EXACT CATEGORIES

by Michael Barr

Introduction

Exact categories, roughly speaking, are categories which satisfy the equation

$$(\text{Abelian}) = (\text{Exact}) + (\text{Additive}).$$

Generally speaking, the axioms of abelian categories were chosen precisely in order to define a good notion of the homology theory of chain complexes of a category If one wishes to remove additivity, there are two possible directions. One direction is to try to axiomatize non-abelian homology. This leads to consideration of pointed categories and then of normal monomorphisms and epimorphisms — those which are kernels and cokernels, respectively. This is essentially the point of view adopted by Brinkmann and Puppe in [BP] and Gerstenhaber-Moore in [Ge]. In essence, it goes back at least as far as Mitchell ([Mi], I.15). Brinkmann and Puppe even use the term exact category to describe the type of categories they are considering. Gerstenhaber does not name the type of categories he is dealing with. His axioms are related to but somewhat different from those of Brinkmann and Puppe. Both suppose as part of their axioms that normal epimorphisms are invariant under pullback. I do not know a single example of a category satisfying that hypothesis unless it also satisfies the hypothesis that every regular epimorphism is normal. A regular epimorphism is one which is the coequalizer of some pair of maps and it is evident that every normal epimorphism is regular, since it is the coequalizer of 0 and whatever it is the kernel of. But the nicest pointed category of all, pointed sets, does not satisfy this assumption, in sharp contrast of the result of Manes [Man], that every additive equational category is abelian. In addition, I have been unable to decide, after a modest expenditure of time, whether the categories of monoids and commutative monoids satisfy the Gerstenhaber-Moore axioms. This is one motivation for ignoring earlier definitions of exactness. A second is the essentially special nature of non-abelian cohomology. Its interest is practically restricted to categories which are more or less like groups. I feel that the term exact is too basic to be used for such a special theory.

The second approach is in the direction of homotopy. By the theorem of Dold-Puppe ([DP], Chapter 3), in an abelian category chain complexes (concentrated in non-negative degrees) are equivalent to simplicial objects. This suggests, at least, that one fruitful direction of inquiry is to find a good theory of homotopy for simplicial objects. It would also be nice if every equational category satisfied the conditions and, of course, if it satisfied the above equation.

The exact categories defined here have precisely these properties. It all began with a theorem of Tierney (unpublished, but see I.(3.11)[1] below) that a category is abelian if and only if and only if it is additive and has finite limits and colimits and universally effective equivalence relations. The definition of exact category given here is a slight weakening of the above, weakened only for technical reasons. An exact category has certain finite limits and colimits and universally effective equivalence relations (see I. (1.2) and I. (1.3) for definitions).

The contents of this paper include the elementary properties of

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[1] A reference of the form N. (a.b) is to Chapter <u>N</u> , paragraph (a.b). A reference of the form (a.b) is the same chapter, paragraph (a.b).

exact categories (I and II), an embedding and meta-theorem which generalize those of Mitchell ([Mi] VI, theorem 1.2) in the abelian case (III), and an application to cohomology and Baer addition of extensions (IV and V). The simplicity of the presentation of the Baer sum should be compared with that of Gerstenhaber in [Ge]. The completeness of the results should be compared with those of Chase in [Ch] in which an unpleasant and unnatural assumption ("coflatness") had to be introduced for want of the notion of right exact sequences.

The homotopy theory is not at all developed here. It is possible, given a simplicial object in an exact category, to say when that is a Kan object; and when it is, to define its homotopy. This will be the subject of a subsequent work. The homotopy so defined will be an object of the category in question, rather than a group. It is base-point free and in sets is the usual groupoid (except in dimension 0) of homotopy classes of maps of spheres. The usual homotopy is recovered as soon as a principal component and a base point there are chosen.

There is one more point I would like to mention. A useful axiom which gives a notion intermediate between being exact and being abelian is the supposition that every reflexive subobject of the square of any object is an equivalence relation (see I. (5.5)). This condition is equivalent to every simplicial object being Kan. It is also sufficient to have the theory of group actions of Chapter IV work equally well for monoid actions. The theory of monoid actions also works well in the category of sets, but for an entirely different reason: that category is cartesian closed so that cartesian products commute with all colimits.

Chapter I. The Elementary Theory

1. Definitions and examples.

(1.1) One of the most important tools will be the factorization of every morphism as a regular epimorphism followed by a monomorphism (see (2.3) below). A regular epimorphism is a map which is the coequalizer of some pair of maps, which can be supposed to be its kernel pair, if that exists. We adopt (or adapt) the notation of MacLane [Mac] and we use $\rightarrowtail$ to denote a monomorphism, $\twoheadrightarrow$ to denote a regular epimorphism, and $\xrightarrow{\sim}$ to denote an isomorphism. We will also use these arrows as substantives and say, for example, "f is $\rightarrowtail$" to mean that f is a monomorphism.

(1.2) If $f: X \to X'$ is any map in any category, its kernel pair $X'' \rightrightarrows X$ has the property that $(-,X'') \rightarrowtail (-,X) \times (-,X)$ is a natural equivalence relation on $(-,X)$; two maps to X are identified if and only if their compositions with f are equal. In general, two maps $X'' \rightrightarrows X$ for which $(-,X'') \rightarrowtail (-,X) \times (-,X)$ is a natural equivalence relation on $(-,X)$ will be called on equivalence relation on X. Not every equivalence relation on X need be a kernel pair, any completeness hypothesis notwithstanding. See (1.4) example 5 below. An equivalence relation which is a kernel pair will be called effective.

(1.3) Let $\underline{X}$ be a category. We say that $\underline{X}$ is <u>regular</u> if it satisfies EX1) below and <u>exact</u> if it satisfies EX2) in addition.

(EX1) The kernel pair of every map exist and have a coequalizer; moreover every diagram of the form

$$\begin{tikzcd} \cdot \arrow[dr, two heads] & \\ & \cdot \\ \cdot \arrow[ur] & \end{tikzcd}$$

has a coequalizer which is of the form

$$\begin{tikzcd} & \cdot \arrow[dr, two heads] & \\ \cdot \arrow[ur] \arrow[dr, two heads] & & \cdot \\ & \cdot \arrow[ur] & \end{tikzcd}$$

EX2) Every equivalence relation is effective.

(1.4) The following are examples of regular categories. All are exact except example 5.

1. The category $\underline{S}$ of sets.
2. The category of non-empty sets.
3. For any triple $\mathbb{T}$ on $\underline{S}$, the category $\underline{S}^{\mathbb{T}}$ of $\mathbb{T}$-algebras.
4. Every partially ordered set considered as a category.
5. The category of Stone spaces (compact hausdorff 0-dimensional spaces).
6. Any abelian category.
7. For any small category $\underline{C}$, the functor category $(\underline{C}^{op},\underline{S})$.
8. For any topology on $\underline{C}$, the category $\mathfrak{S}(\underline{C}^{op},\underline{S})$ of sheaves.

(1.5) Remark. It should be noted that unlike the notion of abelianness, exactness is not self-dual. Outside of abelian categories and the categories of sets and pointed sets, the only category that I know of which is tripleable over $\underline{S}$ and both exact and coexact is compact hausdorff spaces (and its dual, C*-algebras).

(1.6) Definition. Let $\underline{X}$ be a regular category. A sequence

$$X' \underset{d^1}{\overset{d^o}{\rightrightarrows}} X \xrightarrow{d} X''$$

is called

a) left exact if (d^o,d^1) is the kernel pair of d;

b) right exact if d is the coequalizer of d^o and d^1, and, moreover the image of (d^o, d^1) in $X \times X$ is the kernel pair of d (see (2.1) and (2.4) below);

c) exact if it is both left and right exact.

(1.7) Definition. Let $\underline{X}$ and $\underline{Y}$ be exact categories. A functor $U: \underline{X} \to \underline{Y}$ is called

a) quasi-exact it it preserves exact sequences;

b) exact if, in addition, it preserves all finite limits;

c) reflexively (quasi) exact if it is (quasi) exact and reflects isomorphisms.

(1.8) Examples. The following are examples of exact functors.

1. For any triple on $\underline{S}$, the underlying functor $\underline{S}^{\mathbb{T}} \longrightarrow \underline{S}$.
2. For any small category $\underline{C}$ and any object of $\underline{C}$, the functor $(\underline{C}^{op}, \underline{S}) \to \underline{S}$ which evaluates a functor at C. Of course this functor preserves all limits and colimits.
3. For any topology on $\underline{C}$, the associated-sheaf functor $(\underline{C}^{op}, \underline{S}) \to \mathfrak{F}(\underline{C}^{op}, \underline{S})$.
4. Any (additive) exact functor between abelian categories.

Of these examples, only 1 is reflexively exact in general.

2. Preliminary results.

(2.1) Throughout this section, $\underset{=}{X}$ denotes a regular category. We will establish some of its basic properties, in particular the factorization.

(2.2) Proposition. Suppose $X \twoheadrightarrow Y \rightarrow Z$ is given. Then $X \times_Z X \rightarrow Y \times_Z Y$ is an epimorphism.

Proof. The diagrams

$$\begin{tikzcd} X \times_Z X \arrow[r, two heads] \arrow[d, "p_1"'] & Y \times_Z X \arrow[d, "p_1"'] \\ X \arrow[r, two heads] & Y \end{tikzcd} \qquad \begin{tikzcd} Y \times_Z X \arrow[r, two heads] \arrow[d, "p_2"'] & Y \times_Z Y \arrow[d, "p_2"'] \\ X \arrow[r, two heads] & Y \end{tikzcd}$$

are each easily seen to be pullbacks, where p_1 and p_2 are the respective coordinate projections. A composite of two $\twoheadrightarrow$ is certainly an epimorphism and, as we will see in (2.8), is $\twoheadrightarrow$.

(2.3) Theorem. Every map has a factorization of the form $\cdot \twoheadrightarrow \cdot \rightarrowtail \cdot$

Proof. Begin with a map $X \rightarrow Z$, form its kernel pair, and let Y be their coequalizer. There is induced a map $Y \rightarrow Z$ and we can form its kernel pair to get

$$\begin{tikzcd} X \times_Z X \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & X \arrow[d, two heads] \arrow[dr] & \\ & & Z \\ Y \times_Z Y \arrow[r, shift left] \arrow[r, shift right] & Y \arrow[ur] & \end{tikzcd}$$

From the fact that $X \rightarrow Y$ coequalizes $X \times_Z X \rightrightarrows X$ and that $X \times_Z X \longrightarrow Y \times_Z Y$ is an epimorphism, it follows that the two projections $Y \times_Z Y \rightrightarrows Y$ are equal and that $Y \rightarrowtail Z$. Thus the map is factored

$$X \twoheadrightarrow Y \rightarrowtail Z.$$

(2.4) Remark. With minor modifications, this is essentially a theorem of Kelly's ([Ke], proposition 4.2). It is clear that to prove it one need only suppose that a pullback of a regular epimorphism is an epimorphism.

(2.5) Proposition. If the composite f.g is $\twoheadrightarrow$, so is f.

Proof. If f.g is the coequalizer of d^o and d^1, than f is the coequalizer of $g.d^o$ and $g.d^1$.

(2.6) Proposition. Every commutative diagram

$$\begin{tikzcd} \cdot \arrow[r, two heads] \arrow[d] & \cdot \arrow[d] \\ \cdot \arrow[r, tail] & \cdot \end{tikzcd}$$

has a diagonal map as indicated so that both triangles commute

$$\begin{tikzcd} \cdot \arrow[r, two heads] \arrow[d] & \cdot \arrow[d] \arrow[dl] \\ \cdot \arrow[r, tail] & \cdot \end{tikzcd}$$

Proof. Consider the diagram

$$\begin{tikzcd} \cdot \arrow[r, shift left] \arrow[r, shift right] & \cdot \arrow[r] \arrow[d, dash] & \cdot \arrow[d, dash] \\ & \cdot \arrow[r, tail] & \cdot \end{tikzcd}$$

in which the top row is a coequalizer.

(2.7) Corollary. Any map which is both $\rightarrowtail$ and $\twoheadrightarrow$ is $\xrightarrow{\sim}$.

Proof. Consider

$$\begin{tikzcd} {} \arrow[r, tail] \arrow[d] & {} \arrow[d] \\ {} \arrow[r, two heads] & {} \end{tikzcd}$$

where the top and bottom are the given map and the vertical maps are identities.

(2.8) <u>Corollary</u>. If $\cdot \xrightarrow{f}\!\!\!\!\rightarrow \cdot \xrightarrow{g}\!\!\!\!\rightarrow \cdot$, then $\xrightarrow{gf}\!\!\!\!\rightarrow$.

Proof. Factor gf as $\cdot \xrightarrow{h}\!\!\!\!\rightarrow \cdot \rightarrowtail^{k} \cdot$ and consider

$$\begin{tikzcd} \cdot \arrow[r, two heads, "f"] \arrow[d, two heads, "h"'] & \cdot \arrow[d, two heads, "g"] \\ \cdot \arrow[r, tail, "k"'] & \cdot \end{tikzcd}$$

The existence of a diagonal presents k as the second factor of a $\twoheadrightarrow$, whence k is $\twoheadrightarrow$ also, by (2.5), and hence an $\xrightarrow{\sim}$.

(2.9) <u>Corollary</u>. The factorization of (2.3) is unique up to a unique $\xrightarrow{\sim}$.

Proof. Two applications of (2.6).

(2.10) <u>Proposition</u>. An exact functor preserves factorizations.

Proof. A right exact functor evidently preserves $\twoheadrightarrow$ and a left exact functor, by preserving the pullback of $\begin{tikzcd} & \cdot \arrow[d, tail, "f"] \\ \cdot \arrow[r, tail, "f"'] & \cdot \end{tikzcd}$ (which has a limit = dom(f) if and only if f is $\rightarrowtail$), preserves $\rightarrowtail$. Thus it takes the $\cdot \twoheadrightarrow \cdot \rightarrowtail \cdot$ factorization into one which by uniqueness is the required factorization.

(2.11) <u>Proposition</u>. Let $\underline{X}$ and $\underline{Y}$ be exact, $X'' \rightrightarrows X \longrightarrow X'$ a left (resp. right) exact sequence, and U an exact functor. Then $UX'' \rightrightarrows UX \longrightarrow UX'$ is left (resp. right) exact.

Proof. The left half of this is pretty clear. As for the right, let $X_0 \rightarrowtail X \times X$ be the image of $X'' \longrightarrow X \times X$. Then we have

$$X'' \twoheadrightarrow X_0; \quad X_0 \rightrightarrows X \twoheadrightarrow X'$$

in which the second is exact. Applying U we have

$$UX'' \twoheadrightarrow UX_0; \quad UX_0 \rightrightarrows UX \longrightarrow UX'$$

in which the second is exact. But this readily implies that

$$UX'' \rightrightarrows UX \longrightarrow UX'$$

is right exact.

(2.12) Remark. It was to make true this proposition (whose proof is the same as of II, proposition 4.3 of [CE]) that the somewhat unusual definition of right exact sequence was chosen.

(2.13) Proposition. In order that $X' \rightrightarrows X \longrightarrow X''$ be exact, it is necessary and sufficient that $X \xrightarrow{f} X''$ (with $\twoheadrightarrow$) and $X' \rightrightarrows X$ be its kernel pair.

Proof. It is clearly necessary. But if f is $\twoheadrightarrow$, then it is evidently the coequalizer of its kernel pair.

(2.14) Corollary. A functor is quasi-exact if it preserves kernel pairs and $\twoheadrightarrow$; it is exact if it preserves all finite limits and $\twoheadrightarrow$.

(2.15) Proposition. If the product of a finite number of exact sequences exists, it is exact.

Proof. Since a product of kernel pairs is a kernel pair, it is sufficient to show that a product of $\twoheadrightarrow$ is again $\twoheadrightarrow$. Suppose $X \twoheadrightarrow X'$ and $Y \twoheadrightarrow Y'$. As soon as $X' \times Y'$ exists, so do $X \times Y'$ and $X \times Y$, since each of the squares below is a pullback. The vertical arrows are the evident coordinate projections,

$$\begin{tikzcd} X \times Y \arrow[r, two heads] \arrow[d] & X \times Y' \arrow[d] \\ Y \arrow[r, two heads] & Y' \end{tikzcd} \qquad \begin{tikzcd} X \times Y' \arrow[r, two heads] \arrow[d] & X' \times Y' \arrow[d] \\ X \arrow[r, two heads] & X'. \end{tikzcd}$$

Composing, we have $X \times Y \twoheadrightarrow X' \times Y'$.

(2.16) <u>Corollary</u>. For any object X of the exact category $\underline{X}$, $X\times -: \underline{X} \longrightarrow \underline{X}$ is a quasi-exact functor (provided it exists).

Proof: $X \rightrightarrows X \longrightarrow X$ (all maps being identity) is exact.

(2.17) <u>Corollary</u>. Let $\underline{X}$ have finite powers. For any finite integer n, the cartesian n-th power functor $\underline{X} \longrightarrow \underline{X}$ is exact.

Proof. Clear from (2.15) and the fact limits commute with each other.

(2.18) <u>Remark</u>. If the cartesian n-~~th~~ power functor exists and preserves $\twoheadrightarrow$ for all cardinals n or for all $n < \aleph_0$, then that functor is exact for all such n.

3. Additive exact categories.

(3.1) This section is devoted to proving Tierney's theorem that a non-empty additive exact category is abelian. Throughout this section $\underline{A}$ denotes such a category; $\underline{Ab}$ denotes the category of abelian groups.

(3.2) Let $A \in \underline{A}$, and consider any 0 map, say $0: A \longrightarrow A$. Since 0 coequalizes any two maps, the kernel pair of this is $A \times A$, which then exists. Let Z be the coequalizer of the projections

$$A \times A \rightrightarrows A \longrightarrow Z.$$

For any $B \in \underline{B}$,

$$(Z,B) \longrightarrow (A,B) \rightrightarrows (A \times A,B) \xleftarrow{\simeq} (A,B) \times (A,B)$$

is an equalizer, which implies, since all these homs take values in $\underline{Ab}$, $(Z,B) = 0$. In an additive category, any initial object is a zero object, and so $Z = 0$. Moreover, A was an arbitrary object and we showed that $A \twoheadrightarrow 0$. Thus we have proved

(3.3) Proposition. There is a zero object 0 and $A \twoheadrightarrow 0$ for any A.

(3.4) Corollary. Finite products exist in $\underline{A}$.

Proof. For any $A,B \in \underline{A}$,

$$\begin{array}{ccc} A \times B & \longrightarrow & A \\ \downarrow & & \downarrow\!\!\downarrow \\ B & \twoheadrightarrow & 0 \end{array}$$

is a pullback.

(3.5) Proposition. Maps in $\underline{A}$ have kernels.

Proof. Let $f: A \longrightarrow A'$. From the kernel pair $A'' \underset{d^1}{\overset{d^0}{\rightrightarrows}} A$ and let $s: A \longrightarrow A''$ be the diagonal map. I claim that $A'' \xrightarrow{d^0-d^1} A$ is a weak kernel. First, $f.(d^0-d^1) = fd^0-fd^1 = 0$. Second, if $g: B \longrightarrow A$ is such that $f.g = 0$, let $k\colon B \longrightarrow A''$ be such that $d^0.k = g$ and $d^1.k = 0$. Then $(d^0 - d^1).k = g$. It is clear that the image of $d^0 - d^1$ must be the kernel.

(3.6) Corollary. $\underline{A}$ has finite limits.

Proof. It is well-known that in an additive category kernels and finite products are enough.

(3.7) Proposition. Let A be an object of $\underline{A}$ and $A' \rightarrowtail A \times A$, containing the diagonal of A. Then A' is an equivalence relation on A.

Proof. The property of being an equivalence relation is defined with respect to the representable functors, which can be considered to take values in $\underline{Ab}$. But then $(-,A') \rightarrowtail (-,A) \times (-,A)$ will still contain the diagonal. In $\underline{Ab}$ the assertion is trivial and the above argument shows it is true for any additive category.

(3.8) Proposition. Every monomorphism of $\underline{A}$ is normal (that is, a kernel).

Proof. Let $A' \overset{f}{\rightarrowtail} A$. Form

$$A' \times A \overset{\binom{f}{1}}{\underset{\binom{0}{1}}{\rightrightarrows}} A.$$

It is easily seen that the induced map $\begin{pmatrix} f & 0 \\ 1 & 1 \end{pmatrix} : A' \times A \rightrightarrows A \times A$ is $\rightarrowtail$ and contains the diagonal, and hence is an equivalence relation and therefore a kernel pair. But it is clear that a map coequalizes $\begin{pmatrix} f \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ if and only if it annihilates f so that that coequalizer of those maps is the cokernel of f. Conversely, $\begin{pmatrix} f \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ being the kernel pair of that cokernel is equivalent to f being its kernel.

Notice that in the course of this proof we have shown that every

$\rightarrowtail$ has a cokernel, which implies, by the standard factorization, that every map does. The finite products are also coproducts. An additive category is cocomplete as soon as it has direct sums and co-equalizers. Thus we have:

(3.9) Proposition. $\underline{A}$ is finitely cocomplete.

(3.10) Proposition. Every epimorphism in $\underline{A}$ is normal.

Proof. Let f be an epimorphism and factor it as $\cdot \xrightarrow{g} \!\!\!\!\to \cdot \rightarrowtail^{h} \cdot$.
Since h is normal, it is the kernel of some k. If k $\neq$ 0, we would have kf = 0, which contradicts f being an epimorphism. Thus h is an isomorphism, which means that f is $\twoheadrightarrow$. In an additive category this implies that f is normal.

(3.11) Theorem. (Tierney). $\underline{A}$ is abelian.

Proof. $\underline{A}$ is additive; it is finitely complete and cocomplete; every map has a factorization as an epimorphism followed by a monomorphism; every monomorphism and every epimorphism is normal.

(3.12) Example. The category of torsion free abelian groups is regular, but not exact.

