those that have a topological embedding into a compact group. In that case, both the dual and the internal hom are topologized by the weak topology and all our results are immediately applicable. In this case the Mackey and weak topologies coincide and every object has a Mackey topology.

Locally compact groups. If for $\mathcal{C}$ we take the category LC of locally compact abelian groups, then SP$\mathcal{C}$ is the category SPLC of subobjects of products of locally compact abelian groups. Locally compact groups are Mackey groups in this case.

Nuclear groups. If we take for $\mathcal{C}$ the category Nuc of nuclear groups, then SPLC = Nuc as well and there are Mackey reflections. Although SPLC $\subseteq$ Nuc, it is not immediately obvious that the inclusion is proper. One point is that the class of nuclear groups is closed inder Hausdorff quotients, which is not known to be the case for SPLC.

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