Factorizations, Generators and Rank

by Michael Barr

Introduction

The notion of a factorization system in a category is nearly as old as the notion of category itself. The first definition seer to have been given by Mac Lane [Mac], [Mac'] under the name "bi-category structure". Even then it was recognized that there will not in general be a "natural" such structure. However the structur considered in the Mac Lane paper were considerably complicated by the attempt to formalize (and hence dualize) the notion of actual inclusion rather than just monomorphism. It can readily be seen that even in the category of sets the dual notion is going to cause trouble. (Decomposition mappings are not closed under composition.) However the notion has been in the process of refinement ever since, notably by J.Isbell ([Is], [Is'], [Is"]), Z.Semadeni ([Sem]), G.M.Kelly ([Kel]) and J.Kennison ([Ken]). At some point - apparently by mutual consent - the term bicategory disappeared and we speak now of factorizations.

The fact that generators were connected with factorizations has also been apparent for a long time (see [Gro] and [Sem]).

The purpose of this paper is two-fold. The first is to collect in one place a distillation of all this development. This is carried out in the first two sections. Probably nothing in those two sections is really new except the organization, but not all of it has appeard previously in print. Section three studies the relation between factorizations and generators and prepares for section 5. Section 4 gives a negative result on the possibility of using general factorization systems for embeddings. Finally in section 5 we consider the case of generators with rank. In this case we show that there is a canonical generating subcategory which has very good properties. This will be central in a forthcoming paper which extends the results of [Ba] and [Ba'] to large categories when they are cocomplete and have a set of generators with rank.

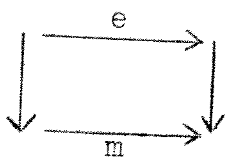
1. Factorization systems.

(1.1) Let $\underline{X}$ be any category. We define the following subcategories of $\underline{X}$. Each contains all the objects of $\underline{X}$ and consists of the morphisms described.

i) $\underline{I}(\underline{X})$ is the class of all isomorphisms of $\underline{X}$.

ii) $\underline{M}_o(\underline{X})$ is the class of all monomorphisms of $\underline{X}$.

iii) $\underline{M}_1(\underline{X})$ is the class of all strong monomorphisms in $\underline{X}$ where we say that m is a strong monomorphism if in any diagram



in which e is an epimorphism, there is a unique morphism from codomain e to domain m making both triangles commute.

iv) $\underline{M}_2(\underline{X})$ consists of all split monos, i.e. those with a left inverse.

v) Dually, we define $\underline{E}_i(\underline{X})$, $i = 0,1,2$, to consist of all, strong, and split epimorphisms, respectively.

(1.2) When $\underline{X}$ is fixed (or understood), we will consistently drop the argument in the above notation. It is clear that $\underline{I}, \underline{M}_o, \underline{M}_2, \underline{E}_o$ and $\underline{E}_2$ are subcategories, i.e. closed under composition and containing the identities (in fact all isomorphisms). That $\underline{M}_1$ and $\underline{E}_1$ are subcategories follows from [Kel], proposition 3.2 .

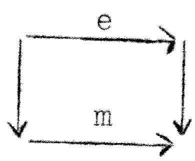
(1.3) The classes of regular monomorphisms and regular epimorphisms (those which are equalizers, resp. coequalizers, of a family of pairs of maps) are also interesting. These, however, are not usually subcategories. When they are, it requires only very mild hypotheses to see that they then coincide with $\underline{M}_1$, resp. $\underline{E}_1$. (see [Kel], proposition 3.8 and section 4. E.g. the existence of kernel pairs and coequalizers of them is sufficient to show that they then coincide.)

(1.4) A pair $(\underline{E}, \underline{M})$ of subcategories of $\underline{X}$ is said to be a factorization system if

i) $\underline{I} \subset \underline{E} \wedge \underline{M}$.

ii) $\underline{X} = \underline{M}.\underline{E}$. That is, every $f \in \underline{X}$ has a factorization as $f = m.e$ with $m \in \underline{M}$ and $e \in \underline{E}$.

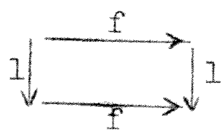
iii) In any commutative square



in which $e \in \underline{E}$ and $m \in \underline{M}$, there is a unique map from the codomain e to the domain m making both triangles commute. This is often called the diagonal fill-in.

(1.5) <u>Proposition.</u> In any factorization system $(\underline{E},\underline{M})$ on $\underline{X}$, $\underline{I} = \underline{E} \wedge \underline{M}$.

Proof. We need only show $\underline{I} \supset \underline{E} \wedge \underline{M}$, since the other inclusion is assumed. If $f \in \underline{E} \wedge \underline{M}$, consider the square

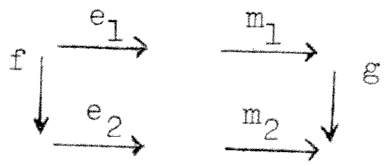


and the existence of a diagonal fill-in gives the result.

(1.6)<u>Proposition</u>. Suppose $(\underline{E},\underline{M})$ is a pair of subcategories satisfying (1.4)i) and ii). Then the following are equivalent.