4. Regular epimorphism sheaves.

(4.1) If $\underline{C}$ is a category, a collection of families $\{U_i \longrightarrow U \mid i \in I\}$ (called coverings) is called a Grothendieck topology on $\underline{C}$ (see [Ar], I, Definition (0.1)), if it satisfies the following conditions.

a) Every $\{U \xrightarrow{f} U'\}$ with f an isomorphism is a covering.

b) If $\{U_i \longrightarrow U \mid i \in I\}$ is a covering and for each $i \in I$, $\{U_{ij} \longrightarrow U_i \mid j \in I_i\}$ is a covering, so is $\{U_{ij} \longrightarrow U \mid i \in I,\ j \in I_i\}$.

c) If $\{U_i \longrightarrow U \mid i \in I\}$ is a covering and $V \longrightarrow U$ is a map, each of pullbacks $U_i \times_U V$ exists and

$$\{U_i \times_U V \longrightarrow V \mid i \in I\}$$

is a covering.

It is easily seen from EX1) and (2.8) that these conditions are satisfied if we take for coverings exactly the $U' \twoheadrightarrow U$. This will be called the regular epimorphism topology. The axiom of a regular category might almost have been chosen with this topology in mind.

(4.2) Given a topology on $\underline{C}$ as above, a sheaf of sets on $\underline{C}$ is a functor $F\colon \underline{C}^{op} \longrightarrow \underline{S}$ such that for every covering $\{U_i \longrightarrow U \mid i \in I\}$,

$$FU \longrightarrow \prod_{i \in I} FU_i \rightrightarrows \prod_{i,j \in I} F(U_i \times_U U_j)$$

is an equalizer. The category of sheaves (with natural transformations as morphisms) is denoted $\mathfrak{F}(\underline{C}^{op}, \underline{S})$. It is equipped with a full faithful embedding $\mathfrak{F}(\underline{C}^{op}, \underline{S}) \longrightarrow (\underline{C}^{op}, \underline{S})$ which has an exact left adjoint. Conversely any coreflective subcategory $\underline{E}$ of a set-valued functor category $(\underline{C}^{op}, \underline{S})$ with an exact coreflector (left adjoint for inclusion) will be a category $\mathfrak{F}(\underline{D}^{op}, \underline{S})$ for some $\underline{D}$ and some Grothendieck topology on $\underline{D}$ for which each of the representable functors is a sheaf. (Such a topology is said to be less fine than the canonical topology; the

the canonical topology is the **finest** topology for which all representable functors are sheaves.[1]) Evidently $\underline{D}$ may be taken to be $\underline{C}$ iff each of the representable functors of $(\underline{C}^{op},\underline{S})$ is in $\underline{E}$. Such an $\underline{E}$ is called a topos. [1]

(4.3) Proposition. Let $\underline{X}$ be a small regular category. Let $\mathfrak{F}(\underline{X}^{op},\underline{S})$ denote the category of set valued sheaves for the regular epimorphism topology described above. Then the canonical embedding $\underline{X} \longrightarrow \mathfrak{F}(\underline{X}^{op},\underline{S})$ is full, faithful and exact.

Proof. It is clear that this topology is less fine than the canonical one, so the Yoneda embedding of $\underline{X}$ takes it into sheaves. The embedding preserves all limits, since the Yoneda embedding does, and it is well known that the embedding of sheaves into all functors creates limits. It is full and faithful for the same reason. Finally, a sheaf F, evaluated at an exact sequence

$$X' \times_X X' \rightrightarrows X' \longrightarrow X,$$

must produce an equalizer

$$FX \longrightarrow FX' \rightrightarrows F(X' \times_X X'),$$

according to the definition of sheaf, By the Yoneda lemma, this is

$$((-,X)F) \longrightarrow ((-,X'),F) \rightrightarrows ((-,X' \times_X X'),F)$$

and that sequence being an equalizer is the some as

$$(-,X' \times_X X') \rightrightarrows (-,X') \twoheadrightarrow (-,X)$$

being a coequalizer in this particular subcategory of the functor category.

(4.5) From this proposition we see that regular categories may be characterized as categories having kernel pairs, pullbacks along regular epimorphisms, coequalizers of kernel pairs, and for every small

1 See Appendix for an improved statement and proof of this result.

full subcategory stable under these operation, a full exact embedding into a topos. The converse is clear. A topos is complete and cocomplete and even exact. If our given category is itself small, we can replace it by its finite limit completion in its embedding into a topos and suppose it has finite limits.

5. Constructions on regular and exact categories.

(5.1) In this section $\underline{X}$ represents a regular (resp. exact) category. We are going to describe two types of constructions which when applied to $\underline{X}$ automatically produce another regular (resp. exact) category.

(5.2) Let $\underline{I}$ be an arbitrary category and $D\colon \underline{I} \longrightarrow \underline{X}$ a functor. We will say that the pair $(D,\underline{I})$ or D alone is a diagram in $\underline{X}$. Note that $\underline{I}$ is not required even to be small. The comma category $(\underline{X},D)$ has for objects pairs (X,α), where X is an object of $\underline{X}$ and α is a natural transformation from (the constant functor whose value is) X to D. A morphism of $(\underline{X},D)$ is a morphism f in $\underline{X}$ giving a commutative triangle

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\alpha"'] & & X' \arrow[dl, "\alpha'"] \\ & D & \end{tikzcd}$$

(5.3) Proposition. The forgetful functor $(\underline{X},D) \longrightarrow \underline{X}$, which takes $(X,\alpha) \longmapsto X$, creates whatever colimits exist in X as well as kernel pairs, pullbacks, finite monomorphic families, and the limit of any diagram $E\colon \underline{J} \longrightarrow \underline{X}$ in which $\underline{J}$ has a terminal object (and in which the limit exists, of course).

Proof. Given a diagram $E\colon \underline{J} \longrightarrow (\underline{X},D)$ which has a colimit in $\underline{X}$, the universal mapping property of colimit will endow that object with a map to D. As for limits, supposing $\underline{J}$ has a terminal object j_0, a functor $E\colon \underline{J} \longrightarrow (\underline{X},D)$ is precisely given by a functor $E\colon \underline{J} \longrightarrow \underline{X}$ together with a natural transformation $Ej_0 \longrightarrow D$. This determines the lifting of E to $(\underline{X}, D)$. The limit $X \longrightarrow E$, when it exists, will equally have a unique map $X \longrightarrow Ej_0 \longrightarrow D$ which lifts X into $(\underline{X},D)$. It is now trivial to see that X is the limit there also. If $f_1,\ldots,f_n\colon X \longrightarrow Y$ is a finite (or for that matter infinite) set of maps, it is called a monomorphic family if for all Z and maps $g.h: Z \longrightarrow X$, $f_i.g = f_i.h$ for $i = 1,\dots,n$ implies that $g = h$. If $Y \longrightarrow D$ is given and $f_1,\dots,f_n: X \longrightarrow Y$ are all maps over D, then they are simultaneously coequalized by $Y \longrightarrow D$. If they do not form a monomorphic family in $\underline{X}$, then there are $g \neq h: Z \longrightarrow X$ with $f_i.g = f_i.h$ for $i = 1,\dots,n$. Then all the composites $Z \underset{g}{\overset{h}{\rightrightarrows}} X \xrightarrow{f_i} Y \longrightarrow D$ are the same. Thus $g \neq h$ as maps over D, and so $\{f_i\}$ is not a monomorphic family in $(\underline{X},D)$ either.

(5.4) Theorem. Let $\underline{X}$ be regular (resp. exact) and $D: \underline{I} \longrightarrow \underline{X}$ a functor. Then $(\underline{X},D)$ is regular (resp. exact).

Proof. Everything except exactness follows from (5.3) and the easily proved (from (5.3)) assertion that $(\underline{X},D) \longrightarrow \underline{X}$ preserves $\twoheadrightarrow$. Exactness (when $\underline{X}$ is exact) also follows from (5.3) if we can show that the underlying functor preserves equivalence relations. To do this we show the following combinatorial characterization of equivalence relations.

(5.5) Proposition. Let $\underline{X}$ be a category which has pullbacks of split epimorphisms. Then $X \underset{d^1}{\overset{d^o}{\rightrightarrows}} Y$ is an equivalence relation if and only if the following conditions are satisfied.

a) $X \underset{d^1}{\overset{d^o}{\rightrightarrows}} Y$ is a monomorphic family.

b) There is an $r: Y \longrightarrow X$ such that $d^o.r = d^1.r = Y(= \mathrm{id}\ Y)$.

c) There is an $s: X \longrightarrow X$ such that $d^o.s = d^1$ and $d^1.s = d^o$.

d) In the diagram below in which Z is a pullback of d^o and d^1, there is a map t as indicated making each of the outside squares commutative.

$$\begin{tikzcd}
 & & & Z \arrow[dlll, "t"'] \arrow[dl, "e^1"] \arrow[dr, "e^0"'] \arrow[drrr, "t"] & & & \\
X \arrow[dr, "d^1"'] & & X \arrow[dl, "d^1"] \arrow[dr, "d^0"'] & & X \arrow[dl, "d^1"] \arrow[dr, "d^0"'] & & X \arrow[dl, "d^0"] \\
 & Y & & Y & & Y &
\end{tikzcd}$$

Proof. I leave it as an exercise to show that in $\underline{S}$, the existence of r,s,t translates the usual reflexive, symmetric, and transitive laws and hence the existence of $(-,r)$, $(-,s)$, $(-,t)$ will show that $(-,X)$ is an equivalence relation on $(-,Y)$. To go the other way, suppose $X \underset{d^1}{\overset{d^0}{\rightrightarrows}} Y$ is an equivalence relation. Then $(Y,X) \longrightarrow (Y,Y)\times(Y,Y)$ must contain the diagonal of (Y,Y), so in particular the diagonal element $(\mathrm{id}Y, \mathrm{id}Y)$ and the $r \in (Y,X)$ mapping to it is the required map. $(X,X) \longrightarrow (X,Y)\times(X,Y)$ is symmetric, and since (d^0,d^1) is in the image of (X,X) (it is the image of the identity map), so must (d^1,d^0) be. The element of (X,X) having those projections is s. Finally letting Z be the pullback as above, we observe that $(Z,X) \longrightarrow (Z,Y)\times(Z,Y)$ is transitive. In particular the images of e^0 and e^1 are $(d^0.e^0,\ d^1.e^0)$ and $(d^0.e^1,\ d^1.e^1)$ respectively, and the equation $d^1.e^0 = d^0.e^1$ implies the existence of t with projections $d^0.e^0$ and $d^1.e^1$, exactly as required.

(5.6) Corollary. Suppose $\underline{X}$ has, and a functor $U: \underline{X} \longrightarrow \underline{Y}$ preserves pullbacks along split epimorphisms; in addition suppose U preserves monomorphic pairs of maps. Then U preserves equivalence relations.

Proof. Trivial.

(5.7) Let $\underline{Th}$ be any finitary algebraic theory. This means $\underline{Th}$ is a category with a functor $n \longmapsto (n)$ from the category of finite sets which preserves coproduct $((n)+(m) = (n+m))$ and is an isomorphism on objects. The category $\underline{S}^{\underline{Th}}$ is the category of product preserving functors $\underline{Th}^{op} \longrightarrow \underline{S}$. Included are all the familiar categories of algebra — in particular groups and abelian groups. If $\underline{X}$ is an arbitrary category, $\underline{X}^{\underline{Th}}$ can be defined as the category whose objects consists of objects $X \in \underline{X}$ together with a lifting of the hom functor $(-,X)$: $\underline{X}^{op} \longrightarrow \underline{S}$ into $\underline{S}^{\underline{Th}}$. A morphism between two such objects is a natural transformation between these functors. Since $\underline{S}^{\underline{Th}} \longrightarrow \underline{S}$ is faithful, this is equivalent, by the Yoneda lemma, to a map between the objects which induces $\underline{S}^{\underline{Th}}$ morphisms on the hom sets. When $\underline{X}$ itself has finite products, it is well known that an algebra is also equivalent to a product preserving functor $\underline{Th}^{op} \longrightarrow \underline{X}$. Moreover this condition is "local" in the sense that in order to recover the equivalence it is only necessary to know the algebra structure for a few objects, namely the powers of $\underline{X}$. For example, a group structure on X is either given by a lifting of $(-,X)$ through the category of groups or by giving morphisms $1 \longrightarrow X$, $X \longrightarrow X$, $X\times X \longrightarrow X$ satisfying laws of a group unit, inverse, and multiplication, respectively (1 denotes the terminal object or 0^{th} power). These morphisms are found by observing that $(1,X)$, (X,X) and $(X\times X,X)$ have group structures. The unit of the first, the inverse (under the group law!) of the identity of X in the second, and the product of the two projections in the third of these groups are the required mappings. However, as the next proposition and its corollary show, when the theory has nullary operations (e.g. groups), then we may as well suppose it has products and the two descriptions coincide. A nullary operation is a map in $\underline{Th}$ of $1 \longrightarrow 0$ and entails for any an algebra X an "element" of $(-,X)$. This means a natural transformation of the constant functor 1 to $(-,X)$. Equivalently it assigns to each Y an $\alpha Y\colon Y \longrightarrow X$ such that for $f\colon Y \longrightarrow Y'$, $\alpha Y'.f = \alpha Y$.

(5.8) <u>Proposition</u>. Let an object $X \in \underline{X}$ admit a constant operation. Then x has a terminal object.

Proof. Choose Y arbitrarily and factor αY as $Y \xrightarrow{\beta Y} T \rightarrowtail X$. If we also factor αX as $X \twoheadrightarrow T_o \rightarrowtail X$, then the diagonal fill-in of the diagram

$$\begin{tikzcd}
Y \arrow[r, two heads] \arrow[d, "\alpha Y"'] & T \arrow[dd, tail] \\
X \arrow[d, "\beta X"'] & \\
T_o \arrow[r, tail] & X
\end{tikzcd}$$

which commutes by naturality of α, gives that $T \rightarrowtail T_o$ and that every object has at least one map to T_o which factors αY. Naturality gives $\alpha T_o.\beta X = \alpha X$. Since we gave αX its unique factorization as βX followed by inclusion of T_o, it follows that αT_o is that inclusion. Finally, for any $f\colon Y \longrightarrow T_o$, $\alpha T_o.f = \alpha Y$, and we may cancel αT_o to conclude that f is $Y \xrightarrow{\beta Y} T \rightarrowtail T_o$, which means that Y has only one map to T_o.

(5.9) <u>Corollary</u>. Every object of $\underline{X}$ has finite powers.

Proof. Once there is a terminal object 1, the kernel pair of $X \longrightarrow 1$ is $X \times X$. Higher products may be constructed by pulling back along coordinate projections

$$\begin{tikzcd}
X^{n+1} \arrow[r] \arrow[d] & X^2 \arrow[d, two heads] \\
X^n \arrow[r, two heads] & X
\end{tikzcd}$$

which are $\longrightarrow\!\!\!\!\rightarrow$ (split by the diagonal map).

(5.10) <u>Proposition</u>. Let $\mathcal{F}(\underline{X}^{op},\underline{S})$ be the category of set valued sheaves in the regular epimorphism topology (4.1). Let <u>Th</u> be a finitary theory. Then the functor $X \longmapsto (-,X)$ preserves <u>Th</u> objects and <u>Th</u> morphisms.

Proof. The inclusion of sheaves into the whole functor category preserves limits, so the products given in the proof are the products as sheaves. If X is a <u>Th</u> object in $\underline{X}$, this means there is, for each $(n) \longrightarrow (m)$ in <u>Th</u>, a map $(Y,X)^m \longrightarrow (Y,X)^n$ which is natural in Y. Corresponding to each commutative diagram

$$\begin{tikzcd} & (m) \arrow[dl] \arrow[dr] & \\ (n) \arrow[rr] & & (p) \end{tikzcd}$$

the diagram

$$\begin{tikzcd} (Y,X)^p \arrow[rr] \arrow[dr] & & (Y,X)^n \arrow[dl] \\ & (Y,X)^m & \end{tikzcd}$$

must also commute. Everything being natural in Y, this means that there is a natural transformation

$$(-,X)^m \longrightarrow (-,X)^n$$

for each $(n) \longrightarrow (m)$ in <u>Th</u> such that diagrams corresponding to the above commute. That is, we have a product preserving functor, $m \longmapsto (-,X)^m$ of $\underline{Th}^{op} \longrightarrow \mathfrak{F}(\underline{X}^{op},\underline{S})$. If X and X' are <u>Th</u> objects, a map $f\colon X \longrightarrow X'$ is a <u>Th</u> morphism if for each Y, the induced map $(Y,X) \longrightarrow (Y,X')$ is a <u>Th</u> morphism, which means that for each $(n) \rightarrow (m)$ in <u>Th</u>,

$$\begin{array}{ccc} (Y,X)^m & \xrightarrow{(Y,f)^m} & (Y,X')^m \\ \downarrow & & \downarrow \\ (Y,X)^n & \xrightarrow{(Y,f)^n} & (Y,X')^n \end{array}$$

commutes. Evidently (using the fact that $\underline{X} \longrightarrow \mathfrak{F}(\underline{X}^{op},\underline{S})$ is full and faithful) this is the same as a natural transformation $(-,X) \xrightarrow{\varphi} (-,X')$ such that there is a commutative diagram

$$\begin{array}{ccc} (-,X)^m & \xrightarrow{\varphi^m} & (-,X')^m \\ \downarrow & & \downarrow \\ (-,X)^n & \xrightarrow{\varphi^n} & (-,X')^n \end{array}$$

corresponding to each $(n) \longrightarrow (m)$ in $\underline{Th}$.

(5.11) Theorem. Let $\underline{X}$ be regular (resp. exact) and $\underline{Th}$ be a finitary theory. Then $\underline{X}^{\underline{Th}}$ is also regular (resp. exact). The underlying $\underline{X}^{\underline{Th}} \longrightarrow \underline{X}$ is a reflexively exact functor.

Proof. It is clear that $\underline{X}^{\underline{Th}} \longrightarrow \underline{X}$ creates all inverse limits which exist in $\underline{X}$ and in particular reflects isomorphisms. The above discussion shows that it is sufficient to consider the case that $\underline{X}$ has finite products. Now suppose that

$$X' \rightrightarrows X \longrightarrow X''$$

is exact in $\underline{X}$ and that X' and X have been equipped with $\underline{Th}$ structures in such a way that $X' \rightrightarrows X$ are morphisms of $\underline{Th}$-algebras (i.e. natural transformations). In that case we have an exact sequence, in particular a coequalizer

$$X'^n \rightrightarrows X^n \longrightarrow X''^n,$$

and corresponding to any map $(1) \longrightarrow (n)$ in $\underline{Th}$ there is a commutative diagram

$$\begin{tikzcd}
X'^n \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & X^n \arrow[r] \arrow[d] & X''^n \arrow[d] \\
X' \arrow[r, shift left] \arrow[r, shift right] & X \arrow[r] & X''
\end{tikzcd}$$

the right hand arrow being induced by the coequalizer. This induces all the operations on X" in such a way that $X \longrightarrow X''$ is a map of algebras as soon as we know that X" is an algebra, i.e. satisfies the equations. To show that, take a commutative triangle

$$\begin{tikzcd}
& (m) \arrow[dl] \arrow[dr] & \\
(n) \arrow[rr] & & (p)
\end{tikzcd}$$

in $\underline{Th}$ and consider

$$\begin{tikzcd}
X^p \arrow[rr] \arrow[dr] \arrow[dd, two heads] & & X^n \arrow[dl] \arrow[dd, two heads] \\
& X^m \arrow[dd, two heads] & \\
X''^p \arrow[rr] \arrow[dr] & & X''^n \arrow[dl] \\
& X''^m &
\end{tikzcd}$$

in which each vertical square and the top triangle commute. Since $X^p \twoheadrightarrow X''^p$, this can be canceled to show that the bottom triangle $\underline{X}^{\underline{Th}} \longrightarrow X$ creates $\twoheadrightarrow$ and hence is exact. In particular, starting with

$$\begin{tikzcd}
\cdot \arrow[dr, two heads] & \\
& \cdot \\
\cdot \arrow[ur] &
\end{tikzcd}$$

in $\underline{X}^{\underline{Th}}$, we can pull it back in $\underline{X}$, and the pullback will automatically be an $\underline{X}^{\underline{Th}}$ algebra and the maps $\underline{X}^{\underline{Th}}$ morphisms. The appropriate arrow will be $\twoheadrightarrow$ in $\underline{X}$, and by the above in $\underline{X}^{\underline{Th}}$ as well. Now suppose that $\underline{X}$ is exact. Given $X' \rightrightarrows X$ in $\underline{X}^{\underline{Th}}$, which is an equivalence relation in $\underline{X}^{\underline{Th}}$, then it follows from (5.6) that it is an equivalence relation in $\underline{X}$ as well. But then it is part of an exact sequence in $\underline{X}$ and the third term can be given a unique $\underline{Th}$ structure so that it is exact in $\underline{X}^{\underline{Th}}$ as well.

(5.12) Theorem. Let $U\colon \underline{X} \longrightarrow \underline{Y}$ be an exact functor and $\underline{Th}$ a finitary theory. Then there is a natural lifting $U^{\underline{Th}}\colon \underline{X}^{\underline{Th}} \longrightarrow \underline{Y}^{\underline{Th}}$ such that

$$\begin{tikzcd} \underline{X}^{\underline{Th}} \arrow[r, "U^{\underline{Th}}"] \arrow[d] & \underline{Y}^{\underline{Th}} \arrow[d] \\ \underline{X} \arrow[r, "U"] & \underline{Y} \end{tikzcd}$$

is commutative. Moreover $U^{\underline{Th}}$ is exact.

Proof. Except for the last line, this is an easy consequence for any U which preserves finite products. The last assertion is also easy, since the other functors in the diagram are exact and $\underline{Y}^{\underline{Th}} \longrightarrow \underline{Y}$ is reflexively exact.

