0.1. THEOREM. *Suppose E is an open subset of $\mathbf{R}^n$, $V \subseteq \mathbf{R}^n$ and $f : E \longrightarrow V$ is a diffeomorphism of class at least C^2. Then for any $x_0 \in E$ there is a $\Delta > 0$ such that for any number δ with $0 < \delta < \Delta$ the image $f(B(x_0, \delta))$ is convex.*

PROOF. Assume without loss of generality that $x_0 = 0$. Choose a compact subset $V^* \subseteq V$ that contains $f(0)$ in its interior. Choose $\epsilon > 0$ such that $B(0, \epsilon) \subseteq f^{-1}(V^*)$. The differentiability implies that there is a constant $K > 0$ such that for x, $u = x + h$ and $v = x - h$ all in $B(0, \epsilon)$,

$$f(u) = f(x) + \sum_{j=1}^{n} h_j \frac{\partial f}{\partial x_j}(x) + R(x, h)$$

$$f(v) = f(x) - \sum_{j=1}^{n} h_j \frac{\partial f}{\partial x_j}(x) + R(x, -h)$$

and $\|R(x, \pm h)\| < K \|h\|^2$. Now we have, on adding the last two equations and dividing by 2 that

$$(f(u) + f(v))/2 = f(x) + (R(x, h) + R(x, -h))/2$$

with $\|(R(x, h) + R(x, -h))/2\| \le K \|h\|^2$ This can be written

$$(f(u) + f(v))/2 = f((u + v)/2) + R^*(u, v) \tag{$*$}$$

where $\|R^*(u, v)\| \le K \|(u - v)/2\|^2$. Since f^{-1} is also differentiable, there is a constant L such that

$$f^{-1}(z + k) = f^{-1}(z) + S(z, k)$$

where $\|S(z, k)\| < L \|k\|$. Combining this with $(*)$, this gives us the formula

$$\begin{aligned} f^{-1}((f(u) + f(v))/2) &= f^{-1}\left(f((u + v)/2) + R^*(u, v)\right) \\ &= f^{-1}\left(f((u + v)/2)\right) + S((f(u) + f(v))/2, R^*(u, v)) \\ &= (u + v)/2 + S((f(u) + f(v))/2, R^*(u, v)) \end{aligned}$$

and an upper bound on the error term inside $B(0, \epsilon)$ is given by

$$\begin{aligned} \|S((f(u) + f(v))/2, R^*(u, v))\| &\le \|LR^*(u, v)\| \\ &\le KL \|(u - v)/2\|^2 \end{aligned}$$

Let Δ be the minimum of ϵ and $1/2KL$. We will show that for all $\delta \le \Delta$, $f(B(0, \delta))$ is convex. To show an open set convex it is sufficient to show it is closed under the operation $x, y \mapsto (x + y)/2$ since that implies that it is closed under the operation x, $y \mapsto \lambda x + (1 - \lambda) y$ for any dyadic rational λ. Other points can be reached by using dyadic convex combinations of points sufficiently near x or y. (A slightly different argument allows one to draw the same inference for closed sets.) So let $\delta \le \Delta$ and choose $x, y \in f(B(0, \delta))$;

we must show that $(x+y)/2 \in f(B(0,\delta))$. Let $u = f^{-1}(x)$ and $v = f^{-1}(y)$. Then $u, v \in B(0,\delta)$ or $\|u\| < \delta$ and $\|v\| < \delta$. We have to show that

$$\left\| f^{-1}((f(u)+f(v))/2) \right\| < \delta$$

We have

$$\left\| f^{-1}((f(u)+f(v))/2) \right\| < \|(u+v)/2\| + 1/(2\delta) \|(u-v)/2\|^2$$

so it suffices to prove that $\|(u+v)/2\| < \delta - 1/(2\delta) \|(u-v)/2\|^2$. Since $\|u\| < \delta$ and $\|v\| < \delta$, the right hand side is positive so we can prove it after squaring both sides. Thus we must show that

$$\|(u+v)/2\|^2 < \left(\delta - \frac{1}{2\delta} \|(u-v/2)\|^2\right)^2$$

Now the parallelogram law says that $\|(u+v)/2\|^2 + \|(u-v)/2\|^2 = \|u\|^2/2 + \|v\|^2/2 < \delta^2$ so that

$$\begin{aligned}
\|(u+v)/2\|^2 &< \delta^2 - \|(u-v)/2\|^2 \\
&< \delta^2 - \|(u-v)/2\|^2 + (1/2 \|(u-v)/2\|)^2 \\
&= (\delta - 1/2 \|(u-v)/2\|)^2
\end{aligned}$$

0.2. Theorem. *Every convex open subset of* $\mathbf{R}^n$ *is* C^∞*-contractible.*

Proof.