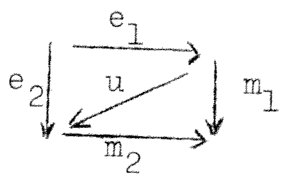
i) $(\underline{E},\underline{M})$ is a factorization system.

ii) The factorization lifts to the morphism category of $\underline{X}$. I.e., in any diagram



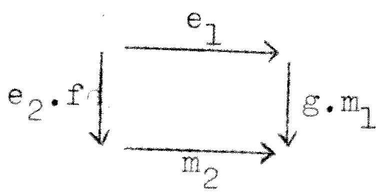
with $e_1, e_2 \in \underline{E}$, $m_1, m_2 \in \underline{M}$, there is a unique h: domain $m_2 \longrightarrow$ domain m_2 making both squares commute.

iii) Any two factorizations of the same map are unique up to a unique isomorphism. That is, given two factorizations $f = e_1 m_1 = e_2 m_2$ of the same map, there is a unique u:domain $m_1 \longrightarrow$ domain m_2 such that both triangles in

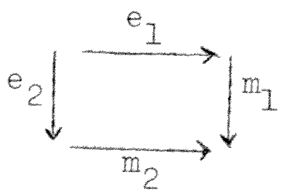


commute and that u is an isomorphism.

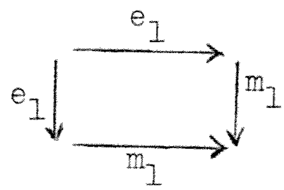
Proof. i) $\Longrightarrow$ ii). Just redraw the diagram



i) $\Longrightarrow$ iii) given a commutative diagram

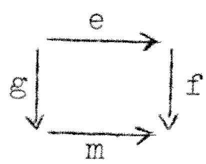


we deduce the existence of maps u:domain $m_1 \longrightarrow$ domain m_2 and v:domain $m_2 \longrightarrow$ domain m_1 such that $m_2 u = m_1$, $ue_1 = e_2$, $m v = m_2$ and $ve_2 = e_1$. Then we see that in the diagram

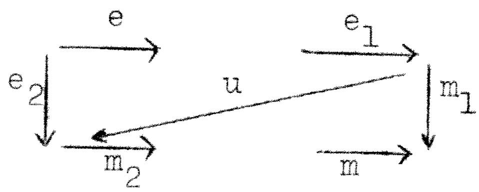


both vu and the identity with make both triangles commute, so by uniqueness $vu = 1$ and similarly $uv = 1$, so that u and v are inverse isomorphisms.

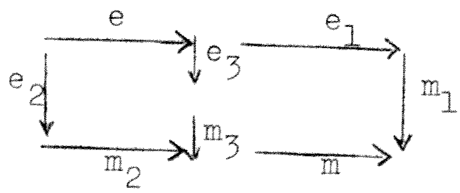
iii) $\Longrightarrow$ i). Given a commutative square



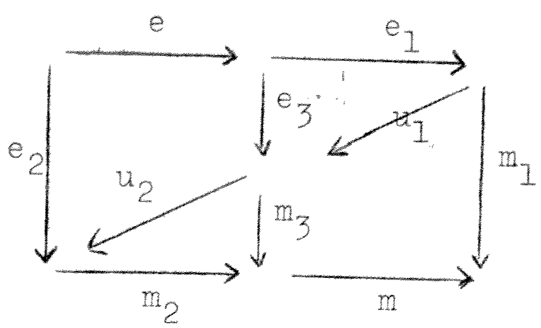
with $e \in \underline{E}$ and $m \in \underline{M}$, let $f = m_1 e_1$ and $g = m_2 e_2$ with $e_1, e_2 \in \underline{E}$, $m_1, m_2 \in \underline{M}$. Then there is a unique map u such that



commutes and then $m_2 u e_1$ is the required map. Now suppose h: domain $e_1 \longrightarrow$ domain m is another such map. Write $h = m_3 e_3$, $m_3 \in \underline{M}$, $e_3 \in \underline{E}$ and consider the diagrar



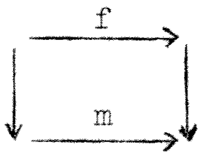
Then there are unique maps u_1 and u_2 as indicated so that the diagram



commutes, and it then follows by uniqueness that $u = u_1.u_2$. Then $h = m_3.e_3 = m_2.u_2.u_1.e_1 = m_2.u.e_1$.

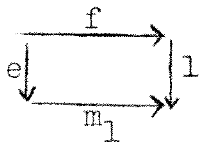
ii) $\Longrightarrow$ i). This is easy, since i) is just a special case of ii) with m_1 and e_2 being the identity.

(1.7) Proposition. Let $(\underline{E},\underline{M})$ be a factorization system on $\underline{X}$ and suppose that $f\in\underline{X}$ is such that for any $m\in\underline{M}$ and any commutative diagram

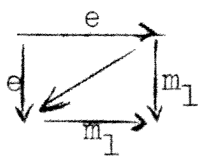


there is a g:codomain f $\longrightarrow$ domain m making both triangles commut Then $f\in\underline{E}$.

Proof. Let $f = m_1 e$ with $m_1\in\underline{M}$, $e\in\underline{E}$. Consider the diagram

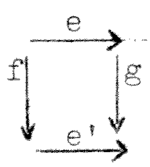


The existence of g with $m_1 g = 1$ and $gf = e$ implies the diagram



can be filled in with either 1 or gm_1 and then uniqueness implies that $gm_1 = 1$. Thus $m_1\in\underline{I}$ and $f = m_1 e\in\underline{E}$.

(1.8) Corollary. If the diagram



is a pushout and $e\in\underline{E}$, then $e'\in\underline{E}$.

Proof. Suppose we have given a commutative diagram

$$\begin{array}{ccc} \cdot & \xrightarrow{e'} & \cdot \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle g'} \\ \cdot & \xrightarrow{m} & \cdot \end{array}$$

with $m \in \underline{M}$. Then there is a map h such that $h.e = f'.f$ and $m.h = g'.g$. The first of these equations implies the existence of a unique h' such that $h'e' = f'$ and $h'g = h$. To show that $mh' = g'$, it is sufficient, by the uniqueness of maps from a pushout, to show that they became equal when composed with e' and g. We have $mh'e' = mf' = g'e'$ and $mh'g = mh = g'g$.