(5.13) Remark. When $\underline{X} = \underline{S}$, (5.11) is true for all theories $\underline{Th}$ (not just finitary ones). This can be easily proved (by the same argument) for any $\underline{X}$ which satisfies the following. The n-th power functor exists and is exact for all cardinal numbers n. For this we need only that n-th powers exist and preserve $\longrightarrow\!\!\!\!\to$. Or these conditions may be valid for all $n < N_o$. In that case, the result bolds for all theories $\underline{Th}$ of rank $< N_o$. Similar remarks apply to (5.12) when $\underline{X}$ and $\underline{Y}$ have, and U preserves all n-th powers, or n-th powers for all $n < N_o$, as the case may be.

Chapter II. Locally Presentable Categories.

1. Definitions.

(1.1) What follows here is a brief description of a more general theory due to Gabriel and Ulmer, as yet unpublished (except as an outline [Ul]) Some of the definitions here differ slightly from theirs in that I restrict consideration to colimits of monomorphic families. I rather think that for exact categories this does not really give a more general theory, although the cardinal numbers used to satisfy some of the definitions might become larger. Throughout this chapter, $\underline{X}$ and $\underline{Y}$ will be two regular categories which are cocomplete.

(1.2) Definition. Let $\underline{I}$ be a partially ordered set and n be a cardinal number. We say that $\underline{I}$ is $\leqslant n$ directed if every set of $\leqslant n$ elements of $\underline{I}$ has an upper bound in $\underline{I}$. An n-filter in $\underline{X}$ is a functor $D\colon \underline{I} \to \underline{X}$ with $\underline{I}$ $\leqslant n$ directed and such that for each $i \leqslant j$ in $\underline{I}$, the value of D at $i \longrightarrow j$, denoted $D(j,i)$, is a monomorphism. Sometimes, for emphasis we will call it a mono-filter. An object $X \in \underline{X}$ is said to have rank $\leqslant n$ if for every n-filter $D\colon \underline{I} \longrightarrow \underline{X}$, $(X, \operatorname{colim} Di) \xrightarrow{\simeq} \operatorname{colim}(X, Di)$.

(1.3) Definition. A set Γ of objects of $\underline{X}$ is said to be a set of generators of $\underline{X}$ if for every $f\colon X \rightarrowtail X'$ which is not an isomorphism there is a $G \in \Gamma$ and a map $G \longrightarrow X'$ which does not factor through f. $\underline{X}$ is said to be locally presentable if it has arbitrary coproducts (denoted $\coprod$) and a set of generators each one of which has rank.

(1.4) Proposition. Let $\underline{X}$ be locally presentable. Then for any $X \in \underline{X}$, there is a $\coprod_{j \in J} G_j \twoheadrightarrow X$ where, for each $j \in J$, $G_j \in \Gamma$.

Proof. Form $\coprod_{G \in \Gamma} \coprod_{(G,X)} G$, the coproduct of one copy of G for each map to X from each $G \in \Gamma$. There is a canonical evaluation $e\colon \coprod \coprod G \longrightarrow X$ defined by $e.\langle u\rangle = u$ where $\langle u\rangle : G \longrightarrow \coprod \coprod G$ is the coordinate injection corresponding to $u\colon G \longrightarrow X$. Factor e as

$$\coprod\coprod G \xrightarrow{e_o} \!\!\twoheadrightarrow X_o \rightarrowtail^{f} X.$$

If $u\colon G \longrightarrow X$ is any map, $e.\langle u\rangle = u$ so that $u = f.e_o.\langle u\rangle$ factors through f. Since this is true for all such u, f must be an isomorphism.

(1.5) It is easy to see that the above characterization could have been taken as the definition of this kind of generator. To distinguish it from the more common kind of generator, whose definition is equivalent (in the presence of coproducts) to the same map being an ordinary epimorphism, these could be called a set of regular generators. Here, however, we will simply call them generators.

(1.6) <u>Proposition</u>. Let $f\colon X \longrightarrow X'$. Then

a) If (G,f) is $\twoheadrightarrow$ for all $G \in \Gamma$, f is $\twoheadrightarrow$.

b) (G,f) is $\rightarrowtail$ for all $G \in \Gamma$ if and only if f is $\rightarrowtail$.

c) (G,f) is $\xrightarrow{\simeq}$ for all $G \in \Gamma$ if and only if f is $\xrightarrow{\simeq}$.

Proof. a) This follows easily from

$$\begin{tikzcd}
\coprod\coprod(G,X) \arrow[r, two heads] \arrow[d, two heads] & \coprod\coprod(G,X') \arrow[d, two heads] \\
X \arrow[r] & X'
\end{tikzcd}$$

b) One way is trivial. If (G,f) is $\rightarrowtail$, consider the diagram

$$\begin{tikzcd}
X''' \arrow[r, "d"] & X'' \arrow[r, shift left, "d^o"] \arrow[r, shift right, "d^1"'] & X \arrow[r, "f"] & X'
\end{tikzcd}$$

in which d^o and d^1 are the kernel pair of f and d is their equalizer, Since $(G,-)$ preserves limits and (G,f) is $\rightarrowtail$, it follows that $(G,d^o) = (G,d^1)$, and then (G,d) is an isomorphism. Since d is a monomorphism, it follows from the definition of generator that d is

$\xrightarrow{\sim}$. But then $d^o = d^1$, which in turn implies that f is $\rightarrowtail$.

c) This is now clear.

(1.7) Remark. It is clear from the above argument that, in particular, the more usual definition of generator is also satisfied.

2. Preliminary results.

Throughout this section $\underline{X}$ is a cocomplete regular category and Γ a set of generators.

(2.1) Proposition. $\underline{X}$ is well-powered.

Proof. For any object X a subobject X_o is determined by those maps from a $G \in \Gamma$ which factor trough X_o. In other words, there are no more subobjects of X than there are subsets of $\cup(G,X)$, the union taken over $G \in \Gamma$.*

(2.2) Corollary. Each object of X has only a set of regular quotients.

Proof. A regular quotient of X is determined by its kernel pair, and that is a subobject of $X \times X$.

(2.3) Proposition. Let $D\colon \underline{I} \longrightarrow \underline{X}$ be a small diagram. Then the set (Γ,D) of all objects $(G,\gamma) \in (\underline{X},D)$ for which $G \in \Gamma$ form a generating set in $(\underline{X},D)$.

Proof. It is a set since each G has only a set of maps to a small diagram. If $X \overset{f}{\rightarrowtail} Y \longrightarrow D$ is a monomorphism, not an isomorphism in $(\underline{X},D)$, then $X \overset{f}{\rightarrowtail} Y$ is a monomorphism as noted in I, ((5.3) above) and clearly not an isomorphism, as the inverse would also be a map of $(\underline{X},D)$. Then there is a map $G \longrightarrow Y$ which does not factor through X, and if we use the composite $G \longrightarrow Y \longrightarrow D$ to lift G into $(\underline{X},D)$ it becomes an element of (Γ,D) with the required property.

(2.4) Theorem. Let $\underline{X}$ be a cocomplete, regular category with a set of regular generators and such that each object has only a set of regular quotients. Then $\underline{X}$ is complete.

Proof. For a diagram $D\colon \underline{I} \longrightarrow \underline{X}$, a limit of D is a terminal object of $(\underline{X},D)$. It is easily seen that cocompleteness is inherited by that

*For nested subobjects, this is clear from the definition of generator. For others, consider the intersection and reduce to the previous case.

category as well as the property of each object having a set of regular quotients. By I. (5.4) and (2.3) the other properties of the statement are also inherited. Hence it suffices to show that such an $\underline{X}$ always has a terminal object. Let Γ be the set of generators, $X = \coprod G$, $G \in \Gamma$, and Q be the colimit of all the regular quotients of X. First I claim that Q is itself a regular quotient of X. It is sufficient to show that every commutative square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Q \arrow[d] \\ Y \arrow[r, tail] & Z \end{tikzcd}$$

has a diagonal fill-in. (Just take $Z = Q$ and Y the image of X in Q.) But by commutativity of the diagram, we have, for each regular quotient $X \twoheadrightarrow X'$,

$$\begin{tikzcd} X \arrow[r, two heads] \arrow[dd] & X' \arrow[d] \arrow[ddl] \\ & Q \arrow[d] \\ Y \arrow[r, tail] & Z \end{tikzcd},$$

giving a family $X' \longrightarrow Y$, obviously coherent and extending to $Q \longrightarrow Y$. Thus Q itself can have no regular quotient, for that would be a further regular quotient of X. For any $Y \in \underline{X}$, there will be a map $\coprod G_i \twoheadrightarrow Y$, and evidently there is a $\coprod G_i \longrightarrow X$, since X is the coproduct of all the $G \in \Gamma$. Pushing out, we get

$$\begin{tikzcd} \coprod G_i \arrow[r] \arrow[d, two heads] & X \arrow[r] & Q \arrow[d, two heads] \\ Y \arrow[rr] & & Q' \end{tikzcd}$$

whence $Q \cong Q'$ and $(Y,Q) \neq \emptyset$. If there were distinct maps $Y \rightrightarrows Q$ for some Y, their coequalizer would be a regular quotient of Q.

(2.5) Remark. It should be noted that this method works for any factorization system and is a form of the special adjoint functor theorem. That is, if there is some factorization system and generators such that the appropriate map is an epimorphism for that system, and if the objects have only a set of quotients in that system, then the special adjoint functor theorem (here in dual form) holds.

(2.6) Proposition. Suppose $\underline{I}$ is some index category; $D\colon \underline{I} \longrightarrow \underline{X}$, $E\colon \underline{I} \longrightarrow \underline{X}$ are functors; and $D \longrightarrow E$ is a natural transformation such that $D_i \twoheadrightarrow E_i$ for all i. Then $\operatorname{colim} D \twoheadrightarrow \operatorname{colim} E$.

Proof. Let $X = \operatorname{colim} D_i$, $Y = \operatorname{colim} E_i$. For each i we have a commutative diagram

$$\begin{tikzcd}
D_i \times_{E_i} D_i \arrow[r, shift left, "d_i^0"] \arrow[r, shift right, "d_i^1"'] \arrow[d] & D_i \arrow[r, two heads] \arrow[d] & E_i \arrow[d] \\
X \times_Y X \arrow[r, shift left, "d^0"] \arrow[r, shift right, "d^1"'] & X \arrow[r] & Y .
\end{tikzcd}$$

Given $X \longrightarrow Z$, which coequalizes d^0, d^1, this induces $E_i \longrightarrow Z$, which coequalizes d_i^0 and d_i^1 and induces a unique $E_i \longrightarrow Z$ making the diagram commute. This family of maps is easily seen to be natural in i, and then there is further induced a map $Y \longrightarrow Z$. Then the outer pentagon of

$$\begin{tikzcd}
D_i \arrow[r] \arrow[d] & E_i \arrow[d] \\
X \arrow[r] \arrow[dr] & Y \arrow[d] \\
& Z
\end{tikzcd}$$

commutes for each i. Since $X = \operatorname{colim} D_i$, this implies that the triangle commutes.

3. Rank.

(3.1) Throughout this section, $\underline{X}$ will denote a locally presentable regular category and Γ a set of generators with rank. We will suppose that n_1 is an infinite cardinal number sufficiently large that $n_1 \geq \#(\Gamma)$ ($\#$ is used to denote cardinality) and $n_1 \geq$ the rank of every object of Γ.

(3.2) Let Γ_1 denote the set of coproducts of n_1 or fewer objects of Γ and Γ_2 denote the set of regular quotients of objects of Γ_1. Let $n_2 = \sup_{X \in \Gamma_2} \#\left(\bigcup_{G \in \Gamma} (G,X)\right)$ and $n = 2^{2^{n_2}}$. Let $\underline{X}_n$ denote the full subcategory of $\underline{X}$ consisting of all objects whose rank $\leq n$.

(3.3) Proposition. With n and $\underline{X}_n$ as above, the objects $X \in \underline{X}_n$ are characterized by each of the following properties.

a) There is a map $\coprod_{i \in I} G_i \twoheadrightarrow X$ with each $G_i \in \Gamma$ and such $\#(I) \leq n$.

b) $\#\left(\bigcup_{G \in \Gamma} (G,X)\right) \leq n$.

This remains true for any power cardinal $\geq n$.

Before giving the proof, we require the following.

(3.4) Proposition. Every object of $\underline{X}$ is a colimit of those subobjects of it which satisfy condition a).

Proof. Let $X \in \underline{X}$ and consider the set of all subobjects of X which satisfy condition a). It follows from (2.6) that the objects satisfying condition a) are closed under n-fold coproducts and, by forming images, that these subobjects form an n-filter. Let X' be its colimit. For $G \in \Gamma$, any map $G \longrightarrow X$ lands in a subobject of X satisfying a), namely its image, and hence factors through X'. Thus $(G,X') \twoheadrightarrow (G,X)$. If two different maps $G \rightrightarrows X'$ are given, each of them, since rank $G \leq n_1 < n$, must factor through one of the given subobjects of X and, by directedness, through some one subobject. Thus, since they factor through a subobject of X, they must remain distinct in X. Thus $(G,X') \rightarrowtail (G,X)$ also, and by (1.6) $X' \xrightarrow{\simeq} X$.

(3.5) Proof of (3.3). Write $X = \operatorname{colim} X_j$ where X_j ranges over the subobject of X satisfying condition a). Now since rank $X \leq n$, the identity map $X \longrightarrow X$, being a map to the colimit of an n-filter, must factor through one of the objects in that filter. This evidently implies that X itself is one of them and so satisfies a). Now suppose an object satisfies a). Then for each $J \subset I$ such that $\#(J) \leq n_1$, let X_J be the image $\coprod_{i \in J} G_i \longrightarrow X$. Then evidently $X_J \in \Gamma_2$, and so $\#(\bigcup_{G \in \Gamma} (G,X_J)) \leq n_2$. The number of such subsets of I is limited by $n^{n_1} = (2^{n_2})^{n_1} = 2^{n_2 \times n_1} = 2^{n_2} = n$. It is clear that the set of all X_J is an n_1-filter on X. Just as above, this permits showing that for each $G \in \Gamma$, $(G, \operatorname{colim} X_J) \rightarrowtail (G,X)$, and hence by (1.6) that $\operatorname{colim} X_J \rightarrowtail X$. On the other hand, each of the $G_i \longrightarrow X$ factors through one of the X_J, and hence we have a factorization

$$\coprod G_i \longrightarrow \operatorname{colim} X_J \longrightarrow X$$

whose composition is $\twoheadrightarrow$, which shows that the second factor is also. Thus $X = \operatorname{colim} X_J$. Now $(G, \operatorname{colim} X_J) = \operatorname{colim}(G,X_J)$, and so

$$\#(\bigcup_{G \in \Gamma} (G,X)) = \#(\bigcup_{G \in \Gamma} \operatorname{colim}(G,X_J))$$

$$\leq \sum_{G \in \Gamma} \#(\operatorname{colim}(G,X_J)) \leq \sum_{G \in \Gamma} \sum_{J \in I} \#(G,X_J)$$

$\leq n_1 \cdot n \cdot n_2 = n$. Thus condition a) implies condition b) and the reverse implication is obvious. Now suppose an object X satisfies condition a) and we have an n-filter $\{Y_j \mid j \in I\}$. We see from $(G,Y_j) \rightarrowtail \operatorname{colim} (G,Y_j) \xrightarrow{\simeq} (G, \operatorname{colim} Y_j)$ and (1.6) that $Y_j \rightarrowtail \operatorname{colim} Y_j$. Now supposing $\coprod_{i \in I} G_i \twoheadrightarrow X$ and $\#(I) \leq n$, we use the readily proved fact that in $\underline{S}$, I-indexed products commute with n-filters and thus

$$(\coprod G_i,\ \mathrm{colim}\ Y_j) \simeq \Pi(G_i,\ \mathrm{colim}\ Y_j)$$

$$\simeq \Pi\ \mathrm{colim}(G_i,Y_j) \simeq \mathrm{colim}\ \Pi(G_i,Y_j)$$

$$\simeq \mathrm{colim}(\Pi G_i,Y_j),$$ which shows that ΠG_i has rank $\leq n$.

The fact that X does follows from a diagonal fill-in in the diagram

$$\begin{tikzcd} \coprod G_i \arrow[r, two heads] \arrow[d] & X \arrow[d] \\ Y_j \arrow[r, tail] & \mathrm{colim}\ Y_j. \end{tikzcd}$$

The last remark about power cardinals $\geq n$ is trivial from the proof.

(3.6) Corollary. $\underline{X}_n$ is n-cocomplete, finitely complete, and closed under sub- and regular quotient objects.

Proof. It is clear that the condition a) above is inherited by n-fold coproducts as well as by regular quotients while condition b) is inherited by subobjects and finite products (in fact, by n_2-fold products).

(3.7) Corollary. Every object of $\underline{X}$ is the colimit of those subobjects of it which belong to $\underline{X}_n$.

(3.8) Corollary. $\underline{X}_n$ is a dense subcategory of $\underline{X}$.

Proof. This means that every $X \in \underline{X}$ is the colimit of the functor $(\underline{X}_n,X)\longrightarrow\underline{X}$ which associates to each $X'\longrightarrow X$ the domain X'. By factoring every such map as $\cdot \twoheadrightarrow \cdot \rightarrowtail \cdot$ and using the fact that $\underline{X}_n$ is closed under regular quotients, we see that the monomorphisms in $(\underline{X}_n,X)$ are cofinal. Thus the colimits are the same and the result is a corollary of (3.7).

(3.9) Proposition. Let $X \in \underline{X}$ and $X' \in \underline{X}_n$. Given any $X \twoheadrightarrow X'$, there is an $\underline{X}_n$ subobject $X'' \rightarrowtail X$ such that the composite $X'' \rightarrowtail X \twoheadrightarrow X'$ is $\twoheadrightarrow$.

Proof. Consider a map $\coprod_{i\in I} G_i \twoheadrightarrow X$. Among all the composites $G_i \longrightarrow \coprod G_i \twoheadrightarrow X \longrightarrow X'$ there can be at most n distinct maps. Choose $J \subset I$ so that the set of such composite maps for $i \in J$ is represented exactly once for $i \in J$. Then $\#(J) \leq n$, while evidently $\coprod_{i\in J} G_i \rightarrow X \longrightarrow X'$ must have the same image in X' and hence is $\twoheadrightarrow$. Then let X'' be the image of $\coprod_{i\in J} G_i \longrightarrow X$.

4. Kan extension of functors.

The purpose of this section is to prove:

(4.1) Theorem: Let $\underline{X}$ and $\underline{Y}$ be locally presentable regular categories and n be a cardinal such that $\underline{X}_n$ satisfies (3.3) and such that $\underline{Y}_n$ contains a set of generators of $\underline{Y}$. Suppose $U: \underline{X}_n \longrightarrow \underline{Y}_n$ is a functor and let $\tilde{U}: \underline{X} \longrightarrow \underline{Y}$ be its Kan extension. Then:

a) If U is reflexively exact, so is $\tilde{U}$.

b) If U is faithful (resp. full and faithful), so is $\tilde{U}$.

(4.2) The rest of this section is devoted to proving this theorem. Without further mention, $\underline{X}$, $\underline{Y}$, n, U, and $\tilde{U}$ will be as in the statement.

(4.3) Proposition. Colimits of n-filters in $\underline{Y}$ commute with finite limits.

Proof. Suppose we are given n-filters $\{Y_i'\}$ and $\{Y_j''\}$ indexed by $i \in I$, $j \in J$, and we let $Y' = \operatorname{colim} Y_i'$, $Y'' = \operatorname{colim} Y_j''$, $Y_{ij} = Y_i' \times Y_j''$, and $Y = \operatorname{colim} Y_{ij}$. Then we want to show that the natural map $Y \xrightarrow{\sim} Y' \times Y''$. We use (1.6) Let Λ be a generating set in $\underline{Y}_n$. For $L \in \Lambda$, $(L,Y) \cong (L, \operatorname{colim} Y_{ij}) \cong \operatorname{colim}(L,Y_{ij}) \cong \operatorname{colim}(L,Y_i' \times Y_j'') \cong$ $\cong \operatorname{colim}((L,Y_i')\times(L,Y_j'')) \cong \operatorname{colim}(L,Y_i') \times \operatorname{colim}(L,Y_j'')$, (since directed colimits commute with finite limits in $\underline{S}$) $\cong (L,\operatorname{colim} Y_i')\times(L,\operatorname{colim} Y_j'')$ $\cong (L,Y')\times(L,Y'') \cong (L,Y' \times Y'')$. The proof for equalizers is similar and we omit it. It is not necessary to have, in that case, maps $Y_i' \rightrightarrows Y_j''$ given for all i,j but only for sufficiently many pairs of indices that the resulting subset of $I \times J$ remain n-directed.

(4.4) Proposition. Let $X', X'' \in \underline{X}$. Then the set of maps $X_i' \times X_j'' \rightarrowtail X' \times X''$, indexed by all $\underline{X}_n$ subobjects $X_i' \rightarrowtail X'$ and all $\underline{X}_n$ subobjects $X_j'' \rightarrowtail X''$, is cofinal among all the $\underline{X}_n$-subobjects of $X' \times X''$.

Proof. Given $X_k \rightarrowtail X' \times X''$ with $X_k \in \underline{X}_n$, we let X'_k be the image of $X_k \longrightarrow X' \times X'' \longrightarrow X'$ and similarly X''_k the image in X''. Then, since products of $\rightarrowtail$ are certainly $\rightarrowtail$, and from the universal mapping property of products, we have

$$X_k \rightarrowtail X'_k \times X''_k \rightarrowtail X' \times X''.$$

(4.5) Proposition. Let $X' \longrightarrow X \rightrightarrows X''$ be an equalizer diagram in $\underline{X}$. Then each $\underline{X}_n$ subobject $X'_i \rightarrowtail X'$ appears at least once among the possible equalizer diagrams

$$X'_i \longrightarrow X_j \rightrightarrows X''_k$$

in which X_j and X''_k are $\underline{X}_n$ subobjects of X and X'' respectively.