0.3. Theorem. *Let* M *be a paracompact manifold of class* C^m*,* $m \geq 2$*. Then* M *has an* m*-simple open cover, that is one in which each finite intersection is either empty or has a contracting homotopy of class* C^m*.*

Proof. Let n be the dimension of the manifold. Choose a locally finite open cover $\mathcal{U}$ that refines an atlas. Thus for each $U \in \mathcal{U}$, there is a diffeomorphism (of class C^m) ψ_U of U onto an open subset $E_U \subseteq R^n$. Choose an open cover $\mathcal{W} = \{W_U \mid U \in \mathcal{U}\}$ such that $\overline{W}_U \subseteq U$, which is possible in a paracompact manifold.

Now fix $x \in M$. Let V''_x be an open neighborhood that intersects only finitely many sets in $\mathcal{U}$. Since V''_x meets only finitely many U there is an open subset V'_x with the following properties:

1. $x \in V'_x \subseteq V''_x$;
2. $x \in U \in \mathcal{U}$ implies $V'_x \subseteq U$;
3. $x \in W \in \mathcal{W}$ implies $V'_x \subseteq W$;
4. $x \notin W \in \mathcal{W}$ implies $V'_x \cap \overline{W} = \emptyset$.

The last condition is also finite since it need be imposed only for the finitely many W_U that meet V_i''. Let $\mathcal{U}_x = \{U \in \mathcal{U} \mid x \in U\}$. Fix some $U_0 \in \mathcal{U}_x$. Let $\psi_0 = \psi_{U_0}$, $x_0 = \psi_0(x)$, $E_0 = \psi_0(V_x')$ and $V_U = \psi_U(V_x')$, the last for $U \in \mathcal{U}_x$. Let $f_U : E_0 \longrightarrow V_U$ be the composite across the top of the diagram, whose vertical arrows are inclusions:

$$\begin{tikzcd}
E_0 \arrow[r] \arrow[d] & V_0' \arrow[r, "="] \arrow[d] & V_0' \arrow[r] \arrow[d] & V_U \arrow[d] \\
\mathbf{R}^n \arrow[r, "{\psi_0}^{-1}"'] & U_0 & U \arrow[r, "\psi_U"'] & \mathbf{R}^n
\end{tikzcd}$$

Then f_U is a diffeomorphism. For all $U \in \mathcal{U}_x$ there is a $\Delta_U > 0$ such that $0 < \delta \leq \Delta_U$ implies that $f_U(B(x_0, \delta))$ is convex. Since $\mathcal{U}_x$ is finite, it follows that there is a single Δ such that $0 < \delta \leq \Delta$ implies that $f_U(B(x_0, \delta))$ is convex for all $U \in \mathcal{U}_x$. Let $V_x = {\psi_0}^{-1}(B(x_0, \Delta))$. Then V_x has the following properties:

1. $x \in V_x \subseteq V_x'$;
2. $x \in U \in \mathcal{U}$ implies $V_x \subseteq U$;
3. $x \in W \in \mathcal{W}$ implies $V_x \subseteq W$;
4. $x \notin W \in \mathcal{W}$ implies $V_x' \cap \overline{W} = \emptyset$.
5. $U \in \mathcal{U}_x$ implies $\psi_U(V_x)$ is convex.

Then I claim that $\{V_x \mid x \in M\}$ is a simple cover. For $x \in U \in \mathcal{U}$, we have $x \in V_x \subseteq U$, so that $\{V_x \mid x \in M\}$ is a refinement of $\mathcal{U}$. Now suppose x_0, x_1, ..., x_n are points of M such that $V_{x_0} \cap V_{x_1} \cap \cdots \cap V_{x_k} \neq \emptyset$. Choose $U \in \mathcal{U}$ so that $x_0 \in W_U$. Then $V_{x_0} \subseteq W_U$. For $0 \leq j \leq k$, $x_j \in U$. Otherwise

$$V_{x_j} \cap V_{x_0} \subseteq V_{x_j} \cap W_U \subseteq V_{x_j} \cap \overline{W}_U = \emptyset$$

from the fourth point above. Thus $V_{x_j} \subseteq U$ for all $0 \leq j \leq k$ and $\psi_U(V_{x_j})$ is convex. Therefore $\psi(V_{x_0} \cap V_{x_1} \cap \cdots \cap V_{x_k}) = \psi(V_{x_0}) \cap \psi(V_{x_1}) \cap \cdots \cap \psi(V_{x_k})$ is also convex and therefore smoothly contractible. ∎