(1.9) Corollary. If $\{e_i : X_i \longrightarrow Y_i\}$ is such that each $e_i \in \underline{E}$ and $e = \coprod e_i : \coprod X_i \longrightarrow \coprod Y_i$ exists, then $e \in \underline{E}$.

Proof. Similar to above.

2. Right and left factorization systems

(2.1) We say that a factorization system $(\underline{E}, M)$ is a right (resp. left) factorization system if $\underline{M} \supset \underline{M}_0$ (resp. $\underline{E} \supset \underline{E}_0$). If both of these conditions hold, we call it a bifactorization system.

(2.2) We say that $\underline{M}$ has left cancellation (resp. $\underline{E}$ has right cancellation) if $fg \in \underline{M} \Longrightarrow g \in \underline{M}$ (resp. $fg \in \underline{E} \Longrightarrow f \in \underline{E}$).

(2.3) Theorem. Suppose $(\underline{E}, \underline{M})$ is a factorization system in $\underline{X}$. Then of the following statements, i) $\Longrightarrow$ ii) $\Longleftrightarrow$ iii) $\Longleftrightarrow$ iv) . If $\underline{X}$ has kernel pairs (or even weak kernel pairs) then all are equivalent.

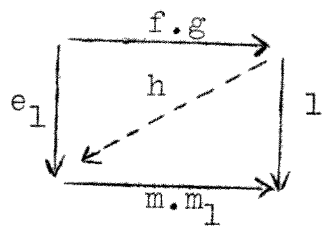
i) $\underline{M} \supset \underline{M}_o$, i.e. $(\underline{E},\underline{M})$ is a right factorization system.

ii) $\underline{E}$ has right cancellation.

iii) $f.g \in \underline{E}$, $g \in \underline{I} \Longrightarrow f \in \underline{E}$.

iv) $f.g = 1 \Longrightarrow f \in \underline{E}$, i.e. $\underline{E}_{\partial} \subset \underline{E}$.

Proof. i) $\Longrightarrow$ ii) For let $f.g \in \underline{E}$. Write $f = m.e$ and then $e.g = m_1.e_1$ with $m, m_1 \in \underline{M}$, $e, e_1 \in \underline{E}$. Then in the diagram

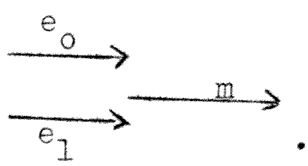


we can find an h with $m.m_1.h = 1$. Then $m.m_1.h.m = m$ and m is a mono, so that $m_1.h.m = 1$ also, which shows that m is an isomorphism and that $f = m.e \in \underline{E}$.

iv) $\Longrightarrow$ ii) Exactly the same except that from $m.m_1.h = 1$ we conclude that $m \in \underline{E}$ and so $f = m.e \in \underline{E}$.

ii) $\Longrightarrow$ iii) $\Longrightarrow$ iv): Trivial.

iv) + weak kernel pairs $\Longrightarrow$ i): Let $m \in \underline{M}$, and suppose there is a weak kernel pair diagram



Then the properties of weak kernel pairs imply the existence of an s such that $e_o.s = e_1 s = 1$. This in turn gives that $e_o, e_1 \in \underline{E}$. Now $m.e_o = m.e_1$ being two factorizations of the same map, there must be an isomorphism u such that $m.u = m$ and $u.e_o = e_1$. But then $u = ue_o.s = e_1 s = 1$, so that $e_o = e_1$. Thus m is mono, for if $m.f_o = m.f_1$ there must exist an f with $f_o = f.e_o = f.e_1 = f_1$.

(2.4)Example. To show that assumption of weak kernel pairs is necessary, let $\underline{X}$ be the category with three objects X,Y,Z with maps generated by $e_o, e_1: X \longrightarrow Y$, $m: Y \longrightarrow Z$ and $u: Y \longrightarrow Y$, subject to the identities $me_o = me_1$, $mu = m$, $ue_o = e_1$, $ue_1 = e_o$, and $u^2 = 1$. Let the class $\underline{E}$ consist of e_o, e_1 and all isomorphisms, and $\underline{M}$ consist of m and all isomorphisms. $\underline{E}_2 = \underline{I}$, so that $\underline{E}_2 \subset \underline{E}$, while $\underline{M}_o \not\supset \underline{M}$, since clearly $m \notin \underline{M}_o$. It is clear from the fact that u is an isomorphism that $(\underline{E}, \underline{M})$ satisfies (1.6), iii) and hence is a factorization system.

3.Generators

(3.1) Here we define the notion of generator with respect to a right factorization system $(\underline{E}, \underline{M})$. With this we will be able to clarify proposition 4.6 of [Kel]. None of the argument seems to apply to factorization systems which are not right factorization systems. In particular, even the definition of generator seems reasonable only in this case.

(3.2)Definition. Let $(\underline{E}, \underline{M})$ be a right factorization system. An $(\underline{E}, \underline{M})$ generator Λ is a set of objects of $\underline{X}$ such that for any $m \in \underline{M} - \underline{I}$ there is a $Y \in \Lambda$ with (Y,m) not an isomorphism.

(3.3) If Λ is any set of object of $\underline{X}$ and has the appropriate coproducts, there is a natural Λ-induced cotriple $\mathbb{G} = (G, \varepsilon, \delta)$ where $GX = \coprod_{Y \in \Lambda} \coprod_{(Y,X)} Y$. See [B-B], section 10.1 for details. GX is equipped with a map $\langle u \rangle : Y \longrightarrow GX$ corresponding to each possible $u: Y \longrightarrow X$, $Y \in \Lambda$ and a map from GX is prescribed and uniquely by giving its composite with each $\langle u \rangle$. For example $\varepsilon X : GX \longrightarrow X$ is given by $\varepsilon X . \langle u \rangle = u$.