Proof. Let $X_j = X'_j$ itself and X''_k be the image in X'' of the equal maps

$$X'_i \longrightarrow X' \rightrightarrows X''.$$

(4.6) Remark. The implication of these last two propositions is that for $X = X' \times X''$, the functor which associates to $X'_i \rightarrowtail X'$ and $X'_j \rightarrowtail X''$, $X'_i \times X''_j \rightarrowtail X' \times X''$ is cofinal. Similarly, suppose $X' \longrightarrow X \rightrightarrows X''$ is an equalizer diagram. Then the functor which, to each pair $X_j \rightarrowtail X$, $X''_k \rightarrowtail X''$ for which the restrictions take X_j into X''_k, associates the equalizer of these restrictions is cofinal.

(4.7) Proposition. Given $X \rightrightarrows X''$ as above, let $\{X_j | j \in J\}$ and $\{X''_k | k \in K\}$ be the n-filters of $\underline{X}_n$ subobjects of X and X'' respectively. Let L be the subset of $J \times K$ of those pairs (j,k) for which the restrictions of the given maps each take X_j into X''_k. Then L is an n-directed set.

Proof. Given n or fewer indices of L, we can find j greater than any of the first coordinates and k' greater than any of the second. We have morphisms

$$\begin{tikzcd}
X_j \arrow[dr] & \\
X_j \arrow[r] & X'' \\
X''_{k'} \arrow[ur, tail] &
\end{tikzcd}$$

where X_j and $X''_{k'}$ both belong to $\underline{X}_n$. Let + denote coproduct and X_k be the image of $X_j + X_j + X''_{k'} \longrightarrow X''$. Clearly the domain of that map belongs to $\underline{X}_n$ and $(j,k) \in L$ dominates each of the given indices.

(4.8) Corolloary. If U preserves finite limits, so does $\tilde{U}$.

(4.9) Proposition. If U preserves $\twoheadrightarrow$, so does $\tilde{U}$.

Proof. Let $X \twoheadrightarrow X'$. For any $\underline{X}_n$ subobject $X'_o \rightarrowtail X'$, we pull back to get

$$\begin{tikzcd}
X_1 \arrow[r, two heads] \arrow[d, tail] & X'_o \arrow[d, tail] \\
X \arrow[r, two heads] & X'
\end{tikzcd}$$

and let $X_o \rightarrowtail X_1$, be an $\underline{X}_n$ subobject, whose existence is guaranteed by (3.9), such that $X_o \twoheadrightarrow X'_o$. Then $UX_o \twoheadrightarrow UX'_o$. Now if I and J are the index sets for the $\underline{X}_n$-subobjects of X and X' respectively, what we have is a map $j \longmapsto i(j)$ of $J \longrightarrow I$ such that $X_{i(j)} \twoheadrightarrow X'_i$. Then colim $UX_{i(j)} \longrightarrow$ colim $UX_i \longrightarrow$ colim UX'_j is such that the composite is $\twoheadrightarrow$ by (2.6). This implies that the second is also. This second map is just $\tilde{U}X \twoheadrightarrow \tilde{U}X'$.

(4.10) Proposition. If U reflects monomorphisms, so does $\tilde{U}$.

Proof. Let $f\colon X \longrightarrow X'$ be a map such that $Uf\colon \tilde{U}X \rightarrowtail \tilde{U}X'$. If f is not $\rightarrowtail$, then there are two maps $X'' \underset{d^1}{\overset{d^o}{\rightrightarrows}} X \longrightarrow X'$ which are co-equalized by f and, as observed in (1.7), there is a $G \in \Gamma$ and a map $G \longrightarrow X''$ which does not equalize d^o and d^1. Let X''_o be the image of G in X'' and X_o be the image of $G + G \longrightarrow X$. Then we have

$$\begin{tikzcd}
X''_o \arrow[r, shift left, "e^o"] \arrow[r, shift right, "e^1"'] \arrow[d, tail] & X_o \arrow[d, tail] & \\
X'' \arrow[r, shift left, "d^o"] \arrow[r, shift right, "d^1"'] & X \arrow[r, "f"] & X'
\end{tikzcd}$$

with X''_o and X_o in $\underline{X}_n$ and $e^o \neq e^1$. Now apply $\tilde{U}$ to get

$$\begin{tikzcd}
UX''_o \arrow[r, shift left, "Ue^o"] \arrow[r, shift right, "Ue^1"'] \arrow[d, tail] & UX_o \arrow[d, tail] & \\
\tilde{U}X'' \arrow[r, shift left, "\tilde{U}d^o"] \arrow[r, shift right, "\tilde{U}d^1"'] & \tilde{U}X \arrow[r, "\tilde{U}f"] & \tilde{U}X'.
\end{tikzcd}$$

Now U reflects isomorphisms and is faithful, so that $Ue^o \neq Ue^1$, which implies that $Ud^o \neq Ud^1$; while $Uf.Ud^o = Uf.Ud^1$ contradicts Uf being $\rightarrowtail$.

(4.11) Proposition. If U reflects isomorphisms, so does $\tilde{U}$.

Proof. First I claim that U reflects $\rightarrowtail$. If $f: X \longrightarrow X'$ is such that $Ug: UX \rightarrowtail UX'$, consider

$$X''' \longrightarrow X'' \rightrightarrows X \xrightarrow{f} X'$$

where $X'' \rightrightarrows X$ is the kernel pair of f and $X''' \longrightarrow X''$ is the equalizer of them. Apply U and reason as in the proof of (1.6). Now suppose that $\tilde{U}f: \tilde{U}X \xrightarrow{\sim} \tilde{U}X'$. By (4.10), $f: X \rightarrowtail X'$. If this is not an $\xrightarrow{\sim}$, there is a map $G \longrightarrow X'$ which does not factor through f. If we let X'_o be the image of $G \longrightarrow X'$ and X_o be the pullback in

$$\begin{tikzcd} X_o \arrow[r, tail, "f_o"] \arrow[d, tail] & X'_o \arrow[d, tail] \\ X \arrow[r, tail, "f"] & X', \end{tikzcd}$$

it is clear that $X'_o \in \underline{X}_n$, and X_o, being a subobject of $\underline{X}'$, is also. Now apply $\tilde{U}$ to get the diagram

$$\begin{tikzcd} UX_o \arrow[r, tail, "Uf_o"] \arrow[d, tail] & UX'_o \arrow[d, tail] \\ \tilde{U}X \arrow[r, tail, "\tilde{U}f"] & \tilde{U}X'. \end{tikzcd}$$

If $\tilde{U}f$ is an isomorphism, so is Uf_o, since the diagram remains a pullback; and then f_o: $X_o \xrightarrow{\simeq} X'_o$. But this implies that the given map $G \longrightarrow X'$ really does factor through f, and we have a contradiction.

(4.12) Proposition. Let U be faithful (resp. full and faithful). Then $\tilde{U}$ is also.

Proof. Write $X = \operatorname{colim} X_i$, $X' = \operatorname{colim} X'_j$, each colim taken over the diagram of $\underline{X}_n$ subobjects of X and X' respectively. Of course from the properties of $\underline{X}_n$ it is clear that these diagrams are n-directed. Then $(X,X') \simeq (\operatorname{colim} X_i, \operatorname{colim} X'_j) \simeq \lim(X_i, \operatorname{colim} X'_j) \underset{①}{\simeq} \lim \operatorname{colim} (X_i,X'_j)$
$\underset{②}{\simeq} \lim \operatorname{colim} (UX_i,UX'_j) \underset{③}{\simeq} \lim (UX_i,\operatorname{colim} UX'_j) \simeq$
$\simeq (\operatorname{colim} UX_i, \operatorname{colim} UX'_j) \simeq (\tilde{U}X, \tilde{U}X')$. The arrows labeled ① and ③ are isomorphisms because X_i and UX_i are objects of rank $\leq n$ in $\underline{X}$ and $\underline{Y}$ respectively. If U is faithful (resp. full and faithful), then the arrow labeled ② is for each i and j a monomorphism (resp. isomorphism) and both directed colimit and arbitrary limit preserve monomorphisms, while, of course, everything preserves isomorphisms. Hence $\tilde{U}$ will also be faithful (resp. full and faithful).

5. Toposes.

(5.1) We have already seen how every small regular category has a full exact embedding into a topos. Moreover, every regular category has a full exact embedding into an illegimate topos. In this section we will show that every cocomplete locally presentable exact category has a full exact embedding into a topos, while, conversely, a topos is itself a locally presentable exact category. We begin with the latter.

(5.2) Theorem: Every topos is locally presentable.

Proof. Let $\underline{E}$ be a topos, and write $\underline{E} = \mathfrak{F}(\underline{C}^{op},\underline{S})$ for some small category $\underline{C}$ and some topology on $\underline{C}$ which is less fine than the canonical topology. Let n be an infinite cardinal number sufficiently large that no covering in the topology on $\underline{C}$ has more than n-elements. Then, as is well known, the objects of $\underline{C}$ (i.e. the representable functors) form a set of generators. I claim that each $C \in \underline{C}$ has rank $\leq n$ in $\underline{E}$. Since in the whole functor category, $(-,C)$ commutes with all colimits (by the Yoneda lemma, $((-,C), \operatorname{colim} G_i) = \operatorname{colim} G_iC = \operatorname{colim}((-,C),G_i))$, it is sufficient to show that if $D\colon \underline{I} \longrightarrow \underline{E}$ is a functor with $\underline{I}$ an n-directed index set, then the colim D_i is the same in $\underline{E}$ as in $(\underline{C}^{op},\underline{S})$; or, which is the same thing, to show that an n-directed colimit of sheaves is a sheaf. So suppose $\{C_j \longrightarrow C \mid j \in J\}$ is a covering of C and I is an n-directed set. In $\underline{S}$, n-directed colimits commute with $\leq$ n-fold products and, since n is infinite, with equalizers. If $F = \operatorname{colim} D_i$, we have that

$$FC \longrightarrow \Pi FG_j \rightrightarrows \Pi F(C_{j_1} \times_C C_{j_2})$$

is isomorphic to

$$\operatorname{colim} D_i(C) \longrightarrow \Pi \operatorname{colim} Di(C_j) \rightrightarrows \Pi \operatorname{colim} Di(C_{j_1} \times_C C_{j_2})$$

which is isomorphic to

$$\operatorname{colim} Di(C) \longrightarrow \operatorname{colim} \Pi Di(C_j) \rightrightarrows \operatorname{colim}(\Pi Di(C_{j_1} \times_C C_{j_2}))$$

which, since each Di is a sheaf, is a directed colimit of equalizers and again an equalizer.

(5.3) Corollary. Every cocomplete locally presentable regular category has a full exact embedding into a topos.

Proof. Let $\underline{X}$ be such a category and find a cardinal n such that $\underline{X}_n$ satisfies (3.3). Let $\underline{C} = \underline{X}_n$, and we have an embedding of $\underline{X}_n \longrightarrow \mathfrak{F}(\underline{C}^{op},\underline{S})$ which, since the cardinality of each covering of the topology is 1, embeds $\underline{X}_n$ as objects of finite rank. Then the hypotheses of (4.1) are satisfied.

Chapter III. The Embedding

1. Statements of result.

(1.1) Theorem. Every locally presentable category has a full exact embedding into a functor category.

(1.2) Theorem. Every topos has a full exact embedding into a functor category.

(1.3) Theorem. Every small regular category has a full exact embedding into a functor category.

(1.4) Theorem. Every small, finitely complete regular category has a full exact embedding into objects of finite rank of a functor category.

(1.5) Except for the last clause of (1.4), it is clear from I. (4.4) II. (4.1) and II. (5.2) that these statements are all equivalent. That last clause could also be derived from the previous theorems, but since we have to prove something, we will prove (1.4). In fact, we will prove something even stronger. Recall that an object $\emptyset$ of a category is an empty object if it is initial and if every map to it is an isomorphism. Let us denote the terminal object of $\underline{X}$ by 1. Then,

(1.6) Theorem: Let $\underline{X}$ be a small finitely complete regular category. Then there is a small category $\underline{C}$, whose objects may be identified with the non-empty subobjects of 1, and a full exact embedding $\underline{X} \longrightarrow (\underline{C}^{op},\underline{S})$ which sends each object of X to a regular quotient of a representable functor.

(1.7) Proposition. A regular quotient of a representable functor has finite rank.

Proof. As observed above (in the proof of II. (5.2)), any representable functor has finite rank - its hom commutes with all colimits. If $\{F_i\}$ is a monofilter (cf. II. (1.2)) of functors and $F = \operatorname{colim} F_i$, then for each representable functor $(-,C)$,

$$((-,C),F) = \operatorname{colim}((-,C),F_i).$$

The filter of sets $((-,C),F_i)$ is still a monofilter, which implies that $((-,C),F_i) \rightarrowtail ((-,C),F)$ and by II.(1.6) that $F_i \rightarrowtail F$. Now suppose $E \in (\underline{C}^{op},\underline{S})$ is a regular quotient of $(-,C)$. To see that $\operatorname{colim}(E,E_i) \xrightarrow{\simeq} \operatorname{colim}(E,F)$, first observe that by the above, the natural map is 1-1. To show it is onto, consider a map $E \longrightarrow F$. The composite $(-,C) \longrightarrow E \longrightarrow F$ must factor through some F_i and the result is obtained from the diagram

$$\begin{tikzcd}
(-,C) \arrow[r, two heads] \arrow[d] & E \arrow[d] \\
F_i \arrow[r, tail] & F
\end{tikzcd}$$

by filling in the diagonal.

(1.8) Corollary. Let $\underline{X}$ be a small, finitely complete regular category in which the terminal object has no non-empty subobject. Then there is a monoid C and full exact embedding $\underline{X} \longrightarrow \underline{S}^C$.

(1.9) Corollary (Mitchell). Let $\underline{A}$ be a small, finitely complete regular additive category (or locally presentable or an $\underline{Ab}$-topos). Then $\underline{A}$ has a full exact embedding into a category of modules.

Proof. Take an embedding into $\underline{S}^C$ as above (there aren't any subobjects of 1 in the additive case). Since it preserves finite products, it lifts to a still exact (additive) embedding into $\underline{Ab}^C$, the category of ZC-modules.

(1.10) The remainder of this chapter is devoted to proving (1.6). Throughout this chapter with the exception of section (2.12)-(2.16),

$\underline{X}$ denotes a small, finitely complete regular category.

2. Support.

(2.1) Choose $X \in \underline{X}$ and factor the terminal map $X \longrightarrow 1$ as $X \twoheadrightarrow S \rightarrowtail 1$. The map $X \longrightarrow S$ is constant, which means that it coequalizes every pair of maps to S. This is because $X \longrightarrow S$ and $X \longrightarrow 1$ have the same kernel pair, $X \times X$. This S is called the support of X and we will write $S = \text{supp } X$.

(2.2) When $\underline{X} = (\underline{C}^{op}, \underline{S})$ and $X \in \underline{X}$, supp X is that functor whose value is 1 wherever the value of X is non-empty and whose value is $\emptyset$ where X's is. Thus supp X is the "characteristic functor" of what would normally be called the support of X.

(2.3) An object $S \in \underline{X}$ will be called a partial terminal object if every map to it is constant.

(2.4) Proposition. Let S be an object of $\underline{X}$. Then the following are equivalent.

a. S is a partial terminal object.

b. The projections $p_1, p_2 \colon S \times S \longrightarrow S$ are equal.

c. The projections $p_1, p_2 \colon S \times S \longrightarrow S$ are equal.

d. The diagonal $s \colon S \longrightarrow S \times S$ is an isomorphism.

Proof. Trivial.

(2.5) Proposition. Let $f \colon S \twoheadrightarrow T$ where S is a partial terminal object. Then f is an isomorphism.

Proof. Consider the kernel pair.

(2.6) Proposition. Let $f \colon X \twoheadrightarrow S$ be constant. Then S is a partial terminal object and $S = \text{supp } X$.

Proof. As any constant map factors through supp X, we have $X \twoheadrightarrow \text{supp } X \twoheadrightarrow S$, the second being $\twoheadrightarrow$ by I (2.5). Now apply

(2.5).

(2.7) Let Supp $\underline{X}$ denote the full subcategory of $\underline{X}$ whose objects are the partial terminal objects. There is at most one map between any two objects of Supp $\underline{X}$ and we will often write $S \leqslant S'$ for $S \longrightarrow S'$.

(2.8) <u>Proposition</u>. supp: $\underline{X} \to \text{Supp}\,\underline{X}$ is left adjoint to inclusion.

Proof. We must show that for $S \in \text{Supp}\,\underline{X}$, $(X,S) \neq \emptyset$ if and only if $(\text{supp } X, S) \neq \emptyset$. The "if" part is clear from the map $X \longrightarrow \text{supp } X$. and the other follows from the fact that any constant map from X factors through supp X.

(2.9) <u>Proposition</u>. The functor supp preserves finite products.

Proof. Since $X \twoheadrightarrow \text{supp } X \rightarrowtail 1$ and $Y \twoheadrightarrow \text{supp } Y \rightarrowtail 1$, we have, by (2.14),

$$X \times Y \twoheadrightarrow \text{supp } X \times \text{supp } Y \rightarrowtail 1 \times 1 = 1.$$

Thus supp X × supp Y enjoys the characteristic property of supp(X×Y).

(2.10) <u>Proposition</u>. Let X and Y be objects of $\underline{X}$. Then supp X = = supp Y if and only if there is an object Z and maps $Y \twoheadleftarrow Z \twoheadrightarrow X$.

Proof. Given such maps, we conclude from $Z \twoheadrightarrow X \twoheadrightarrow \text{supp } X$ that supp Z = supp X and similarly supp Z = supp Y. Conversely, given supp X = supp Y = S we have

$$\begin{tikzcd}
X \times Y \arrow[r, two heads] \arrow[d, two heads] & Y \arrow[d, two heads] \\
Y \arrow[r, two heads] & S
\end{tikzcd}$$

(2.11) <u>Proposition</u>. Let $\underline{X}$ be regular, $X \in \underline{X}$. $X \times -: \underline{X} \longrightarrow \underline{X}$ reflects isomorphisms if and only if supp X is a terminal object of $\underline{X}$.

Proof. First observe that $X \times \text{supp } X \longrightarrow X$ by product projection is an isomorphism, since each map to X induces a unique map to supp X. For each $S \in \text{Supp } \underline{X}$, $\text{supp } X \times S = \text{supp}(X \times S)$. Moreover $S \times \text{supp } X \longrightarrow S$ gives $X \times \text{supp } X \times S \longrightarrow X \times S$, which is evidently an isomorphism. Thus if $X \times -$ reflects isomorphisms, we have $S \times \text{supp } X = S$ or $S \leqslant \text{supp } X$ for all $S \in \text{Supp } \underline{X}$. Since every object maps to some $S \in \text{Supp } \underline{X}$, every object has a map, necessarily unique to supp X, which means that it is terminal. On the other hand, suppose supp X is the terminal object, which we will denote 1, and suppose that $Y \xrightarrow{f} Y'$ is any map with $X \times Y \xrightarrow{X \times f} X \times Y'$ an isomorphism. We first show that f must be $\longrightarrow\!\!\!\!\!\rightarrow$.

The diagram

$$\begin{tikzcd} X \times Y' \arrow[r, two heads] \arrow[d] & Y' \arrow[d] \\ X \arrow[r, two heads] & 1 \end{tikzcd}$$

is a pullback, whence $X \times Y' \twoheadrightarrow Y'$, which together with the commutative diagram

$$\begin{tikzcd} X \times Y \arrow[r, "\sim"] \arrow[d] & X \times Y' \arrow[d, two heads] \\ Y \arrow[r] & Y' \end{tikzcd}$$

and I. (2.5) implies that $Y \twoheadrightarrow Y'$.

Now form

$$Y''' \xrightarrow{d} Y'' \underset{d^1}{\overset{d^o}{\rightrightarrows}} Y \xrightarrow{f} Y'$$

in which $Y'' \underset{d^1}{\overset{d^o}{\rightrightarrows}} Y$ is the kernel pair of f and $Y''' \xrightarrow{d} Y''$ is their equalizer. Exactly as in the proof of I (2.16), $X \times -$ preserves kernel pairs and equalizers, and so

$$X \times Y''' \longrightarrow X \times Y'' \rightrightarrows X \times Y \longrightarrow X \times Y'$$

is a sequence of the same type But now $X \times f \xrightarrow{\sim} \Longrightarrow X \times d^o = X \times d^1$ implies that $X \times d$ is $\xrightarrow{\sim}$. By the above, this implies that d is $\twoheadrightarrow$, which implies $d^o = d^1$ and then that f is $\rightarrowtail$. By the uniqueness of the factorization, only an isomorphism can be both.

(2.12) Definition. Let $\underline{X}$ be a regular category with a terminal object 1. An object $X \in \underline{X}$ is said to have full support or to be fully supported if $X \twoheadrightarrow 1$. $\underline{X}$ is called fully supported if every object of $\underline{X}$ is. This is equivalent to the existence of only one partial terminal object, since the existence of a terminal object is enough to show that supports exist.

(2.13) It is clear from the results of this section that the functor Supp is a fibration, that the fibres are fully supported regular categories (and exact if the total category is), and that the transition functors are exact. This last follows from the fact that the transition functor from the fibre over S for $S \leqslant S'$ is given by $S \times -$. This functor preserves all projective limits, since $S^n = S$ for all cardinals n. Conversely, any partially ordered $\underline{P}$ together with a functor $\underline{P}^{op}$ to the category of regular (resp. exact) categories and exact functors can be pasted together to make a regular (resp. exact) category.

(2.14): Proposition. Every map in $\underline{X}$ may be factored f = g.h where supp h is an identity and f is a cartesian map in the fibration.

Proof. This is the essence of a fibration. Given $f: X \longrightarrow Y$, we factor it as $X \longrightarrow \text{supp } X \times Y \longrightarrow Y$. The existence of f implies supp $X \leqslant$ supp Y, so supp(supp $X \times Y$) = supp X. The second factor is exactly a cartesian map.

(2.15) <u>Proposition</u>. Let $\underline{S}$ be a full subcategory of supp $\underline{X}$. Then the full subcategory of $\underline{X}$ consisting of those objects whose support lies in $\underline{S}$ is regular (and exact when $\underline{X}$ is).

Proof. Trivial.

3. Diagrams

(3.1) Let $\underline{I}$ be an (index) category and $D\colon \underline{I} \longrightarrow \underline{X}$ be a functor. Then we will often say that the functor D, or for emphasis, the pair $(\underline{I},D)$, is a diagram in $\underline{X}$.