(3.4)Theorem: Let $\mathfrak{G}$ be the Λ-induced cotriple on a category $\underline{X}$ with a right factorization system $(\underline{E},\underline{M})$. Then the following are equivalent.

i) For all X, there is a family $\{Y_i\}$, $i\varepsilon I$ of objects of Λ and an $\underline{E}$-morphism.
$e: \coprod Y_i \longrightarrow X$.

ii) For all X, $\varepsilon X \epsilon \underline{E}$.

iii) Λ is an $(\underline{E},\underline{M})$ set of generators.

Proof. i)$\Longrightarrow$ii): Suppose there is a map $e: \coprod Y_i \longrightarrow X$. Suppose $y_i: Y_i \longrightarrow \coprod Y_i$ is the coproduct injection and let $f: \coprod Y_i \longrightarrow GX$ be given by $f.y_i = \langle e.y_i \rangle$. Then $\varepsilon X.f.y_i = \varepsilon X.\langle ey_i \rangle = e.y_i$ for each $i\varepsilon I$, so that $\varepsilon X.f = e$, and since $e\varepsilon\underline{E}$, it follows from (2.3) that $\varepsilon X \epsilon \underline{E}$.

ii)$\Longrightarrow$iii): Suppose $m: X \longrightarrow X' \epsilon \underline{M}$ is such that (Y,m) is an isomorphism for all $m\epsilon\underline{M}$. It follows that Gm is an isomorphism. But $m.\varepsilon X = \varepsilon X'.\,Gm$ gives, by the cancellation property of (2.3), that $m\epsilon\underline{E}$. Since also $m\epsilon\underline{M}$, m is an isomorphism.

iii)$\Longrightarrow$ii): Factor the map $\varepsilon X = m.e$ with $m\epsilon\underline{M}$, $e\epsilon\underline{E}$. Then it is sufficient to show that (Y,m) is an isomorphism for all $Y\epsilon\Lambda$. But (Y,m) is 1-1, since $m\epsilon\underline{M}_o$. To see that it is onto, let $u: Y \longrightarrow X$. Then $e.\langle u \rangle$ is a map with $m.e.\langle u \rangle = u$, so that (Y,m) is an isomorphism.

ii)$\Longrightarrow$i) is, of course, trivial.

(3.5)Theorem: Let $\underline{X}$ have a factorization $(\underline{E},\underline{M})$ and an $(\underline{E},M)$ generating set Λ. Then if $\underline{X}$ has coproducts (of families of objects of Λ), $\underline{X}$ has arbitrary intersections of $\underline{M}$-subobjects and pullbacks along maps of $\underline{M}$.

Proof. Let $\{X_i\}_{i\epsilon I}$ be a family (we could even permit that I be a proper class, except that one consequence of even finite intersections, is that $\underline{M}$-subobject lattices are small) of $\underline{M}$-subobjects of X.

Consider $Z = \coprod_{Y \in \Lambda} \coprod Y$, the second coproduct indexed by the set of maps $Y \longrightarrow X$ which factor through each X_i. Then Z is a cofactor of GX and we can factor the composite $Z \longrightarrow X \longrightarrow X$ as $Z \xrightarrow{e} X_o \xrightarrow{m} X$ with $e \in \underline{E}$ and $m \in \underline{M}$. Now from the definition of Z, it is clear that the map $Z \longrightarrow GX$ factors through each X_i, and so we have a diagram

$$\begin{array}{ccc} Z & \longrightarrow & X_o \\ \downarrow & & \\ GX_i & & \downarrow \\ \downarrow & & \\ X_i & \longrightarrow & X \end{array}$$

which is easily seen to be commutative by putting in the arrows $GX_i \longrightarrow GX \longrightarrow X$. Then the diagonal fill-in shows that $X_o \subset X_i$ for all $i \in I$. Conversely, let $X' \longrightarrow X$ factor through each X_i. Factoring the map, we may suppose that X is an $\underline{M}$-subobject of X. Then any $Y \longrightarrow X'$ with $Y \in \Lambda$ is a map $X \longrightarrow X$ which factors through each X_i, so that $GX' \longrightarrow GX$ factors through Z. Then consider the diagram

$$\begin{array}{ccc} GX' & \longrightarrow & X' \\ \downarrow & & \\ Z & & \downarrow \\ \downarrow & & \\ X_o & \longrightarrow & X \end{array}$$

which is similarly shown to be commutative, and again the diagonal fill-in gives that $X' \subset X_o$.

(3.6) <u>Proposition</u>. Let $\underline{X}$ have a factorization system $(\underline{E},\underline{M})$ and an $(\underline{E},\underline{M})$ generating set Λ. If finite intersections exist, then $\underline{X}$ is $\underline{M}$-well-powered.

Proof. See the proof of proposition 4.6 of [Kel]. In fact, the finite intersections allow the construction of a monomorphism from the class of subobjects of X to the set of subsets of $\bigcup_{Y \in \Lambda} (Y,X)$.

(3.7) <u>Theorem</u>. Let $\underline{X}$ have a bifactorization system $(\underline{E},\underline{M})$ and an $(\underline{E},\underline{M})$ generating set Λ, and be cocomplete and $\underline{E}$-co-well-powered (or else have the Isbell property that the $\underline{E}$ quotient lattices be complete). Then $\underline{X}$ is complete.

Proof. First we construct products. Given a family $\{X_i\}_{i\in I}$ of objects of $\underline{X}$, let $Z = \coprod_{Y\in\Lambda} \coprod Y$, where the second coproduct is indexed by the set $\prod_{i\in I}(Y,X_i)$. If $\{u_i:Y\longrightarrow X_i\}$ is an i indexed family, let $\langle\{u_i \mid i\in I\}\rangle$ denote the corresponding coproduct injection. Then define $q_i:Z\longrightarrow X_i$ by letting $q_i.\langle\{u_i \mid i\in I\}\rangle = u_i:Y\longrightarrow X_i$. Now consider all $\underline{E}$-quotients of Z through which every q_i factors. This class is closed under any cointersections which exist, and the hypothesis of the theorem quarantee that the cointersection of all them exists. Let $e:Z\longrightarrow X$ be the $\underline{E}$-quotient mapping and let $p_i:X\longrightarrow X_i$ be the maps such that $p_i.e = q_i$ for all i. Now if $p_i^!:X'\longrightarrow X_i$ is given for all i, there is induced a map $f:GX'\longrightarrow Z$ by $f.\langle u\rangle = \langle\{p_i^!.u\}\rangle$. Form the pushout

$$\begin{array}{ccc} GX' & \xrightarrow{\ f\ } & Z \\ {\scriptstyle \varepsilon X'}\downarrow & & \downarrow{\scriptstyle e'} \\ X' & \xrightarrow{\ f'\ } & Z' \end{array} \quad .$$

By (1.8), $e'\in\underline{E}$. Moreover, for all $i\in I$ and $u:Y\longrightarrow X'$, $q_i.f.\langle u\rangle = q_i\langle\{p_i^!.u\}\rangle = p_i^!.u = p_i^!.\varepsilon X.\langle u\rangle$, so that $q_i.f = p_i^!.\varepsilon X'$; and there is induced a unique $g_i:Z'\longrightarrow X_i$ such that $g_i.e' = q_i$ and $g_i.f' = p_i^!$. But then since X was the cointersection of all such Z', there is a map $e'':Z'\longrightarrow X$ such that $e''.e' = e$. Finally $e''.f':X'\longrightarrow X$ satisfies $p_i.e''.f'.\varepsilon X = p_i.e''.e'.f = p_i.e.f = q_i.f = g_i.e'.f = g_i.f'.\varepsilon X = q_i.\varepsilon X$. By assumption εX is an epimorphism and so $p_i.e''.f' = q_i$. Thus X together with the p_i is at least a weak limit. If two maps $X'\longrightarrow X$ factored all the q_i, we could form their coequalizer. Since $\underline{M}\subset\underline{M}_0$, one can easily see that any coequalizer must satisfy (1.7) and be in $\underline{E}$. Then this coequalizer would be a further $\underline{E}$-quotient of Z through which all the q_i would factor, which would be a contradiction. Hence X together with the p_i is the product. The construction of equalizers is similar. Given two maps $X \overset{f}{\underset{g}{\rightrightarrows}} X'$, we form $Z = \coprod_{Y\in\Lambda} \coprod Y$, the second coproduct indexed by all $Y \longrightarrow X$ which equalize f and g. Factoring the map $Z \longrightarrow GX \longrightarrow X$ as $Z \xrightarrow{e} X_o \xrightarrow{m} X$ with $e\in\underline{E}$ and $m\in\underline{M}$, we use the fact that $e\in\underline{E}_o$ to conclude $f.m=g.m$. If $X' \longrightarrow X$ equalizes f and g, we may, by factoring it, suppose it is a map in $\underline{M}$. From here, the proof proceeds exactly as the proof of (3.5).

(3.8)<u>Remark.</u> The assumption that this is a bifactorization system, while obviously not a necessary result of completeness, is clearly needed for this proof, since the equalizers constructed are all in $\underline{M}$. In particular $\underline{M}_2 \subset \underline{M}$, which by the dual of (2.3) implies that $\underline{E} \subset \underline{E}_o$. The assumption that $\underline{X}$ is co-well-powered is also necessary. Consider the category of ordinals as an ordered set. Take $\underline{M} = \underline{I}$ and $\underline{E} = \underline{E}_o$ = all maps. This is evidently a factorization system and o is a generator. Cocompleteness is clear but the category lacks a terminal object.

(3.9) Gabriel and Ulmer have shown however, in the case $\underline{M} = \underline{M}_o$, $\underline{E} = \underline{E}_1$ where the generators have rank (defined in section 5), that the category is co-well-powered and complete. When the class $\underline{E}_1$ consists of regular epimorphisms, this was also a result of Kelly([Kel], proposition 4.6) based on the observation that a regular epimorphism e with domain X is determined by those pairs of maps $Y \rightrightarrows X$, $Y\in\Lambda$ which are coequalized by e. This result does not depend on rank. Incidentally, the generator in the above example has rank trivially.

4. Embedding

(4.1) Suppose that $\underline{X}$ is a category with finite limits, coequalizers of kernel pairs, and a factorzation system $(\underline{E},\underline{M})$. If $\underline{E}$ is closed under pullbacks, then it would seem that Lubkin's construction as described in [Ba I and II] would work. It is true that the construction will work and provide a left exact functor $U:\underline{X} \longrightarrow \underline{S}^A$ where A is the discrete category of nonempty subobjects of the terminal object of $\underline{X}$. It will be faithful, for example, provided $\underline{E} \subset \underline{E}_o$. It cannot, however, have all the nice properties described in those papers without being actually an instance of the theory described there, more precisely, we prove.

(4.2) Theorem. Suppose $U:\underline{X} \longrightarrow \underline{S}^A$ has the property that U preserves finite limits, $U(\underline{E}) \subset \underline{E}_o(\underline{S}^A)$, $U(\underline{M}) \subset \underline{M}_o(\underline{S}^A)$, $U^{-1}(\underline{E}_o(\underline{S}^A)) \subset \underline{E}$ and $U^{-1}(\underline{M}_o(\underline{S}^A)) \subset \underline{M}$. Then $\underline{M} = \underline{M}_o(\underline{X})$ and $\underline{E} = \underline{E}_1(\underline{X})$ is the set of regular epimorphisms.