(3.2) If $(\underline{I},D)$ is a diagram in $\underline{X}$ and X is an object, let (D,X) denote the set $\operatorname{colim}(Di,X)$, the colimit being taken over $i \in \underline{I}$. Then an element of (D,X) is represented by an object $i \in \underline{I}$ together with a map $f\colon Di \longrightarrow X$. We may denote this (i,f) and its class by $\|i,f\|$. Then $\|i,f\| = \|j,g\|$ if $f\colon Di \longrightarrow X$ and $g\colon Dj \longrightarrow X$ are the same in the colimit. In the special case when $\underline{I}$ is filtered (the only type of diagram we will have - in fact they will all be directed sets), this means that there is a $k \in \underline{I}$ and $\alpha\colon k \longrightarrow i$, $\beta\colon k \longrightarrow j$ in $\underline{I}$ such that

$$\begin{tikzcd} Dk \arrow[r, "D\alpha"] \arrow[d, "D\beta"'] & Di \arrow[d, "f"] \\ Dj \arrow[r, "g"'] & X \end{tikzcd}$$

commutes. When $\underline{I}$ is not filtered, take the equivalence relation generated by that relation.

(3.3) More generally, if $(\underline{I},D)$ and $(\underline{J},E)$ are diagrams, we define (D,E) as $\lim(D,Ej)$, the limit taken over $j \in \underline{J}$. In effect, an element of (D,E) is represented by choosing for each $j \in \underline{J}$ a $\sigma j \in \underline{I}$ and a map $fj\colon Dj \longrightarrow Ej$ such that for $\alpha\colon\ j_1 \longrightarrow j_2$ in $\underline{J}$, $\|\sigma j_1, E\alpha . fj_1\| =$ $= \|\sigma j_2, fj_2\|$ in (D,Ej_2). Then two families $(\sigma,\{fj\})$ and $(\tau,\{gj\})$ represent the same element of (D,E) if for each $j \in \underline{J}$, $\|\sigma j, fj\| =$ $= \|\tau j, gj\|$ as maps of $D \longrightarrow Ej$. The composition of two such families is obvious and gives a category. Diag $\underline{X}$, of diagrams in $\underline{X}$.

(3.4) <u>Proposition</u>. If $(\underline{I},D)$ and $(\underline{J},E)$ are two diagrams, then $(D,E) = $
$= \lim_{j \in \underline{J}} \operatorname{colim}_{i \in \underline{I}} (Di,Ej)$

Proof. This is just a shorthand form of the above discussion.

(3.5) If $X \in \underline{X}$, we let X also denote the diagram $(\underline{I},D)$ where I has exactly one object i and one map and $Di = X$. Then this embedding is obviously full and faithful. In fact, it can be easily seen that Diag $\underline{X}$ is just $(\underline{X},\underline{S})^{op}$ and that this embedding is the Yoneda embedding. However, this fact is not needed here, as we will work directly with diagrams. On account of this, we will call such a diagram either representable or the diagram represented by X.

(3.6) From now on, all diagrams will be over partially ordered sets, in fact, over inverse directed sets. In terms of functor categories, this means that we are restricting our attention to the category of finite-limit-preserving functors. If, for $i,j \in \underline{I}$ there is a map $j \longrightarrow i$, i.e. if $j \leqslant i$, we use (i,j) to denote it; and then, of course, $D(i,j)$: $Dj \longrightarrow Di$ is the corresponding map in the diagram.

(3.7) Recall that every $f: X \longrightarrow Y$ can be factored in the form

$$X \xrightarrow{h} X \times \operatorname{supp} X \xrightarrow{g} Y.$$

We will say that f is special if h is $\twoheadrightarrow$.

(3.8) <u>Proposition</u>: Special morphisms are stable under composition and pullbacks.

Proof. Let $X \longrightarrow Y$ and $Y \longrightarrow Z$ be special. Then $X \twoheadrightarrow \operatorname{supp} X \times Y$ and $Y \twoheadrightarrow \operatorname{supp} Y \times Z$ give $\operatorname{supp} X \times Y \twoheadrightarrow \operatorname{supp} X \times \operatorname{supp} Y \times Z = \operatorname{supp} X \times Z$. This, together with I. (2.8), gives the first result. As for the second, if $X \longrightarrow Y$ is special and we form a pullback

$$\begin{tikzcd} X \arrow[r, two heads] \arrow[d] & \operatorname{supp} X \times Y \times_Y Y' \arrow[r] \arrow[d] & Y' \arrow[d] \\ X \arrow[r, two heads] & \operatorname{supp} X \times Y \arrow[r] & Y \end{tikzcd}$$

then $\operatorname{supp} X \times Y \times_Y Y' \cong \operatorname{supp} X \times Y'$.

(3.9) Given a diagram $(\underline{I},D)$, we define a new diagram $(\underline{I}_S,S_S)$ for any $S \in \operatorname{Supp} \underline{X}$ by letting $\underline{I}_S = \{i \mid \operatorname{supp} Di \geq S\}$ and $D_S i = Di \times S$. We see that D_S can be thought of as being a functor $\underline{I}_S \longrightarrow \underline{X}_S$, where the latter denotes the full subcategory of all objects whose support is S.

(3.10) Given a diagram $(\underline{I},D)$ we say it is P-diagram if it satisfies:

P1) $\underline{I}_S$ is an inf semilattice for all $S \in \operatorname{Supp} \underline{X}$.

P2) For any $i \in \underline{I}$ and any special morphism $f\colon X \longrightarrow Di$, there is a $j \leq i$ with $D(i,j) = f$ (and of course $Dj = X$).

The diagram $(\underline{I},D)$ is called an A-diagram if it satisfies:

A1) = P1).

A2) For any $i < j$, the interval $(i,j] = \{k \mid i < k \leq j\}$ is finite.

A3) For any $i < j$, the natural map $Di \longrightarrow \lim(D|(i,j])$ is special.

(3.11) It should be noted that these definitions are not isomorphism invariant and should be supplemented by saying that a diagram isomorphic to one of the above type is of that type also. It would be useful to discover, purely in terms of the functors represented, what these definitions mean.

(3.12) Proposition. Let $(\underline{I},D)$ be a P- diagram (resp. A-diagram) in $\underline{X}$.

Then $(\underline{I}_S, D_S)$ is a P-diagram (resp. A-diagram) in $\underline{X}_S$.

Proof. The condition P1) = A1) is evidently designed to be inherited in this way. If $f: X \longrightarrow D_S i$ is special, supp X = S clearly is equivalent to $X \twoheadrightarrow D_S i$. There must exist $j < i$ with $D(i,j) = f$. We have supp $Dj = S$, so $j \in I_S$ and $D_S j = Dj$. Thus P2) is inherited. If $(\underline{I}, D)$ is an A-diagram, $(\underline{I}_S, D_S)$ satisfies A1 as above and A2 is clear. Then $Di \longrightarrow \lim D|(i,j]$ being special implies that

$$Di \twoheadrightarrow \text{supp } Di \times \lim D|(i,j],$$

and if supp $Di \geqslant S$,

$$\begin{aligned} S \times Di &\twoheadrightarrow S \times \text{supp } Di \times \lim D|(i,j] \\ &= S \times \lim D|(i,j] \\ &= \lim D_S|(i,j], \end{aligned}$$

since supp $Dk \geqslant S$ for all $k > i$ and $S \times -$ is an exact functor.

(3.13) <u>Proposition</u>. Let $(\underline{I}, D)$ be an A-diagram. Then $D(j,i)$ is special for $i < j$. Also $D_S(j,i)$ is $\twoheadrightarrow$ for all $i < j$ such that supp $Di \geqslant S$.

Proof. Since the interval $(i,j]$ is finite, there is a finite chain $i = i_o < i_1 < \ldots . < i_n = j$ such that each $(i_r, i_{r+1}]$ has only one element, namely i_{r+1}, and then A3 implies that $Di_r \longrightarrow Di_{r+1}$ is special. Then $D(j,i)$, being the composite of these, is special also. The last statement is obvious, since a special morphism between two objects of the same support is $\twoheadrightarrow$.

(3.14) <u>Proposition</u>. Let $(\underline{I}, D)$ be a P-diagram. Then for any $S \in \text{Supp } \underline{X}$,

$$(D_S, -) : \underline{X} \longrightarrow \underline{S}$$

is exact.

Proof. Since $\underline{I}_S$ is inverse directed, it evidently preserves finite limits. If $f: X \twoheadrightarrow Y$, then supp X = supp Y. Let $\| i,g \|: D_S \longrightarrow Y$ be a map. Since the pullback of

$$\begin{array}{ccc} & & X \\ & & \big\downarrow \\ D_S i & \xrightarrow{g} & Y \end{array}$$

comes equipped with a $\twoheadrightarrow D_S i$, it is represented in the diagram, so there is a commutative diagram

$$\begin{array}{ccc} D_S j & \xrightarrow{h} & X \\ {\scriptstyle D_S(i,j)}\big\downarrow & & \big\downarrow \\ D_S i & \xrightarrow{g} & Y. \end{array}$$

Then $\| j,h \|: D_S \longrightarrow X$ is a map such that $(D_S,f)\| j,h \| = \| j,g.D_S(i,j) \| =$
$= \| i,g \|$, which implies that (D_S,f) is onto.

(3.15) Proposition. Let $(\underline{I},D)$ be a P-diagram. For each $i \in \underline{I},S$, $\| i, D_S i \|: D_S \longrightarrow D_S i$ is an epimorphism.

Proof. As pointed out in (3.13), every map in the diagram D_S is $\twoheadrightarrow$. If $f,g: Di \longrightarrow X$ are distinct, then for all $j < i$, $D(i,j)f \neq D(i,j).g$. Evidently every diagram is the limit of representable diagrams and an inverse limit of monomorphisms is a monomorphism.

4. <u>The Lubkin completion process</u>.

(4.1) In this section we show how to "complete" a given diagram to a P-diagram. This construction was first described by Lubkin in his original proof of the abelian category imbedding, [Lu]. As a matter of fact, Lubkin observed then that there was nothing inherently abelian in his proof. Lubkin even stated a non-abelian embedding theorem, although based on the notion of ordinary, rather than regular, epimorphisms.

(4.2) Let $(\underline{I},D)$ be a diagram, $i_o \in \underline{I}$ and $f\colon X \longrightarrow Di_o$ be a map in $\underline{X}$. We describe a new diagram $\mathrm{Lub}(\underline{I},D,i_o,f) = (\underline{I}',D')$ as follows. Let $\underline{I}^*$ be a partially ordered set disjoint from and order isomorphic to $\{i \in \underline{I} \mid i \leqslant i_o\}$, by a map $i \longleftrightarrow i^*$. Let $\underline{I}'$ denote $\underline{I} \cup \underline{I}^*$, in which each component has its own order and moreover $i^* < j$ if and only if $i \leqslant j$. In particular, $i^* < i$, and the order is generated by that relation together with the orders in $\underline{I}$ and $\underline{I}^*$. We define D' by $D'|\underline{I} = D$, $D'i_o^* = X$, $D'(i_o,i_o^*) = f$, and for $i \leqslant i_o$, $D'i^*$ is defined so that the diagram

$$\begin{tikzcd}
D'i^* \arrow[r, "D'(i_o^*,i^*)"] \arrow[d, "D'(i,i^*)"'] & X = D'i \arrow[d, "f"] \\
Di \arrow[r, "D(i_o,i)"'] & Di_o
\end{tikzcd}$$

is a pullback. D' is defined on maps $i^* \longrightarrow i_o^*$ and $i^* \longrightarrow i$ as shown. For $i \leqslant j \leqslant i_o$, $D'(j^*,i^*)$ is uniquely induced by a pullback and $D'(j,i^*)$ is defined as $D'(j,j^*)$. $D'(j^*,i^*) = D(j,i).D'(i,i^*)$. This last equality is a consequence of the definition of $D'(j^*,i^*)$ as a map into a pullback.

(4.3) Let $(\underline{I},D)$ and $(\underline{I}',D')$ be diagrams. We say that $(\underline{I}',D')$ is a Lubkin-extension of $(\underline{I},D)$ if there is some $i_o \in \underline{I}$ and $f\colon X \longrightarrow Di$

with $(\underline{I}',D') = \mathrm{Lub}(\underline{I},D,i_o,f)$. In particular, this means that $\underline{I} \subset \underline{I}'$ and $D'|\underline{I} = D$.

(4.4) Let n be an ordinal number. A sequence $\{(\underline{I}_m,D_m)|m \leqslant n\}$ of diagrams is called a Lubkin-sequence if for each m, $(\underline{I}_{m+1},D_{m+1})$ is a Lubkin-extension of $(\underline{I}_m,D_m)$ and if for each limit ordinal m, $\underline{I}_m = \bigcup_{p<m} \underline{I}_p$; $D_m|\underline{I}_p = D_p$.

(4.5) Let $(\underline{I},D)$ be a diagram. If n is an ordinal number and $\{f_m|m < n\}$ is a sequence of morphisms $f_m: X_m \longrightarrow Di_m$, we define a Lubkin-sequence by letting $(\underline{I}_o,D_o) = (\underline{I},D)$, and for each m, $(\underline{I}_{m+1},D_{m+1}) = \mathrm{Lub}(\underline{I}_m,D_m,i_m,f_m)$, while for each limit ordinal m, $\underline{I}_m = \bigcup_{p<m} \underline{I}_p$, $D_m|\underline{I}_p = D_p$.

(4.6) Let $(\underline{I},D)$ be a diagram. Let n_1 be an ordinal such that there is a 1-1 correspondence $m \longmapsto f_m$ between the ordinals $m < n$ and the set of all special morphisms whose codomain is a Di for $i \in \underline{I}$. Then applying the above construction, we get a diagram $(\underline{I}_{n_1},D_{n_1})$. This diagram has the property that given $i \in \underline{I}$ and $f: X \longrightarrow Di$ special, there is some $j \in \underline{I}_{n_1}$ such that $j < i$ and $f: X \longrightarrow Di = D_{n_1}(i,j): D_{n_1}j \longrightarrow Di$. Now let n_2 be an ordinal such that there is a 1-1 correspondence $m \longmapsto f_m$ between all the ordinals $n_1 \leqslant m < n_2$ and the set of all special morphisms whose domain is a $D_{n_1}i, i \in \underline{I}_{n_1}$. Extend the Lubkin-sequence $\{(\underline{I}_m,D_m)|m \leqslant n_1\}$ to one defined for $m \leqslant n_2$ by applying the process of (4.5) beginning with $(\underline{I}_{n_1},D_{n_1})$. Then we may continue in this way with ordinals $n_2,n_3,\ldots$. Let $n = \sup\{n_i|i \in \omega\}$. By letting $\underline{I}_n = \bigcup_{m<n} \underline{I}_m$, $D_n|\underline{I}_m = D_m$, we construct a Lubkin sequence $\{(\underline{I}_m,D_m)|m \leqslant n\}$ with the property that for all special $f: X \longrightarrow Di$, $i \in \underline{I}_n$, there is a $j < i$ in $\underline{I}_n$ such that $f: X \longrightarrow Di = D(i,j): Dj \longrightarrow Di$.

The diagram $(\underline{I}_n, D_n)$ will be called a Lubkin completion of $(\underline{I}, D)$.

(4.7) Proposition. Let $(\underline{I}, D)$ be a diagram in which $\underline{I}_S$ is an inf semilattice for each $S \in \text{Supp } \underline{X}$. Then a Lubkin completion of it is a P-diagram.

Proof. P1) is an inductive property, so it suffices to consider a single Lubkin extension. Let $(\underline{I}, D)$ satisfy P1) and $(\underline{I}', D') = $ $= \text{Lub}(\underline{I}, D, i_o, f)$. Let $i \wedge j$ denote the inf of two elements of $\underline{I}_S$. If $i < i_o$ and $i \in \underline{I}_S$, then $i_o \in \underline{I}_S$ also and supp $Di^* = $ supp $Di \cap$ supp X, where X is the domain of f. If supp X is not $\geqslant S$, then $\underline{I}'_S = \underline{I}_S$. If supp $X \geqslant S$, then supp $Di^* \geqslant S$ if and only if supp $Di \geqslant S$. Now if $i,j \in \underline{I}_S$, $i \wedge j \in \underline{I}'_S$, being the same as in $\underline{I}_S$. If $i,j \in \underline{I}_S$, $i \leqslant i_o$, $i^* \wedge j = (i \wedge j)^*$ and is in $\underline{I}_S$ when i^*_o is. If also $j \leqslant i_o$, $i^* \wedge j^* =$ $(i \wedge j)^*$ as well. As for P2), this is what the Lubkin completion is all about. Supposing that $i \in \underline{I}_n$ and $f: X \longrightarrow Di$ is special, then $i \in \underline{I}_{n_r}$ for some $r \in \omega$ and $f = f_m$ for some ordinal m such that $n_r < m < n_{r+1}$. Then f is represented in the diagram $(\underline{I}_m, D_m)$ and thereafter.

(4.8) Proposition. Suppose $(\underline{I}, D)$ is an A-diagram. Then any Lubkin extension of it is an A-diagram.

Proof. Let $(\underline{I}', D') = \text{Lub}(\underline{I}, D, i_o, f)$. We have just seen that A1) = P1) is preserved by Lubkin extension. As for A2), if $i,j \in \underline{I}$, $(i,j]$ is the same in $\underline{I}$ and $\underline{I}'$. If $i,j \in \underline{I}$, $i < i_o$, $(i^*, j] = (i^*, (j \wedge i)^*] \cup [i,j]$ and the first term is order isomorphic to $(i, j \wedge i]$. If $j < i_o$ also, $(i^*, j^*]$ is order isomorphic to $(i,j]$. To show A3) is satisfied, we consider the cases.

Case 1. $i < j$ in $\underline{I}$. This follows directly from the fact that $D'|\underline{I} = D$.

Case 2. $i^* < j^*$. This case is a simple application of the fact that limits commute with limits to show that

$$\begin{tikzcd} D'i^* \arrow[r] \arrow[d] & \lim D'|(i^*,j^*] \arrow[d] \\ Di \arrow[r] & \lim D|(i,j] \end{tikzcd}$$

is a pullback. Then since the bottom arrow is special, so is the top.

Case 3. $i^* < j$ but $i = i_o \wedge j$. In this case, $(i^*,j] = [i,j]$ and so $\lim D'|(i^*,j] = Di$. Then since f is special, so is $D'i^* \longrightarrow Di$.

Case 4. $i^* < j$ and $i < i_o \wedge j$. I claim that in this case Di^* is the limit under consideration. To see this let $j_o = j \wedge i_o$, and suppose we are given $g(k): Y \longrightarrow Dk$ for each $k \in [i,j]$ and $g(k^*)$: $Y \longrightarrow Dk^*$ for each $k \in (i^*,j_o^*]$, which constitute a coherent family. Then $D'(j_o,j_o^*).g(j_o^*) = D'(j_o,i).g(i)$, so that since

$$\begin{tikzcd} D'i^* \arrow[r] \arrow[d] & D'j_o^* \arrow[d] \\ Di \arrow[r] & Dj_o \end{tikzcd}$$

is a pullback, there is a unique $g: Y \longrightarrow D'i^*$ such that $D'(i,i^*).g =$ $= g(i)$ and $D'(j_o^*,i^*).g = g(j_o^*)$. If $k \in [i,j]$, then $g(k) = D(k,i).g(i)$, so that $D'(k,i^*).g = D(k,i).D'(i,i^*).g = D(k,i).g(i) = g(k)$. If $k^* \in (i,j_o^*]$, then to show that $D'(k^*,i^*).g = g(k^*)$, we use the fact that

$$\begin{tikzcd} D'k^* \arrow[r] \arrow[d] & D'j_o^* \arrow[d] \\ Dk \arrow[r] & Dj_o \end{tikzcd}$$

is a pullback. We have $D'(j_o^*,k^*).D'(k^*,i^*).g = D'(j_o^*,i^*).g = g(j_o^*) =$ $= D'(j_o^*,k^*).g(k^*)$ and $D'(k,k^*).D'(k^*,i^*).g = D'(k,i^*).g =$

$= D(k,i).D'(i,i^*).g = D(k,i).g(i) = g(k) = D'(k,k^*).g(k^*)$.

(4.9) <u>Corollary</u>. A Lubkin completion of an A-diagram is simultaneously an A- and P-diagram.

5. The embedding.

(5.1) We are now ready to describe the embedding. The functor $\underline{X}(1,-)$ is represented by the diagram $D_o\colon \underline{I}_o \longrightarrow \underline{X}$ in which $\underline{I}_o$ has one object and D_o at that object is the terminal object 1. This is evidently an A-diagram and we let $(\underline{I},D)$ be a Lubkin completion of it. We let $\underline{C}$ be the category whose objects are the non-empty subobjects of 1, and whose morphisms are defined by

$$\underline{C}(S_1,S_2) = (D_{S_1},D_{S_2}):$$

that is, morphisms (as defined in (3.3)) between the diagrams $(\underline{I}_{S_1},D_{S_1})$ and $(\underline{I}_{S_2},D_{S_2})$. This is equivalent to natural transformations between the functors represented by the diagrams. Composition in $\underline{C}$ is just the composition of natural transformations. Note that $\underline{C}(S_1,S_2) = \emptyset$ unless $S_1 \leqslant S_2$, which means that there is a functor $\underline{C} \longrightarrow \text{Supp}\ \underline{X}$. We define $U\colon \underline{X} \longrightarrow (\underline{C}^{op},\underline{S})$ by $(UX)S = (D_S,X)$, the mapping described in (3.2). Composition of natural transformations (recall that this is really natural transformations between $(X,-)$ and $(D_S,-)$) makes this functorial in $\underline{X}$ and (contravariantly) in $\underline{C}$. Since limits and colimits in functor categories are computed elementwise, it follows that U is exact as long as $(U-)S$ is for each S. That functor is $(D_S,-)$.

(5.2) Proposition. U is exact.

Proof. See (3.14).