Proof. First observe that $\underline{I} = \underline{E} \cap \underline{M}$ together with the other conditions implies U reflects isomorphisms. Then if $f \neq g$, form the equalizer

$$\xrightarrow{\;h\;} \overset{f}{\underset{g}{\rightrightarrows}} \quad .$$

It remains an equalizer when U is applied and Uh not an isomorphism implies that $Uf \neq Ug$. Thus U is faithful. Now if $m \in \underline{M}$ is not mono, there are $f \neq g$ with $mf = mg$. Then $Um.Uf = Um.Ug$ with $Uf \neq Ug$ shows Um not mono, which is a contradiction, so that $\underline{M} \subset \underline{M}_o$. Similarly, $\underline{E} \subset \underline{E}_o$. Now if $f \in \underline{M}_o$, Uf is mono, since U preserves the kernel pair of f, which are equal if and only if f is mono. Thus $\underline{M} \subset \underline{M}_o$ and so $\underline{M} = \underline{M}_o$, which, by (1.7), imples that $\underline{E} = \underline{E}_1$. But if $\underline{E}_1$ is invariant under pullbacks, then every regular epi has a pullback which is an epimorphism, and by [Kel], 4.1. and 5.14, every strong epimorphism is regular.

5. Generators with rank

(5.1) Let α be an ordinal number. We say that a category $\underline{I}$ is α-filtering if every diagram $D:\underline{J}\longrightarrow\underline{I}$ in which the cardinal of the set of morphisms of $\underline{J}$ is $\leq\alpha$ has an upper bound in $\underline{I}$. In practice this means two things.

i) For every α-indexed family $\{i_n \mid n\in\alpha\}$ of objects of $\underline{I}$, there is an i for which there is a map $i_n\longrightarrow i$ for each $n\in\alpha$.

ii) For two objects i,i' and any α-indexed family of maps

$$i \;\begin{array}{c}\longrightarrow\\ \vdots\\ \longrightarrow\end{array}\; i'$$

there is a map $i'\longrightarrow i''$ which simultaneously coequalizes all of them.

In our applications $\underline{I}$ will be a partially ordered set and condition ii) will be vacuous. Then we will call $\underline{I}$ an α-directed set.

(5.2) Let $(\underline{E},\underline{M})$ be a factorization system and α be a cardinal number. An $(\alpha,\underline{M})$ filter in $\underline{X}$ consists of an α-filtering category $\underline{I}$, a functor $D:\underline{I}\longrightarrow\underline{X}$ such that there is an object X of $\underline{X}$, and a map $D\longrightarrow X$ with the property that for all $i\in\underline{I}$, the map $Di\longrightarrow X\in\underline{M}$. If $\underline{M}\subset\underline{M}_0$, then we can as well suppose that $\underline{I}$ is an α-directed set, since for $i'\rightrightarrows i$, the map $Di\longrightarrow X$ must coequalize $Di'\rightrightarrows Di$, and being mono they must have been equal.

(5.3) Let $\underline{X}$ be as above and suppose that every $(\alpha,\underline{M})$ filter in $\underline{X}$ has a colimit. Then we say that a functor $U:\underline{X}\longrightarrow\underline{Y}$ has $\underline{M}$-rank $\leq\alpha$ provided that for every $(\underline{M},\alpha)$-filter $D:\underline{I}\longrightarrow\underline{X}$, $U(\operatorname{colim} D)$ is the colimit of UD. We will also say $\underline{M}$-rank $U\leq\alpha$ (or $\underline{M}$-rank $U=\alpha$ if α is the least such). An object $X\in\underline{X}$ will be said to have $\underline{M}$-rank $\leq\alpha$ provided the hom functor $(X,-):\underline{X}\longrightarrow\underline{S}$ does.

(5.4) A set of ($\underline{E}$,$\underline{M}$) generators with rank is a set Λ of ($\underline{E}$,$\underline{M}$) generators with each $Y \in \Lambda$ having some $\underline{M}$-rank. This is clearly equivalent to the existence of α such that for all $Y \in \Lambda$, $\underline{M}$-rank $Y \leq \alpha$.

(5.5) We list three properties of an object X with respect to an ($\underline{E}$,$\underline{M}$) factorization, an ($\underline{E}$,$\underline{M}$) generating set Λ, and a cardinal α.

(R1α). $\underline{M}$-rank $X \leq \alpha$.

(R2α). Cardinal $\Lambda \leq \alpha$ and for each $Y \in \Lambda$, cardinal $(Y,X) \leq \alpha$.

(R3α). There is an $\underline{E}$-morphism. $\coprod_{i \in I} Y_i \longrightarrow X$ in which $Y_i \in \Lambda$ for all $i \in I$ and cardinal $I \leq \alpha$. We let $\underline{R1\alpha}$, $\underline{R2\alpha}$ and $\underline{R3\alpha}$ respectively denote the full subcategory consisting of those $X \in \underline{X}$ which satisfy that respective condition.

(5.6) Throughout the rest of this section, $\underline{X}$ will be a cocomplete category with a bifactorization system ($\underline{E}_1$,$\underline{M}_o$) which we will still denote by ($\underline{E}$,$\underline{M}$). This is the situation in which Gabriel and Ulmer are working and it seems likely that these results are derivable from theirs, but I have not seen precise statements of them. We will suppose that Λ is a set of ($\underline{E}$,$\underline{M}$) generators with rank. As mentioned above, Gabriel-Ulmer have shown that this implies that $\underline{X}$ is $\underline{E}$-co-well-powered. We also suppose that β is a cardinal numer such that cardinal $\Lambda \leq \beta$ and for each $Y \in \Lambda$ $\underline{M}$-rank $Y \leq \beta$.