(5.3) Proposition. Let $E\colon \underline{J} \longrightarrow \underline{X}_S$ be a P-diagram and $F\colon \underline{K} \longrightarrow \underline{X}_S$ be an A-diagram. Let $k_o \in K$ and

$$E \xrightarrow{\|j_o,f\|} Fk_o$$

be a map. Then it extends to a map $E \longrightarrow F$. This means that there is a map $E \longrightarrow F$ such that

$$\begin{tikzcd} E \arrow[r] \arrow[d, "\|j_o, Ej_o\|"'] & F \arrow[d, "\|k_o, Fk_o\|"] \\ Ej_o \arrow[r] & Fk_o \end{tikzcd}$$

commutes, since always $f.\|j_o, Ej_o\| = \|j_o.f\|$.

Note that we use the name of an object to denote also its identity map.

Proof. First we observe that F (like any diagram based on an inverse directed set) is isomorphic to the diagram gotten by truncating F above k_o: That is, replacing $\underline{K}$ by $\{k|k \geq k_o\}$ and restricting F. This new diagram, moreover, satisfies the conditions for being an A-diagram itself (not merely being isomorphic to one). Thus we may suppose that k_o is terminal in $\underline{K}$. Next we observe that $E = E_S$ represents an exact functor of $\underline{X} \longrightarrow \underline{S}$. This means that the $\underline{S}$ diagram $(\underline{K},\tilde{F})$ defined by $\tilde{F}k = (E,Fk)$ is an A-diagram in $\underline{S}$, since exact functors preserve the properties defining an A-diagram, finite limits as well as regular epimorphisms (which are what special maps reduce to in $\underline{X}_S$). Since $(E,F) = \lim(E,Fk)$, then $(E,F) = \lim \tilde{F}k$, taken over $k \in \underline{K}$. Hence this proposition is reduced to the following special case (when $E = 1$ and $\underline{X} = \underline{S}$).

(5.4) <u>Proposition.</u> Let $(\underline{K},F)$ be an A-diagram in $\underline{S}$ and $k_o \in \underline{K}$ be terminal. Then $\lim F \longrightarrow Fk_o$ is onto.

Proof. We choose a point of Fk_o which we will denote by $p(k_o)$. We consider families $(\underline{L},p(\underline{L}))$ in which $\underline{L}$ is a full subset of $\underline{K}$ that is, a subset with the restricted order) and $p(\underline{L}) = \{p(l)|l \in \underline{L}\}$ is a point of $\lim F/\underline{L}$ subject to the following conditions.

a) $k_o \in \underline{L}$.

b) $p(k_o)$ is the already given point.

c) For $k \in \underline{K}$, $l \in \underline{L}$, $l < k \Longrightarrow k \in \underline{L}$.

This family is partially ordered in the obvious way: $(\underline{L}_1, p(\underline{L}_1)) < (\underline{L}_2, p(\underline{L}_2))$ if $\underline{L}_1 \subset \underline{L}_2$ and $p(\underline{L}_2)|\underline{L}_1 = p(\underline{L}_1)$. This set is inductive; the only thing non-trivial is showing that a union of a nested family has a point of the limit. But the test of whether a point of $\{F\ell \mid \ell \in \underline{L}\}$ is a point of the inverse limit involves only two indices at a time, and in an inductive union the satisfaction of such a test is inherited. Hence there is a maximal $(\underline{L}, p(\underline{L}))$ among the family. We need only show that $\underline{K} = \underline{L}$. If not, there is $k \in \underline{K}$, $k \notin \underline{L}$: Since the interval $(k, k_o]$ is finite and $k_o \in \underline{L}$, there must be some $k \notin \underline{L}$ for which $(k, k_o] \subset \underline{L}$. But since

$$Fk \longrightarrow \lim F|(k, k_o]$$

is onto and $\{p(\ell) \mid \ell \in (k, k_o]\}$ is an element of that inverse limit, there is a $p(k) \in Fk$ such that for all $k' \in (k, k_o]$, i.e. all $k' > k$, $F(k', k)p(k) = p(k')$. By condition c) above, no element of $\underline{L}$ precedes k, so that in fact $p(\underline{L}) \cup \{p(k)\}$ is a point of $\lim F|\underline{L} \cup \{k\}$. Clearly the conditions a), b), and c) above are satisfied and we have constructed a proper extension of $(\underline{L}, p(\underline{L}))$, which is a contradiction.

(5.5) Now for an object $X \in \underline{X}$ with support S. Let $(\underline{I}, D)$ be the diagram constructed in (5.1). Since $X \longrightarrow 1$ factors as $X \twoheadrightarrow S \rightarrowtail 1$, there is some $i_o \in \underline{I}$ with $Di_o = X$. Let $\underline{J} = \{i \in \underline{I}_S \mid i \leq i_o\}$. Let $E = D|\underline{J}$. Evidently $(\underline{J}, E) \cong (\underline{I}_S, D_S)$, and $(\underline{J}, E)$ is easily seen to be both an A- and a P-diagram. Let $F: \underline{J} \longrightarrow \underline{X}$ be the functor whose value at $i \in \underline{J}$ is the kernel pair of $E(i_o, i) = D(i_o, i)$. Since Di and Di_o have the same support, this amounts to saying that

$$Fi \overset{d^o i}{\underset{d^1 i}{\rightrightarrows}} Ei \xrightarrow{E(i_o, i)} Ei_o = X$$

is exact.

(5.6) <u>Proposition.</u> The diagram $(\underline{J}, F)$ is an A-diagram.

Proof. A1) and A2) are obvious. Let $k < j \in \underline{J}$. Since limits commute with limits,

$$\lim F|(k,j] = \lim(E \times_X E)|(j,k] = \lim E|(j,k] \times_X \lim E|(j,k].$$

Since $Ej \twoheadrightarrow \lim E|(j,k]$, the result $Ej \times_X Ej \twoheadrightarrow \lim(E \times_X E)|(j,k]$ follows from I.(2.2).

(5.7) Proposition. The diagram

$$F \underset{d^1}{\overset{d^o}{\rightrightarrows}} E \xrightarrow{\|i_o, X\|} X$$

is a coequalizer.

Proof. Since every diagram is a limit of objects of $\underline{X}$, it is sufficient to show this for maps into them. Suppose $\|j,g\|: E \longrightarrow Y$ is a map coequalizing d^o and d^1. This means that $\|j,g.d^o\| = \|j,g.d^1\|$, and since $F \xrightarrow{\|j,Fj\|} Fj$ is an epimorphism (see (3.15)), it follows that $g.d^o = g.d^1$. But

$$\begin{array}{ccccc} Fj & \underset{d^1 j}{\overset{d^o j}{\rightrightarrows}} & Ej & \xrightarrow{E(i_o,j)} & X \\ & & \downarrow g & & \\ & & Y & & \end{array}$$

is a coequalizer and hence there is induced $f: X \longrightarrow Y$ with $f.E(i_o,j) = g$. Since $E(i_o,j)$ is a map in the diagram, it represents the map $\|i_o,X\|: E \longrightarrow X$. Uniqueness of f follows from (3.15).

(5.8) Proposition. Let $G: \underline{K} \longrightarrow \underline{X}$ be any diagram and F the diagram constructed in (5.5). Given two distinct maps $F \rightrightarrows G$, there is a map $E \longrightarrow F$ with $E \longrightarrow F \rightrightarrows G$ also distinct.

Proof. It is sufficient, as above, to consider the case when G is an object of $\underline{X}$, say G = Y. Let the two maps be $\|i,f\|: F \longrightarrow Y$ and $\|j,g\|: F \longrightarrow Y$. By choosing $k \geqslant i,j$ we may suppose that $i = j$.

Since $Fi \xrightarrow{E(i_o,i).d^o i} \twoheadrightarrow X$, there is some $\ell \in \underline{J}$ such that $E\ell = Fi$. Since F is an A-diagram (see (5.6)), the map $E\ell \longrightarrow Fi$ can be extended to a map $E \longrightarrow F$, giving a commutative diagram

$$\begin{tikzcd} E \arrow[r] \arrow[d] & F \arrow[d] \\ E\ell \arrow[r, "="] & Fi \end{tikzcd}$$

and $E \longrightarrow E\ell$ an epimorphism. Since $Fi \rightrightarrows Y$ are distinct, so are $E \longrightarrow E\ell \rightrightarrows Fi \longrightarrow Y$, and then $E \longrightarrow F \rightrightarrows Y$.

(5.9) <u>Proposition</u>. U is full and faithfull.

Proof. Suppose $X \overset{f}{\underset{g}{\rightrightarrows}} Y$ and $Uf = Ug$. If $Z \xrightarrow{e} X$ is the equalizer, this implies that Ue is an isomorphism. If S = supp X, $(UZ)S \cong (UX)S$ and $(UX)S \neq \emptyset$ implies that $(UZ)S \neq \emptyset$ and that $S \leq \text{supp } Z$, while clearly $\text{supp } Z \leq S$. Now choose a vertex $i \in \underline{I}_S$ with $Di = X$. By the isomorphism, the element $\| i,X \| \in (UX)S$ must come from (UZ)S and be represented by some $\| j,h \|$. By choosing $k = i \wedge j$ and observing that $D_S \longrightarrow D_S k$ is epi (see (3.15)), we have a commutative diagram

$$\begin{tikzcd} Dk \arrow[r, "h.D(j{,}k)"] \arrow[d, "D(i{,}k)"'] & Z \arrow[d, "e"] \\ Di \arrow[r, "="] & X \end{tikzcd}$$

from which we see that e is $\longrightarrow\!\!\!\!\rightarrow$. Since e is also an equalizer, this implies that e is an $\xrightarrow{\sim}$ and that $f = g$.

Now suppose that $\varphi\colon UX \longrightarrow UY$ is a natural transformation of functors. Taking S = supp X, we see that $\varphi S\colon (UX)S \longrightarrow (UY)S$, and since $(UX)S \neq \emptyset$, $(UY)S \neq \emptyset$ and $S \leq \text{supp } Y$. If $s\colon X \twoheadrightarrow S$ is the map (there is only one), then $(\varphi, Us): UX \longrightarrow UY \times US = U(Y \times S)$ is also natural. If we show that $(\varphi, Us) = U(f,s), f: X \longrightarrow Y$, then $(\varphi, Us) = (Uf, Us): UX \longrightarrow UY \times Us$ and $\varphi = p_2.(\varphi, Us) = p_1.(Uf, Us) = Uf$. Hence it is sufficient to consider the case that supp Y = S as well. Let $(\underline{J}, E)$ and $(\underline{J}, F)$ be the diagrams constructed in (5.5) above. Then $(UX)S = (F,X)$ and $(UY)S = (F,Y)$. Let d denote $\|i_o, X\|: E \longrightarrow X$. Then by (5.7),

$$F \underset{d^1}{\overset{d^o}{\rightrightarrows}} E \xrightarrow{\ d\ } X$$

is a coequalizer. Now the map d represents an element, also denoted d, of UX, and is transformed into an element $\varphi(d): E \longrightarrow Y$. If $\varphi(d).d^o \neq \varphi(d).d^1$ as maps $F \longrightarrow Y$, there would exist, by (5.8), a map $g: E \longrightarrow F$ such that $\varphi(d).d^o.g \neq \varphi(d).d^1.g$. But the statement that φ is natural means that for any map $S \longrightarrow S$ in $\underline{C}$, that is to say, any natural transformation $u: E \longrightarrow E$, and for any $h: E \longrightarrow X$, $\varphi(h.u) = \varphi(h).u$. But $d^o.g$ and $d^1.g$ are maps $E \longrightarrow E$, and so we have $\varphi(d).d^o.g = \varphi(d.d^o.g) = \varphi(d.d^1.g) = \varphi(d).d^1.g$, which is a contradiction. Thus $\varphi(d).d^o = \varphi(d).d^1$, and by the property of equalizers, there is induced a map $f: X \longrightarrow Y$ with $f.d = \varphi(d)$. Now suppose $e: E \longrightarrow X$ represents some other element of $(UX)S$. Since E is an A- and P-diagram, $e: E \longrightarrow X$ can be extended to $v: E \longrightarrow E$ such that $d.v = e$. Then $\varphi(e) = \varphi(d.v) = \varphi(d).v = f.d.v = f.e$. Hence $\varphi = Uf$. This completes the proof.

(5.10) Proposition. For each object X of $\underline{X}$, UX is a regular quotient of a representable functor.

Proof. Let S = supp X. Choose an index $i \in \underline{I}_S$ with $D_S i = X$ and let $d = \|i, X\|: D_S \longrightarrow X$. By (5.3), we have for any P-diagram E,

$(E,D_S) \twoheadrightarrow (E,X)$. In particular, this holds for $E = D_{S'}$, and so

$$(D_{S'},D_S) \twoheadrightarrow (D_{S'},X),$$

or

$$\underline{C}(S',S) \twoheadrightarrow (UX)S',$$

which means that $\underline{C}(S',-)$ maps onto UX, or that UX is a regular quotient of $\underline{C}(S',-)$.

With this we have completed the proof of (1.6) as well as of all the other results stated in section 1.

(5.11) Remark. It seems worthwhile to make two additional remarks about this embedding. First, as a colimit of a directed set of representable functors, it does more than merely preserve the finite limits that exist. Rather it will preserve the finite limits in any reasonable finite limit completion of the category, e.g. that described in I.(4.5). The second is that as a consequence of the fact that $D_S \twoheadrightarrow D_S i$ for each i, the functor commutes with intersections of any family of subobjects of an object which have an intersection. This property is apparently a completely accidental consequence of the construction and it is not known what, if any, use it might have.

(5.12) If $\underline{V}$ is an exact closed category with exact direct limits and a faithful underlying functor, then by interpreting the $\underline{S}$ valued functor as taking values in $\underline{V}$, we get a $\underline{V}$-valued exact (not full) embedding which reflects isomorphisms. If $\underline{V}$ is the form $\underline{S}^{\mathbb{T}}$, where $\mathbb{T}$ is a commutative triple of finite rank, this is satisfied and one may even see directly that the full embedding lifts to a full exact embedding into a $\underline{V}$-valued functor category.

6. Diagram chasing.

(6.1) When one has an embedding theorem of this sort, the obvious thing to do with it is to chase diagrams. In the abelian cases this was usually cited as one of the main applications. In fact, however, in the abelian case, most of the diagrams can be chased almost as easily in the original abelian category. In fact most of the diagrams to be chased seem to involve, one way or another, the snake lemma. (I am loosely using the term "diagram-chasing" to include "diagram filling" as well.) As seen in the next two chapters, the non-abelian case offers diagrams of both greater variety and greater difficulty. This seems to be largely because exact sequences involve kernel pairs, rather than kernels; coequalizers, rather than cokernels.

(6.2) One further point, equally valid in the abelian and non-abelian case, should be mentioned here. The embedding theorem is valid for small (or locally presentable) regular categories. There are three possible ways around this difficulty for large categories, of which at least two work and one is set-theoretically unassailable. Taking that one first, any diagram, any set of objects, can be extended to a full regular (resp. exact) subcategory by a more - or - less evident process. Given a set of objects, make a full subcategory. Add to this this

a) the kernel pair of any map,

b) the regular image of any map (equivalent to the coequalizer of its kernel pair), and

c) the pullback of any pair of maps like

$$\begin{tikzcd} \cdot \arrow[dr, two heads] & \\ & \cdot \\ \cdot \arrow[ur] & \end{tikzcd}$$

Each of the processes adds a set of objects whose number is (roughly) the set of maps of the given subcategory.Now iterate this countably many times and take the union. The result will evidently be a full, small, regular (resp. exact) subcategory. If the original category had finite limits we could obviously modify this to give finite limits to this subcategory.

(6.3) A second possibility is to relate everything to Grothendieck universes. If a category is large in one universe, it is small in the next and can be embedded in a functor category there. Or it can first be embedded into a locally presentable category. If $\underline{S}$ is the first universe (which may as well be identified with its category of sets) and $\underline{S}^*$ is an enlargement, the embedding of $\underline{X}$ into all $\underline{S}$-continuous functors of $\underline{X}^{op} \longrightarrow \underline{S}^*$ is evidently $\underline{S}$-continuous and the functor category is locally presentable, since $\underline{X}$ is embedded as generators, each of rank $\leq$ to the cardinal of $\underline{S}$ as an object of $\underline{S}^*$.

(6.4) The final way is more speculative but would be the most satisfactory (or, anyway, the most satisfying) if it worked. It is possible that every regular category $\underline{X}$ possesses a class of exact functors $U: \underline{X} \longrightarrow \underline{S}$, $U \in \underline{U}$, with the following property. Every class $\{\varphi U | U \in \underline{U}\}$ of maps $UX \xrightarrow{\varphi U} UY$ for which each natural transformation $\alpha: U \longrightarrow U'$ gives a commutative diagram

$$\begin{array}{ccc} UX & \xrightarrow{\varphi U} & UX' \\ {\scriptstyle \alpha X}\downarrow & & \downarrow{\scriptstyle \alpha X'} \\ U'X & \xrightarrow{\varphi U'} & U'X' \end{array}$$

implies the existence of a unique $f: X \longrightarrow Y$ such that $\varphi U = Uf$ for all $U \in \underline{U}$. Since a class $\underline{U}$ is a collectively full and faithful family, a diagram can be chased by applying every such U. "Every" is, in this context, the same as "any" and can be supposed for purposes of verification to be just one. It is not known whether such a class $\underline{U}$ always exists.

(6.5) Whichever strategem is adopted doesn't change the fact that certain types of diagram chasing in regular categories can be carried out in functor categories. Strict diagram chasing (that is, not involving filling-in, but only commutativity) can be carried out in $\underline{S}$, since the evaluating functors $(\underline{C}^{op},\underline{S}) \longrightarrow \underline{S}$ given by evaluativy at the objects of $\underline{C}$ form a family of exact functors which are collectively faithful. In fact more is true.

(6.6) Proposition. The evaluation functors $(\underline{C}^{op},\underline{S}) \longrightarrow \underline{S}$ for $C \in \underline{C}$ collectively are faithful, exact, reflect isomorphisms and reflect equivalence relations.

Proof. That they are faithful is clear, since equality of natural transformations is defined that way. The evaluations preserve all limits and colimits (limits and colimits are calculated "pointwise"), so exactness is also clear. For similar reasons they reflect isomorphisms (collectively). Finally suppose $F \longrightarrow G \times G$ is such that FC is an equivalence relation on GC for all $C \in \underline{C}$. First, $FC \rightarrowtail (G\times G)C = GC \times GC$ implies that $F \rightarrowtail G \times G$. Next, the coequalizer $F \rightrightarrows G \longrightarrow H$ is computed pointwise so that $FC \rightrightarrows GC \longrightarrow HC$ is a coequalizer for each $C \in \underline{C}$. But the kernel pair of $GC \longrightarrow HC$ is just FC, which means that $F \rightrightarrows G$ is a kernel pair, *a fortiori* an equivalence relation.

(6.7) Corollary. Let $\underline{X}$ be a small (or locally presentable) regular category. Then there is a family of exact functors $U_i\colon \underline{X} \longrightarrow \underline{S}$, $i \in I$, which collectively are faithful, reflect isomorphisms, and reflect equivalence relations. If, in addition, $\underline{X}$ is exact, then these U_i preserve the coequalizer of any pair of maps $X \overset{d^0}{\underset{d^1}{\rightrightarrows}} Y$ such that the image of $(U_i d^0, U_i d^1)\colon U_i X \longrightarrow U_i Y \times U_i Y$ is an equivalence relation for each $i \in I$.

Proof. If $U\colon \underline{X} \longrightarrow (\underline{C}^{op}, \underline{S})$ is full, faithful, and exact, we let I be the objects of $\underline{C}$ and U_i be U followed by evaluation at the corresponding object. Then every thing but the last statement is clear. To see that, suppose d^0 and d^1 are as above. Then we can factor (d^0, d^1) as $X \twoheadrightarrow Z \rightarrowtail Y \times Y$. By the proposition and the given conditions, UZ is an equivalence relation on Y. If the diagram

$$Z \rightrightarrows Y \longrightarrow Y'$$

is a coequalizer, it is exact. Then for each $i \in I$,

$$U_i X \twoheadrightarrow U_i Z$$

and

$$U_i Z \rightrightarrows U_i Y \longrightarrow U_i Y'$$

is a coequalizer, which implies that

$$U_i X \rightrightarrows U_i Y \longrightarrow U_i Y'$$

is a coequalizer.

(6.8) Metatheorem. Let $\underline{X}$ be a regular category. Then any small diagram chasing argument valid in $\underline{S}$ is valid in $\underline{X}$, provided the data of the diagram involve only finite inverse limits and coequalizers of right exact sequences; if, moreover, the category is exact, these data may also include coequalizers of pairs of maps which, in $\underline{S}$, can be shown to have as image an equivalence relation.

(6.9) Given the somewhat vague statement of this metatheorem, it is hardly susceptible of being proved. To apply it, it is necessary only to verify that the type of diagram to be chased is by its nature susceptible of being proved by applying a family of reflexively exact functors which also reflect equivalence relations.

(6.10) <u>Example</u>. Suppose $\underset{\sim}{X}$ is a regular category and we are given a commutative diagram

$$\begin{tikzcd}
Y' \arrow[r, "f'"] \arrow[d, shift right, "e^{o}"'] \arrow[d, shift left, "e^{1}"] & X' \arrow[d, shift right, "d^{o}"'] \arrow[d, shift left, "d^{1}"] \\
Y \arrow[r, "f"'] \arrow[d, "e"'] & X \arrow[d, "d"] \\
Y'' \arrow[r, "f''"'] & X''
\end{tikzcd}$$

in which both columns are exact and the square

$$\begin{tikzcd}
Y' \arrow[r, "f'"] \arrow[d, "e^{o}"'] & X' \arrow[d, "d^{o}"] \\
Y \arrow[r, "f"'] & Y
\end{tikzcd}$$

is a pullback (which is equivalent to the square with e^1 and d^1 being a pullback). Then the square

$$\begin{tikzcd}
Y \arrow[r, "f"] \arrow[d, "e"'] & X \arrow[d, "d"] \\
Y'' \arrow[r, "f''"'] & X''
\end{tikzcd}$$

is also a pullback.