(5.7) Theorem. There is an α with $\underline{R1\alpha} = \underline{R2\alpha} = \underline{R3\alpha}$.
The proof will be given in three steps. The first is the trivial observation that for any α , $\underline{R2\alpha} \subset \underline{R3\alpha}$.

(5.8) Proposition. For all $\alpha \geq \beta$, $\underline{R3\alpha} \subset \underline{R1\alpha}$.
Proof. Suppose there is an $\underline{E}$-morphism $\coprod_{i \in I} Y_i \longrightarrow X$ with cardinal $I \leq \alpha$, and $\{Z_n\}_{n \in N}$ is an ($\underline{M}$,α) filter of $\underline{M}$-subobjects of Z.

Let $X \longrightarrow \text{colim } Y_n$ be given. Then for each $i \in I$ there is an $n_i \in N$ such that the map $Y_i \longrightarrow \coprod Y_i \longrightarrow X \longrightarrow \text{colim } Z_n$ factors through Z_{n_i}. Since this is an α-filter and cardinal $I \leq \alpha$, there is some uniform n such that every $Y_i \longrightarrow \coprod Y_i \longrightarrow X \longrightarrow \text{colim } Z_n$ factors through Z_n . This means that $\coprod Y_i \longrightarrow X \longrightarrow \text{colim } Z_n$ factors through Z_n, and then the result follows by considering the diagonal fill-in in the diagram

$$\begin{array}{ccc} \coprod Y_i & \longrightarrow & X \\ & & \downarrow \\ \downarrow & & \text{colim } Z_n \\ & & \downarrow \\ Z_n & \longrightarrow & Z \end{array} .$$

(5.9) Proposition. Let $\{Z_n\}$ be an $(\underline{M},\beta)$ filter of $\underline{M}$-subobjects of Z. Then the induced map $f:\text{colim } Z_n \longrightarrow Z$ is in $\underline{M}$.
Proof. Since $\underline{M} = \underline{M}_o$, it is necessary to show only that the map is mono. If not, there are maps $g \neq h$ such that $f.g = f.h$, and they have an equalizer which is in $\underline{M}$-$\underline{I}$, so there is a map k with domain in Λ such that $g.k \neq h.k$. Thus we have

$$Y \underset{h.k}{\overset{g.k}{\rightrightarrows}} \text{colim } Z_n$$

distinct maps coequalized by f. But there is an n such that each factors through Z_n, and then since $Z_n \longrightarrow Z$ is mono, we would have

$$Y \rightrightarrows Z_n \longrightarrow Z$$

also distinct.

(5.10) Proposition. For all $\alpha \geq \beta$, $\underline{R1\alpha} \subset \underline{R3\alpha}$.
Proof. Consider $X \in \underline{R1\alpha}$ and let $\{X_n \mid n \in N\}$ denote the set of all subobjects of X which are in $\underline{R3\alpha}$. This is clearly β-directed. For if $N_o \subset N$ is a subset of N of carinality $\leq \beta$, choose for $n \in N_o$ an $\underline{E}$-morphism $\coprod_{i \in I_n} Y_i \longrightarrow X_n$. Then the image X_o of $\coprod Y_i \longrightarrow X$,

where the coproduct is indexed by $\bigcup_{n\in N_o} I_n$, contains each X_n . In fact, just fill in the diagonal in the diagram

$$\begin{array}{ccc}
\coprod\{Y_i \mid i\in I_n\} & \longrightarrow & X_n \\
\downarrow & & \\
\coprod\{Y_i \mid i\in \bigcup I_n\} & & \downarrow \\
\downarrow & & \\
X_o & \longrightarrow & X .
\end{array}$$

Now the map colim $X_n \longrightarrow X \in \underline{M}$, while on the other hand, every $Y \longrightarrow X$, $Y\in\Lambda$, factors through a subobject of X which is in $\underline{R1\alpha}$ (in fact $\underline{R11}$), and thus through colim X_n, so that $(Y,\text{colim } X_n) \simeq (Y,X)$ for all $Y\in\Lambda$. By definition (3.2) this implies that colim $X_n \xrightarrow{\simeq} X$. Since $X\in\underline{R1\alpha}$, every map $X \longrightarrow X$ factors through some X_n. In particular the identity map does, so we have $X \longrightarrow X_n \longrightarrow X$ whose composite is the identity. Since the second factor is in $\underline{M}$, they are isomorphisms, and $X\in\underline{R3\alpha}$ since X_n is.

(5.11)Proposition. For any α , $\underline{R3\alpha}$ is small.
Proof. Of course, this really means that it has a small skeleton. Since Λ is a set, the class of object $\coprod_{i\in I} Y_i$ where cardinal $I\leq\alpha$ and $Y_i\in\Lambda$ is also small and since the category is $\underline{E}$-co-well-powered, the class of $\underline{E}$-quotients of these is also small.

(5.12) Now consider a γ sufficiently large that $\underline{R3\beta}\subset\underline{R2\gamma}$. This clearly exists, since $\underline{R3\beta}$ is a small category and $\underline{X}$ is locally small.

(5.13)Proposition. If $\underline{R3\beta}\subset\underline{R2\gamma}$, $\gamma\geq\beta$, and $\alpha=2^{\gamma}$, then $\underline{R3\alpha}\subset\underline{R2\alpha}$.
Proof. Let $X\in\underline{R3\alpha}$ and consider an $\underline{E}$-morphism $\coprod_{i\in I} Y_i \longrightarrow X$ with each $Y_i\in\Lambda$ and cardinal $I\leq\alpha$. Consider all subsets of I of

cardinal $\leq\beta$. Let these be denoted by $\{I_n \mid n\in N\}$. Each of these determines a map of $\beta \longrightarrow \alpha$, and so there are at most $\alpha^{\beta} = 2^{\gamma \times \beta} = 2^{\gamma} = \alpha$. For each n, let X_n denote the image of $\coprod_{i\in I_n} Y_i \longrightarrow X$. Just as in the proof of (5.10), the X_n are an $(\underline{M},\beta)$ filter on X and colim $X_n = X$. For any $Y\in\Lambda$, $(Y,X) = \operatorname{colim}(Y,X_n)$ which is of at most α sets, each of size at most γ, and so $(Y,X) \leq \alpha$. Since also $\beta < \alpha$, it follows that $X \in \underline{R2\alpha}$.

(5.14) The smallest cardinal α for which $\underline{R1\alpha} = \underline{R2\alpha} = \underline{R3\alpha}$ is a characteristic of the generating set which might be called the rank of Λ (since $\underline{M} = \underline{M}_0$, it is now appropriate to omit its name from these notions). Among all generating sets one might then choose the one for which α is as small as possible. Forming $\underline{R1\alpha}$ for that α gives a canonical generating set for $\underline{X}$. Notice that it will contain all the minimum rank generating sets. This α might be described as the rank of $\underline{X}$. The existence of such a canonical generating set answers a question raised by Lawvere.

(5.15) Proposition. When α is as above, $\underline{R1\alpha}$ is α cocomplete and γ complete for any γ such that $2^{\gamma} < \alpha$. In particular it is finitely complete.

Proof. Let $\{X_i \mid i\in I\}$ be an I indexed family of $X_i \in \underline{R1\alpha}$, with cardinal $I \leq \alpha$.

Since $\underline{R1\alpha} = \underline{R3\alpha}$, we can choose, for each $i\in I$, an index set J_i with cardinal $J_i \leq \alpha$ and an $\underline{E}$-morphism $\coprod_{j\in J_i} Y_j \longrightarrow X_i$. Then by (1.9), with $J = \cup J_i$, the map

$$\coprod_{j\in J} \longrightarrow \coprod_{i\in I} X_i$$

is also in $\underline{E}$ and cardinal $J \leq \alpha$. Coequalizers are trivial since every regular epi is in $\underline{E}$. As for limits, if $\{X_i \mid i\in I\}$ where 2 cardinal $I \leq \alpha$ and $X_i \in \underline{R1\alpha}$ for all $i\in I$, then for all $Y\in\Lambda$, $(Y, \prod X_i) =$

$\prod(Y,X_i)$ has cardinality $\leq \alpha^{\beta} = \alpha$. Equalizers are clear by similar reasoning.

(5.16)Proposition. Let α be as above and suppose $\coprod_{i\in I} X_i \longrightarrow X$ is an $\underline{E}$-morphism and $X \in \underline{R1\alpha}$. Then there is some $I_o \subset I$ with cardinal $I_o \leq \alpha$ such that the composite

$$\coprod_{i\in I_o} X_i \longrightarrow \coprod_{i\in I} X_i \longrightarrow X$$

is still in $\underline{E}$.

Proof. For each $I_n \subset I$ of cardinal $\leq \alpha$, let X_n be the image of $\coprod_{i\in I_n} X_i$. This is an $(\underline{M},\alpha)$ filter on X and by (5.9) and the fact that $X \in \underline{R1\alpha}$, we have $(X,X) \xrightarrow{\simeq} (X, \operatorname{colim} X_n) \simeq \operatorname{colim} (X,X_n)$. Then the identity map factors throug some X_n, which implies that $X = X_n$.

(5.17)Proposition. With the same α as above, the category $\underline{R1\alpha}$ is closed under formation of $\underline{E}$-quotients and $\underline{M}$-subobjects.

Proof. For it is evident that $\underline{R2\alpha}$ is closed under $\underline{M}$-subobjects and that $\underline{R3\alpha}$ is closed under $\underline{E}$-quotients.

References

[Mac'] S.Mac Lane, *Duality for groups*, Bull.Amer.Math.Soc. *56* (1950), 485-516.

[Mac] S.Mac Lane, *Groups, categories, and duality*, Proc.Nat. Acad.Sci.U.S.A., *34* (1948), 263-267.

[Is] J.R.Isbell, *Some remarks concerning categories and subspaces*, Canad.J. Math. *9* (1957), 563-577.

[Is'] J.R.Isbell, *Subobjects, adequacy, completeness and categories of algebras.* Rozprawy Mat. *36* (1964).

[Is"] J.R.Isbell, *Structure of categories*, Bull.Amer.Math. Soc., *72* (1966), 619-655.

[Sem] Z.Semadeni, *Projectivity, injectivity and duality,* Rozprawy Mat.*35* (1963).

[Kel] G.M.Kelly, *Monomorphismus, epimorphisms and pullbacks*, J.Austral.Math.Soc.*9* (1969), 124-142.

[Ken] J.F.Kennison, *Full reflective subcategories and generalized covering spaces,* Ill.J.Math. *12* (1968), 353-365.

[Gro] A.Grothendieck, *Sur quelques points d'algèbre homologique,* Tôhoku Math.J. Ser. 2 *9* (1957), 119-221.

[Ba] M.Barr, *Non-abelian full embedding, I*, (to appear).

[Ba'] M.Barr, *Non-abelian full embedding, II,* (to appear).

[B-B] M.Barr and J.Beck, *Homology and standard constructions*, in "Seminar on Triples and Categorical Homology Theory", Lecture notes no.80 (1969), Springer-Verlag, Berlin.