Proof. Even in the category of sets this is moderately difficult to prove. In an arbitrary regular category it follows from the metatheorem. I am indebted to Anders Kock for suggesting this example. It arises in the theory of elementary toposes and also in descent theory.

Chapter IV. Groups and Representations

1. Preliminaries.

(1.1) Throughout this chapter and the next, $\underline{X}$ denotes a fixed exact category. From I(5.11) both Gp$\underline{X}$ and Ab$\underline{X}$, the categories of groups and abelian groups in $\underline{X}$, respectively, form exact categories. The latter, in particular, is abelian.

(1.2) Let $G \in \mathrm{Gp}\underline{X}$, and $u\colon 1 \longrightarrow G$, $i\colon G \longrightarrow G$, and $m\colon G \times G \longrightarrow G$ be the unit, inverse, and multiplication maps, respectively. A pair (X,a) where $X \in \underline{X}$ and $a\colon G \times X \longrightarrow X$ is called a left representation of G or a left G-object if the following diagrams commute:

$$\begin{tikzcd} G \times G \times X \arrow[r, "G \times a"] \arrow[d, "m \times X"'] & G \times X \arrow[d, "a"] \\ G \times X \arrow[r, "a"] & X \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "\cong"] \arrow[ddr, equal] & 1 \times X \arrow[d, "u \times X"] \\ & G \times X \arrow[d, "a"] \\ & X \end{tikzcd} .$$

A morphism $f\colon X \longrightarrow X'$ is a morphism of G-objects $(X,a) \longrightarrow (X',a')$ provided

$$\begin{tikzcd} G \times X \arrow[r, "G \times f"] \arrow[d, "a"'] & G \times X' \arrow[d, "a'"] \\ X \arrow[r, "f"] & X' \end{tikzcd}$$

commutes.

Note that all these products exist, since, for example,

$$\begin{tikzcd} G \times X \arrow[r] \arrow[d] & X \arrow[d] \\ G \arrow[r, two heads] & 1 \end{tikzcd}$$

is a pullback.

The left G-objects and their morphisms evidently form a category $\underline{LO}(G)$ which has an evident underlying functor $\underline{LO}(G) \longrightarrow \underline{X}$. Turning every thing around, we can define the category $\underline{RO}(G)$ of right G-objects and their morphisms. Finally, we say that a 3-tuple (X,a,a') where $(X,a) \in \underline{LO}(G)$ and $(X,a') \in \underline{RO}(G)$ is a 2-sided G-object if

$$\begin{tikzcd} G \times X \times G \arrow[r, "G \times a'"] \arrow[d, "a \times G"'] & G \times X \arrow[d, "a"] \\ X \times G \arrow[r, "a'"] & X \end{tikzcd}$$

commutes. The category of these objects and morphism which are simultaneously in $\underline{LO}(G)$ and $\underline{RO}(G)$ is called $\underline{BO}(G)$. It is clear that one could define G^{op} and show that $\underline{LO}(G^{op})$ is the same as $\underline{RO}(G)$ and $\underline{LO}(G \times G^{op})$ is the same as $\underline{BO}(G)$.

(1.3) Theorem. Let $\underline{X}$ be a regular category (resp. exact). Then $\underline{LO}(G)$ is regular (resp. exact) and the functor $\underline{LO}(G) \longrightarrow \underline{X}$ is a reflexively exact functor.

Proof. That it reflects isomorphisms is trivial. Now consider an exact sequence

$$X' \underset{d^1}{\overset{d^o}{\rightrightarrows}} X \xrightarrow{d} X''$$

in which (X',a') and (X,a) are left G-objects and d^o, d^1 are G-morphisms.

Then the top row of

$$\begin{tikzcd} G \times X' \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "a'"] & G \times X \arrow[r] \arrow[d, "a"] & G \times X'' \arrow[d, dashed, "a''"] \\ X' \arrow[r, shift left] \arrow[r, shift right] & X \arrow[r] & X'' \end{tikzcd}$$

is still exact and hence a" is induced as indicated. From here the proof proceeds exactly as in I.(5.11).

(1.4) Corollary. $\underline{RO}(G)$ and $\underline{BO}(G)$ and their underlying functors to $\underline{X}$ enjoy the same properties.

Proof. This can be either proved the same way or made to follow as a corollary via the remark preceding (1.3).

(1.5) Theorem: Let $U: \underline{X} \longrightarrow \underline{Y}$ be exact. Then there is induced, for each $G \in \underline{X}$ an exact functor

$$\underline{LO}(G) \longrightarrow \underline{LO}(UG)$$

such that

$$\begin{array}{ccc} \underline{LO}(G) & \longrightarrow & \underline{LO}(UG) \\ \downarrow & & \downarrow \\ \underline{X} & \longrightarrow & \underline{Y} \end{array}$$

commutes

Proof. Recall that according to I.(5.11), UG will be a group object in $\underline{Y}$. That U takes G-objects to UG-objects follows easily from the fact that U preserves products. The exactness is a consequence of the reflexive exactness of $\underline{LO}(UG) \longrightarrow \underline{Y}$.

(1.6) Corollary. $\underline{RO}(G)$ and $\underline{BO}(G)$ enjoy the same properties.

(1.7) Lemma: Suppose (X,a,a') is an object of $\underline{BO}(G)$ and $s: G \times X \longrightarrow X \times G$ is the map which interchanges the factors. Then the immage of $G \times X \xrightarrow{(a,a'.s)} X \times X$ is an equivalence relation on X. That is, if X' is defined as the coequalizer in the diagram

$$G \times X \underset{a'.s}{\overset{a}{\rightrightarrows}} X \longrightarrow X',$$

then this sequence is right exact.

Proof. If $\underline{X}$ is small, choose $U: \underline{X} \longrightarrow \underline{S}$ which is reflexively exact and reflects equivalence relations. Then UG is an ordinary group and UX is a 2-sided UG-object. Thus it suffices to consider the case of ordinary groups operating on ordinary sets by a 2-sided operation. So we have $G \times X \longrightarrow X \times X$ by a map taking $(g,x) \longmapsto (gx,xg)$ and we want to show the image is an equivalence relation on X. It is reflexive as $(1,x) \longmapsto (x,x)$ and symmetric as $(g^{-1},gxg) \longmapsto (xg,gx)$. If (gx,xg) and $(g'x',x'g')$ satisfy $xg = g'x'$, $(gg',x'g^{-1}) \longmapsto$
$\longmapsto (gg'x'g^{-1},x'g') = (gxgg^{-1},x'g') = (gx,x'g')$, and so the image is transitive. When $\underline{X}$ is large, use an appropriate modification (cf. III.(6.4)).

2. Tensor products.

(2.1) Proposition. Let G be a group in $\underline{X}$, $(X,a) \in \underline{LO}(G)$ and $X' \in \underline{X}$. Then $(X \times X', a \times X') \in \underline{LO}(G)$ also.

Proof. Trivial.

(2.2) Of course $X' \times X \cong X \times X'$, so that $X' \times X \in \underline{LO}(G)$. If $(X',a') \in \underline{RO}(G)$, $X' \times X$ has the structure of a left G-object from X and of a right G-object from X'.

(2.3) Proposition. $X' \times X$ with this structure is an object of $\underline{BO}(G)$.

Proof. Trivial.

(2.4) Definition. Let $X \in \underline{LO}(G)$, $X' \in \underline{RO}(G)$. We define $X' \otimes_G X$ as the coequalizer in the diagram

$$X' \times G \times X \overset{a' \times X}{\underset{X \times a}{\rightrightarrows}} X' \times X \longrightarrow X' \otimes_G X.$$

Note that though $X' \times X$ is a left and right G-object, it is most convenient to put G in the middle. It follows from (1.7) that the sequence is right exact and thus remains right exact (in particular a coequalizer) when any right exact functor is applied.

(2.5) Proposition. $- \otimes_G -$ is a functor $\underline{RO}(G) \times \underline{LO}(G) \longrightarrow \underline{X}$.

Proof. If $(X,a) \xrightarrow{f} (Y,b)$ is a map of left G-objects, the diagram

$$\begin{tikzcd}
X' \times G \times X \arrow[r, shift left, "a' \times X"] \arrow[r, shift right, "X' \times a"'] \arrow[d, "X' \times G \times f"'] & X' \times X \arrow[r] \arrow[d, "X' \times f"] & X' \otimes_G X \arrow[d, dashed] \\
X' \times G \times Y \arrow[r, shift left, "a' \times Y"] \arrow[r, shift right, "X' \times b"'] & X' \times Y \arrow[r] & X' \otimes_G Y
\end{tikzcd}$$

commutes, whence $X' \otimes f$ is induced from the coequalizer.

(2.6) <u>Proposition</u>. Suppose $X' \in \underline{LO}(H \times G^{op})$ (This means that it is a left H, right G, bi-object) and $X \in \underline{LO}(G)$. Then $X' \otimes_G X)$ has the natural structure of a left H object.

Proof. The top row of

$$\begin{tikzcd}
H \times X' \times G \times X \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "b \times G \times X"'] & H \times X' \times X \arrow[r] \arrow[d, "b \times X"'] & H \times (X' \otimes_G X) \arrow[d, dashed] \\
X' \times G \times X \arrow[r, shift left] \arrow[r, shift right] & X' \times X \arrow[r] & X' \otimes_G X
\end{tikzcd}$$

is still a coequalizer. Here $b\colon H \times X' \longrightarrow X'$ is, of course, the H'-structure map and the commutativity of one the squares at the left is exactly the fact of X' being a bi-object. The induced map $H \times (X' \otimes X) \longrightarrow X' \otimes X$ is easily shown to be a structure map, using, for example, that

$$H \times H \times X' \times X \longrightarrow\!\!\!\!\!\rightarrow H \times H \times (X' \otimes_G X).$$

(2.7) It is clear that G with its left and right multiplication maps belongs to $\underline{BO}(G)$. If $f\colon H \longrightarrow G$ is a morphism of group objects, there is an obvious functor $f^*\colon \underline{LO}(G) \longrightarrow \underline{LO}(H)$, in which $(X,a) \longmapsto (X, a.(f \times X))$. There is also included a functor $f_!\colon \underline{LO}(H) \longrightarrow$ $\longrightarrow \underline{LO}(G)$ which takes a H-object X to $G \otimes_H X$, evidently a G-object from the above remark.

(2.8) <u>Theorem</u>. The functor $f_! \dashv f^*$.

Proof. The inner adjunction is the map $X \xrightarrow{(u.t,X)} G \times X \longrightarrow G \otimes_H X$ in which $X \xrightarrow{t} I \xrightarrow{u} G$ is the terminal map of X followed by the unit of G. The outer adjunction is induced by

$$\begin{tikzcd} G \times H \times X \arrow[r, shift left] \arrow[r, shift right] & G \times X \arrow[r] \arrow[d, "a"] & G \otimes_H X \arrow[dl, dashed] \\ & X & \end{tikzcd}$$

That the first is H linear, the second exists and is G-linear, and the two satisfy the laws of an adjunction may be easily verified by applying the metatheorem.

(2.9) <u>Corollary</u>. For any G, the underlying functor $\underline{BO}(G) \longrightarrow \underline{X}$ has a left adjoint, $X \longmapsto G \times X$.

Proof. Apply the above to $G \longrightarrow 1$. It is evident that $G \otimes_1 X = G \times X$.

(2.10) <u>Theorem</u>. Let $X \in \underline{LO}(G{\times}H^{op})$, $Y \in \underline{LO}(H{\times}K^{op})$, $Z \in \underline{LO}(K \otimes L^{op})$. Then there is a canonical map

$$(X \otimes_H Y) \otimes_K Z \longrightarrow X \otimes_H (Y \otimes_K Z)$$

such that the diagram

$$\begin{tikzcd} & X \times Y \times Z \arrow[dl] \arrow[dr] & \\ (X \otimes_H Y) \otimes_K Z \arrow[rr] & & X \otimes_H (Y \otimes_K Z) \end{tikzcd}$$

commutes (see the proof for the definition of these vertical maps), and that map is an isomorphism.

Proof. The vertical maps in the diagram are gotten by letting $t(X,Y)$ denote the canonical projection $X \times Y \longrightarrow X \otimes_H Y$. Then the one map is $t(X \otimes_H Y,Z).t(X,Y) \otimes Z$ and the other is similar. One way of proving this is to first prove it in $\underline{S}$ (trivial). Then use the meta-theorem to show that in the diagram

$$\begin{tikzcd} X \times Y \times K \times Z \arrow[r, shift left] \arrow[r, shift right] & X \times Y \times Z \arrow[r] \arrow[d] & X \times (Y \otimes_K Z) \arrow[dl, dashed] \\ & (X \otimes_H Y) \otimes_K Z & \end{tikzcd}$$

the vertical arrow coequalizes the two maps on the left. Since the row is a right exact, it is a coequalizer, and there is induced $X \times (Y \otimes_K Z) \longrightarrow (X \otimes_H Y) \otimes_K Z$ with the appropriate property. Another use of the metatheorem shows that in the diagram

$$\begin{tikzcd} X \times H \times (Y \otimes_K Z) \arrow[r, shift left] \arrow[r, shift right] & X \times (Y \otimes_K Z) \arrow[r] \arrow[d] & X \otimes_H (Y \otimes_K Z) \arrow[dl, dashed] \\ & (X \otimes_H Y) \otimes_K Z & \end{tikzcd}$$

the vertical arrow again coequalizes the two arrows on the left and the required map is the one induced. That it is an isomorphism may be readily verified by a third use of the embedding.

(2.11) <u>Theorem</u>: If $X \in \underline{LO}(G)$, $G \otimes_G X \cong G$; and if $Y \in \underline{RO}(G)$, $Y \otimes_G G \cong Y$.

Proof. These can be derived either directly from adjointness or from arguments similar to (but simpler than) the above.

(2.12) <u>Theorem</u>: The associativity and unit of the previous two theorems are jointly coherent.

Proof. Prove it in $\underline{S}$ and use the metatheorem.

(2.13) <u>Corollary</u>. If $g\colon K \longrightarrow H$, $f\colon H \longrightarrow G$, then $(f.g)_! = f_!.g_!$.

Proof. From the previous theorems we have for $X \in \underline{LO}(K)$, $f_!(g_!X) = G \otimes_H (H \otimes_K X) \cong (G \otimes_H H) \otimes_K X \cong G \otimes_K X = (fg)_!(X)$.

(2.14) Remark. Later on, when G is commutative (and then $\underline{LO}(G)$ and $\underline{RO}(G)$ are equivalent to the same full subcategory of $\underline{BO}(G)$, namely the subcategory of symmetric objects), there will be a commutativity isomorphism as well, which by the same reasoning will be jointly coherent with the above.

(2.15) Proposition. Let $U\colon \underline{X} \longrightarrow \underline{Y}$ be an exact functor, $G \in \underline{X}$, $X_1 \in \underline{RO}(G)$, and $X_2 \in \underline{LO}(G)$. Then

$$U(X_1 \otimes_G X_2) \cong UX_1 \otimes_{UG} UX_2.$$

Proof. Exact functors preserve both products and right exact sequences. Apply U to

$$X_1 \times G \times X_2 \rightrightarrows X_1 \times X_2 \longrightarrow X_1 \otimes_G X_2.$$

3. Principal objects.

(3.1) Definition. Let G be a group in $\underline{X}$. A left G-object X will be called a principal left G-object if

a) $X \twoheadrightarrow 1$.

b) $G \times X \xrightarrow{(a,p_2)} X \times X$ is an isomorphism. Here $a\colon G \times X \longrightarrow X$ is the structure while $p_2\colon G \times X \longrightarrow X$ is the second coordinate projection. We let $\underline{PLO}(G)$ denote the full subcategory of these objects.

(3.2) The definition is, in view of III(2.11), exactly the same as Chase's [Ch] which goes back, in turn, to Beck [Be]. Much of the preliminary material in this section is special cases of results proved by Chase, His proofs, however, were generally much more complicated because he had no metatheorem available.

(3.3) Proposition. Let $U\colon \underline{X} \longrightarrow \underline{Y}$ be exact. Then $U(\underline{PLO}(G)) \subset \underline{PLO}(UG)$.

Proof. U preserves $\twoheadrightarrow$, finite products, and (like any functor) isomorphisms.

(3.4) Proposition. Let G be a group (in $\underline{S}$). Then $\underline{PLO}(G)$ consists (up to isomorphism) of the single object G, and the morphisms, all $\xrightarrow{\sim}$, consist of the right multiplications by the elements of G.

Proof. Let $X \in \underline{PLO}(G)$. Condition i) of (3.1) says that $X \neq \emptyset$. Condition ii) says that the map $G \times X \longrightarrow X \times X$, which takes $(g,x) \longmapsto (gx,x)$ for $g \in G$ and $x \in X$, is an isomorphism. This amounts to saying that if x is held fixed, there is for each $x' \in X$ a unique solution in G to $gx = x'$. In other words, if $x \in X$ is fixed, the mapping $G \longrightarrow X$ by $g \longmapsto gx$ is an isomorphism. The rest of the proposition is trivial.

(3.5) Proposition. $\underline{PLO}(G)$ is a groupoid (that is every map is $\xrightarrow{\sim}$).

Proof. If $X \longrightarrow X'$ is a map in $\underline{PLO}(G)$ choose an embedding and apply the last proposition.

(3.6) Proposition $X \in \underline{PLO}(G)$ is isomorphic to G if and only if there is a map $1 \longrightarrow X$ in $\underline{X}$. In fact, $\underline{PLO}(G)(G,X) \cong \underline{X}(1,X)$.

Proof. $\underline{PLO}(G) \subset \underline{LO}(G)$ is full and faithful. Hence this follows from adjointness:

$$\underline{LO}(G)(G,X) = \underline{LO}(G)(G \times 1,X) \cong \underline{X}(1,X).$$

(3.7) Theorem: Let $U\colon \underline{X} \longrightarrow \underline{S}$ range over a family of exact embeddings which collectively reflect isomorphisms. Then $\underline{PLO}(G)$ consists of those X for which $UX \cong UG$ as UG-objects.

Proof. If $UX = UG$, then the canonical map $(Ua,p_2)\colon UG \times UX \longrightarrow UX \times UX$ is an isomorphism, which means that $U(a,p_2)\colon U(G \times X) \longrightarrow U(X \times X)$ is also, and finally that $(a,p_2)\colon G \times X \longrightarrow X \times X$ is. On the other hand, by (3.3) and (3.4), $X \in \underline{PLO}(G)$ implies $UX \cong UG$.

(3.8) Theorem: Let $f\colon H \longrightarrow G$ be a morphism of groups. Then $f_!(\underline{PLO}(H)) \subset \underline{PLO}(G)$.

Proof. For any exact $U\colon \underline{X} \longrightarrow \underline{S}$, $U(G \otimes_H X) \cong UG \otimes_{UH} UX \cong UG \otimes_{UH} UH \cong$ $\cong UG$. Note that $f_!$ is not in general exact, so that (3.3) does not apply here.

(3.9) Proposition. Suppose $f\colon H \longrightarrow G$ is the trivial map, $H \longrightarrow 1 \xrightarrow{u} G$. Then for $X \in \underline{PLO}(H)$, $f_!(X) \cong G$.

Proof. It is sufficient to show that there is a G-morphism of $f_!(X) \longrightarrow G$. In the diagram

$$\begin{tikzcd}
G \times H \times X \arrow[r, shift left] \arrow[r, shift right] & G \times X \arrow[r] \arrow[d, "p_1"] & X \arrow[dl, dashed] \\
& G &
\end{tikzcd}$$

the vertical map coequalizes the two maps on the left (the structure $G \times H \longrightarrow G$, is in this case just the projection) and induces $X \longrightarrow G$, evidently a G-morphism.

4. Structure of groups.

(4.1) In this section we derive a few results about the relation between kernels and kernel pairs. We continue to let $\underline{X}$ denote an exact category.

(4.2) We know from I.(5.11) that the underlying functor from Gp $\underline{X} \longrightarrow \underline{X}$ is exact and hence preserves limits and regular epimorphisms. Since the category is also pointed, the notions of normal monomorphisms and epimorphisms also arise. It is evident that a normal epimorphism is always regular, but in general (e.g. in pointed sets) the converse is not always true. Here we will show that it is.

(4.3) Proposition. Gp $\underline{X}$ has finite products.

Proof. The terminal map $G \longrightarrow 1$ of any group is $\longrightarrow\!\!\!\!\rightarrow$, being split by the unit. Then the pullback

$$\begin{tikzcd} G \times H \arrow[r, two heads] \arrow[d, two heads] & G \arrow[d, two heads] \\ H \arrow[r, two heads] & 1 \end{tikzcd}$$

exists.

(4.4) Proposition. Gp $\underline{X}$ has finite limits.

Proof. It is necessary only to show that equalizers exist. During this argument we will denote the composition of morphisms by a dot, as f.g, while the multiplication of two morphisms to some group will be denoted simply by juxtaposition, as fg. The inverse, under the group law, will be denoted f^{-1}. This latter is particularly ambiguous but none of the maps arising in the proof will be isomorphisms (except accidently) and the inverse in the category will not be used. Of course neither f^{-1} nor fg will generally be morphisms of Gp $\underline{X}$ when f and g are. Now suppose we are given two maps $f,g\colon G \longrightarrow H$. We let

$u\colon 1 \longrightarrow G$, $1 \longrightarrow H$ denote interchangeably the unit morphisms. In particular $f.u = u$, $g.u = u$ and $fg^{-1}.u = (f.u)(g^{-1}.u) = (f.u)(g.u)^{-1}) = = uu^{-1} = uu = u$. If X is the image of $fg^{-1}\colon G \longrightarrow H$, this shows that $u\colon 1 \longrightarrow H$ factors through X via fg^{-1}. Now let K be the pullback in the diagram

$$\begin{tikzcd}
K \arrow[r, "t"] \arrow[d, "h"'] & 1 \arrow[d] \arrow[dr, "u"] & \\
G \arrow[r, two heads] & X \arrow[r] & H
\end{tikzcd}$$

Once this pullback exists, it follows that

$$\begin{tikzcd}
K \arrow[r] \arrow[d, "h"'] & 1 \arrow[d] \\
G \arrow[r, "fg^{-1}"] & H
\end{tikzcd}$$
is also a pullback.

Now K is a group, and in particular $h\colon K \rightarrowtail G$ is a subgroup, if and only if $(X,K) \overset{(X,h)}{\rightarrowtail} (X,G)$ is a subgroup for each X. Applying $(X,-)$, we still get a pullback in $\underline{S}$

$$\begin{tikzcd}
(X,K) \arrow[r] \arrow[d] & (X,1) = 1 \arrow[d] \\
(X,G) \arrow[r, "(X{,}f)\,(X{,}G)^{-1}"'] & (X,H)
\end{tikzcd}$$

and (X,K) really is the equalizer of the two group homomorphisms (X,f) and (X,g), and hence is a subgroup.

(4.5) Proposition. Every regular epimorphism is normal.

Proof. We use the same conventions as in the proof above. The underlying functor $Gp\ \underline{X} \longrightarrow \underline{X}$ preserves finite inverse limits. It preserves, in particular, kernels, since the kernel of a map is the equalizer of that map and the trivial map. As in (3.9), we let u also denote this trivial map between any two groups. Now suppose that

$$G' \underset{e}{\overset{d}{\rightrightarrows}} G \xrightarrow{f} G''$$

is a coequalizer and $H \xrightarrow{g} G$ is the kernel of f. We want to show that f is the cokernel of h, and it clearly suffices to show that for any $h\colon G \longrightarrow K$, $h.g = u$ implies $h.e = h.d$. But g is also the equalizer of f and u as maps in $\underline{X}$. Now $f.de^{-1} = (f.d)(f.e^{-1}) = (f.d)(f.e)^{-1} = (f.d)(f.d)^{-1} = u$. Hence there is map $k\colon G' \longrightarrow H$ such that $g.k = de^{-1}$. Now for any $h\colon G \longrightarrow K$ with $h.g = u$, $u = h.g.k = h.de^{-1}$ = (as above) $(h.d)(h.e)^{-1}$, and on multiplying this by u, which is the unit of (G,K), we have $h.e = h.d$, which completes the proof.

Chapter V. Cohomology.

1. Definitions.

(1.1) In this chapter we will define cohomology sets of $\underline{X}$ with coefficients in a group in $\underline{X}$. Only H^0 and H^1 will be defined here. There are several suggestions for higher sets; these are being investigated currently. The "cohomology sets" are covariant functors of the coefficients. What they are contravariant functors of is suggested by the classical examples (cf. section 4). If $\underline{X}$ is exact, so is $(\underline{X},X)$ for any $X \in \underline{X}$ by I.(5.4); and if $X \longrightarrow X'$ is a map, there is induced $(\underline{X},X') \longrightarrow (\underline{X},X)$ by pulling back, provided the pullbacks exist. Even if they don't, they do for all $Y \twoheadrightarrow X'$, and that is all the cohomology is concerned with. If G is a group in $(\underline{X},X)$, it also is in $(\underline{X},X')$, and there is induced $H^i(X',G) \longrightarrow H^i(X,G)$, $i = 0,1$. In the discussion below, the X is suppressed and we write $H^i(G)$, which should actually be $H^i(1,G)$. (X is terminal in $(\underline{X},X)$ and the cohomology of X is the cohomology of that terminal object.)

(1.2) Throughout this chapter we will keep certain notational conventions. In addition to $\underline{X}$ being exact, we suppose that it has a terminal object 1 and that $t: X \longrightarrow 1$ denotes the terminal map of every object. Each group comes equipped with its multiplication m, its inverse i, and its unit u. For any object X and group G, we will also use $u: X \longrightarrow G$ to denote the composite $X \xrightarrow{t} 1 \xrightarrow{u} G$. The maps denoted t form a right ideal with respect to all the objects and those denoted by u form a left ideal with respect to groups and group homomorphisms. In addition, for this section we fix an exact sequence of groups and group homomorphisms

$$1 \xrightarrow{u} G' \xrightarrow{f} G \xrightarrow{f'} G'' \xrightarrow{t} 1.$$

(1.3) The cohomology will be relative to an underlying functor U: $\underline{X} \longrightarrow \underline{Y}$. Although the functor U and the category $\underline{Y}$ are usually exact, it seems desirable to develope the relative theory without those assumptions. Accordingly we will suppose only that U preserves finite limits. The absolute, or unrelativized, theory may be recovered by letting U be an exact functor to a category $(\underline{C},\underline{S})$ where $\underline{C}$ is discrete, for in that category every epimorphism splits and every principal G-object is isomorphic to G. The desirability of considering such a relative theory was pointed out by Jon Beck.

(1.4) Definition. Let G be a group in $\underline{X}$ and $X \in \underline{PLO}(G)$. We say that X is split by a functor U if $UX \cong UG$ as a UG object.

(1.5) Proposition. With U,X and G as above, X is split by U if and only if there is a morphism $1 \longrightarrow UX$.

Proof. Of course in the case in which $\underline{Y}$ is exact, this follows from IV.(3.6). But we have not supposed that. In any event, $(1,UG) \neq \emptyset$, so one direction is trivial. To go the other way, let $H = UG$ and $Y = UX$, and suppose there is a map $s: 1 \longrightarrow Y$. Now H is a group, Y is an H object, and $H \times Y \xrightarrow{\sim} Y \times Y$. This implies that the representable functor $(-,H)$ is a group, $(-,Y)$ is an H-object, and

$$(-,H) \times (-,Y) \xrightarrow{\sim} (-,Y) \times (-,Y).$$

Then for any Y' such that $(Y',Y) \neq \emptyset$, (Y',Y) is a principal (Y',G) object. This implies that $(Y',G) \xrightarrow{\sim} (Y',Y)$ by the map that, associates to a fixed $f_o: Y' \longrightarrow Y$ and to an arbitrary map $g: Y' \longrightarrow G$, the map

$$Y' \xrightarrow{(g,f_o)} G \times Y \longrightarrow Y,$$

the second map being the structure. If we take for f_o the composite

$$Y' \xrightarrow{t} 1 \xrightarrow{s} Y,$$

this defines a natural $(-,G)$ equivalence $(-,G) \xrightarrow{\sim} (-,Y)$ which must be induced by a G equivalence $G \xrightarrow{\sim} Y$.

(1.6) Definition. We know that $\underline{PLO}(G)$ is a groupoid (IV.(3.5)). In addition, there is a distinguished component in $\underline{PLO}(G)$, the one containing G. We define H^oG to be the set of automorphisms of G, and given $U: \underline{X} \longrightarrow \underline{Y}$, we define $H^1(U,G)$ to be the set - or maybe class - of all components of $\underline{PLO}(G)$ split by U. That means those components containing a representative split by U. Since the distinguished component is clearly split by U, this may be considered as a pointed set - or class - with the distinguished component as base point. In the case that the functor U is exact and takes values in $\underline{S}$, whence every $X \in \underline{PLO}(G)$ splits, the resultant set $H^1(U,G)$ is simply the set of connected components of $\underline{PLO}(G)$ and is denoted $H^1(G)$. This is the "absolute" cohomology.

(1.7) Proposition. Let $f: G' \longrightarrow G$ be a group homomorphism. Then if $X \in \underline{PLO}(G')$ is U split, so is $f_!(X) \in \underline{PLO}(G)$.

Proof. There is a map $X \longrightarrow f_!(X)$ (essentially the front adjunction) and a map $1 \longrightarrow UX$ gives one $1 \longrightarrow UX \longrightarrow Uf_!(X)$.

(1.8) Theorem (Beck). Suppose $\underline{X}$ is exact and $U: \underline{X} \longrightarrow \underline{Y}$ is a tripleable underlying functor. Then for $G \in \text{Gp}\,\underline{X}$, $H^o(G)$ and $H^1(U,G)$ are the zeroth and first (non-abelian) triple cohomology sets of the object 1 with coefficients in G.

The proof is rather long and is given in [Be]. If F is left adjoint to U and the front and back adjunctions are given by $\eta: \underline{Y} \longrightarrow UF$ and $\varepsilon: FU \longrightarrow \underline{X}$, then the triple sets are computed from the complex

$$1 \longrightarrow \underline{X}(FU1,G) \rightrightarrows \underline{X}(FUFU1,G) \rightrightarrows\!\!\!\!\!\rightarrow \underline{X}(FUFUFU1,G),$$

the arrows induced by such things as εFU and $FU\varepsilon$ and similar maps at the next stage. The fact, standard in tripleable categories, that

$$FUFUX \underset{UF\varepsilon X}{\overset{FU\varepsilon X}{\rightrightarrows}} FUX \xrightarrow{\varepsilon X} X$$

is a coequalizer, implies easily, if X is taken as 1, that the zeroth cohomology is $\underline{X}(1,G)$.

(1.9) Corollary. Suppose $U: \underline{X} \longrightarrow \underline{S}$ is tripleable. Then U is exact and the zeroth and first triple cohomology of the object 1 with coefficients in a group object G are exactly $H^0(G)$ and $H^1(G)$.

Proof. The exactness of U in this case is well-known (in fact is the direct ancestor of the definition of exactness used in this paper) and the rest then follows from the preceding theorem.

2. <u>The exact sequence</u>.

(2.1) If $U\colon \underline{X} \longrightarrow \underline{Y}$ is a finite limit preserving functor and

$$1 \longrightarrow G' \xrightarrow{f} G \xrightarrow{f'} G'' \longrightarrow 1$$

is an exact sequence in Gp $\underline{X}$, we say that it is a U-split exact sequence if Uf' is a split epimorphism. Thus

$$1 \longrightarrow UG' \xrightarrow{Uf} UG \xrightarrow{Uf'} UG'' \longrightarrow 1$$

is a split exact sequence.

(2.2) <u>Theorem</u>. Let $U\colon \underline{X} \longrightarrow Y$ preserve finite limits and

$$1 \longrightarrow G' \longrightarrow G \longrightarrow G'' \longrightarrow 1$$

be a U-split exact sequence. Then there is a natural map δ: $H^oG'' \longrightarrow H^1(U,G')$ such that the resulting sequence

$$\begin{tikzcd}
1 \arrow[r] & H^oG' \arrow[r] & H^oG \arrow[r] & H^oG'' \arrow[dll] \\
& H^1(U,G') \arrow[r] & H^1(U,G) \arrow[r] & H'(U,G'')
\end{tikzcd}$$

is exact, the last four terms being exact as a sequence of pointed sets.

Proof. One can easily show that $1 \longrightarrow G' \longrightarrow G \longrightarrow G''$ being an exact sequence in Gp $\underline{X}$ is equivalent to

$$1 \longrightarrow (-,G') \longrightarrow (-,G) \longrightarrow (-,G'')$$

being an exact sequence of group valued functors on $\underline{X}$ (cf. I.(5.10)). In particular, evaluated at 1, we get

$$1 \longrightarrow (1,G') \longrightarrow (1,G) \longrightarrow (1,G'')$$

is exact, which gives the exactness of half of the sequence. The next step is to give the connecting map. Suppose $d\colon 1 \longrightarrow G''$ is given (we identify $(1,G'')$ with Aut G''). Let X be the pullback in the diagram

$$\begin{tikzcd} X \arrow[r, "t"] \arrow[d, "g"'] & 1 \arrow[d, "d"] \\ G \arrow[r] & G'' \end{tikzcd} .$$

Since $G \longrightarrow G''$ is a U-split epimorphism and U preserves pullback, $X \longrightarrow 1$ is also a U-split epimorphism. A map

$$a\colon G' \times X \longrightarrow X$$

is defined by $t.a = t$ and $g.a = (f.p_1)(g.p_2)$. Recall that t denotes everybody's terminal map, p_1 and p_2 are coordinate projections, and q.a is to be the product in the group $\underline{X}(G' \times X,G)$ of $(f.p_1)$ and $(q.p_2)$ We see that a is well defined from

$f'.(f.p_1)(q.p_2) = (f'.f.p_1)(f'.q.p_2) = (u.p_1)(d.t.p_2) = u(d.t) = d.t.$ Here we use the fact that f' is a homomorphism of group objects. To see that this gives X the structure of a a principal G-object — evidently U-split — it suffices to consider the situation in $\underline{S}$. There d picks out a point of G" and X is the inverse image of that point, operated on by left translation by G'. It is evidently isomorphic to G' in that case and so, in general, is a principal G'-object whose class we denote by $\delta(d)$.

(2.3) <u>Proposition</u>. The sequence

$$H^{o}G \longrightarrow H^{o}G'' \longrightarrow H^{1}(U,G')$$

is exact.

Proof. Refering to the definition of $\delta(d)$ above, we see that if d lifts to a map $1 \longrightarrow G$, this gives a splitting of $X \longrightarrow 1$ by the pullback property. The converse is trivial.

(2.4) <u>Proposition</u>. The sequence

$$H^{o}G'' \longrightarrow H^{1}(U,G') \longrightarrow H^{1}(U,G)$$

is exact.

Proof. If $d\colon 1 \longrightarrow G''$ is given, and X is a principal G'-object representing $\delta(d)$, X comes equipped with a map $X \xrightarrow{q} G$, easily seen to be G'-linear. From the adjointness

$$\mathrm{Hom}_{G'}(X,G) \xrightarrow{\sim} \mathrm{Hom}_G(G \otimes_{G'} X, G)$$

we see that there is a map $G \otimes_{G'} X \longrightarrow G$ and so they are isomorphic. Conversely, if they are isomorphic, there is a map $X \xrightarrow{q} G$. Consider the diagram

$$\begin{tikzcd} G' \times X \arrow[r, shift left, "a"] \arrow[r, shift right, "p_2"'] & X \arrow[r] \arrow[d, "q"] & 1 \\ & G \arrow[r, "f'"] & G''. \end{tikzcd}$$

Since $(a,p_2)\colon G' \times X \xrightarrow{\sim} X \times X$ and $X \twoheadrightarrow 1$, the top row is a coequalizer. The facts that $f'.f = u$ and q is a G'-linear morphism imply that $f'.q.a = f'.q.p_2$ (e.g., use the metatheorem) and hence a map $d\colon 1 \longrightarrow G''$ is induced making the square commute. If $\delta(d)$ is represented by an $X' \in \underline{PO}(G')$, the properties of pullback give a map $X \longrightarrow X'$, easily seen to be a G-morphism and hence an isomorphism.

(2.5) Proposition. The sequence

$$H^1(U,G') \longrightarrow H^1(U,G) \longrightarrow H^1(U,G'')$$

is exact.

Proof. The composite map is $f_!.f'_! = (f'.f)_! = u_!$, which is trivial by IV(3.9). To go the other way, suppose that $G'' \otimes_G X \cong G''$. The front adjunction gives a map $X \longrightarrow G'' \otimes_G X$ and we see from the commutative diagram

$$\begin{tikzcd}
G \times X \arrow[r] \arrow[d, two heads] & G' \times X \arrow[d, two heads] \\
X \arrow[r] & G' \otimes_G X
\end{tikzcd}$$

that $X \twoheadrightarrow G' \otimes_G X$ Then we may pull this back along any $1 \longrightarrow G' \otimes_G X$ to obtain

$$\begin{tikzcd}
X' \arrow[r, two heads] \arrow[d] & 1 \arrow[d] \\
X \arrow[r, two heads] & G' \otimes_G X .
\end{tikzcd}$$

The map $G' \times X' \longrightarrow G \times X \longrightarrow X$ gives X' the structure of a G' object. Applying U, we get a pullback square

$$\begin{tikzcd}
UX' \arrow[r] \arrow[d] & 1 \arrow[d] \\
UG \arrow[r] & UG'' .
\end{tikzcd}$$

Since $UG \longrightarrow UG''$ is a split epimorphism, so is $UX' \longrightarrow 1$. Similarly, we may use the metatheorem to see that $X' \in$ PLO(G'). Finally, the map $X' \longrightarrow X$, easily seen to be a G'-morphism, gives a G-isomorphism $G \otimes_{G'} X' \xrightarrow{\sim} X$. This completes the proof of (1.2).

3. Abelian groups.

(3.1) In this section we consider the special case of the theorem (2.2) in which G is abelian. To emphasize this fact, we use A instead of G throughout this section to denote an abelian group object of $\underline{X}$. Ab $\underline{X}$ denotes the category of abelian group objects of $\underline{X}$ and morphisms of groups. The first observation we have is an immediate consequence of I.(3.11) and I.(5.11).

(3.2) Theorem: Let $\underline{X}$ be an exact category. Then Ab $\underline{X}$ is abelian.

(3.3) When A is abelian $\underline{LO}(A)$ can be embedded as a full subcategory of $\underline{BO}(A)$ as the subcategory of symmetric objects. Namely, given an $a: A \times X \longrightarrow X$ making X into a left A-object, X becomes a right A-object, indeed a 2-sided A-object, via the composite

$$X \times A \longrightarrow A \times X \xrightarrow{a} X,$$

in which the first morphism is the switching isomorphism. Via this embedding we may consider the tensor product as defining a functor

$$- \otimes - : \underline{LO}(A) \times \underline{LO}(A) \longrightarrow \underline{BO}(A).$$

(3.4) Proposition. The image of the isomorphism above is contained in $\underline{LO}(A)$.

Proof. In sets, a symmetric 2-sided A-object X satisfies $ax = xa$. In $X \otimes_A Y$, we have $a(x \otimes y) = ax \otimes y = xa \otimes y = x \otimes ay = x \otimes ya =$ $(x \otimes y)a$, given that both X and Y are symmetric. Now use the meta-theorem.

(3.5) Proposition. The image of $- \otimes -$ restricted to $\underline{PLO}(A) \times \underline{PLO}(A)$ is contained in $\underline{PLO}(A)$.

Proof. Using IV.(2.11), IV.(2.15) and IV.(3.7), we have, for X, $Y \in \underline{PLO}(A)$, and for exact $U: \underline{X} \longrightarrow \underline{S}$,

$$U(X \otimes_A Y) \cong UX \otimes_{UA} UY \cong UA \otimes_{UA} UA \cong UA,$$

whence by again applying IV.(3.7) $X \otimes_A Y \in \underline{PLO}(A)$.

(3.6) Proposition. The functor $- \otimes_A - : \underline{LO}(A) \times \underline{LO}(A) \longrightarrow \underline{LO}(A)$ is associative, commutative, and unitary up to jointly coherent isomorphism.

Proof. Prove it in $\underline{S}$ and use the metatheorem.

(3.7) Corollary. The set $H^1(A)$ is an abelian monoid, the product being induced by $- \otimes_A -$.

(3.8) Theorem. $H^1(A)$ is an abelian group with respect to the tensor product.

Proof. We need only show that there are inverses. Let $X \in \underline{LO}(G)$ have structure map $a: A \times X \longrightarrow X$ and $i: A \longrightarrow A$ be the inverse map of A, a homomorphism since A is commutative. Let $X^{\#}$ denote X with structure map

$$A \times X \xrightarrow{i \times X} A \times X \xrightarrow{a} X.$$

An application of the embedding shows that it is principal. Let $b: X \times X \longrightarrow A$ be the composite

$$X \times X \xrightarrow{(a,p_2)^{-1}} A \times X \xrightarrow{p_1} A$$

from which $(a,p_2)^{-1} = (b,p_2)$. Now consider

$$\begin{tikzcd} X \times A \times X^{\#} \arrow[r, shift left] \arrow[r, shift right] & X \times X^{\#} \arrow[r] \arrow[d, "b"] & X \otimes_A X^{\#} \arrow[dl, dashed] \\ & A & \end{tikzcd}$$

which makes sense since X and $X^{\#}$ are the same object of $\underline{X}$. In sets,

$A \cong X$, and we may suppose $A = X$. In that case, $a\colon A \times A \longrightarrow A$ is addition and we may easily check that $b\colon A \times A \longrightarrow A$ is subtraction, $p_1 - p_2$. Then b coequalizes the two maps $X \times A \times X^{\#}$ to $X \times X^{\#}$. Then there is induced a map $X \otimes_A X^{\#} \longrightarrow A$, easily seen to be an A-morphism, hence an isomorphism. The metatheorem allows us to pull this argument back to $\underline{X}$.

(3.9) <u>Proposition</u>. If $U\colon \underline{X} \longrightarrow \underline{Y}$ preserves finite limits, $H^1(U,A)$ is a subgroup of $H^1(A)$.

Proof. If UX_1 and UX_2 are split, then we have a map $1 \longrightarrow UX_1 \times UX_2 \cong U(X_1 \times X_2) \longrightarrow U(X_1 \otimes_A X_2)$, the latter being this image under U of the natural projection $X_1 \times X_2 \longrightarrow X_1 \otimes_A X_2$. If $X \otimes_A X^{\#} \cong A$, then X and $X^{\#}$ are isomorphic in $\underline{X}$, so UX splits if and only if $U(X^{\#})$ does. Finally, the trivial class, that of A, splits already in $\underline{X}$.

(3.10) <u>Theorem</u>: Let $U\colon \underline{X} \longrightarrow \underline{Y}$ preserve finite limits and $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ be a U-split exact sequence in Ab $\underline{X}$. Then the sequence of (2.2) is an exact sequence of abelian groups.

Proof. $0 \longrightarrow H^0(A') \longrightarrow H^0(A) \longrightarrow H^0(A'')$ is obviously exact in $\underline{Ab}$. For $g\colon B \longrightarrow B'$, the induced map $H^1(U,B) \longrightarrow H^1(U,B_1)$ is given by $X \longmapsto B' \otimes_B X$. Using (3.6), we have $(B' \otimes_B X_1) \otimes_{B'} (B' \otimes_B X_2)$ $\cong ((B' \otimes_B X_1) \otimes_{B'} B') \otimes_B X_2) \cong (B' \otimes_B X_1) \otimes_B X_2 = B' \otimes_B (X_1 \otimes_B X_2)$ so that the induced map $H^1(U,B) \longrightarrow H^1(U,B')$ is an abelian group homomorphism. In particular

$$H^1(U,A') \longrightarrow H^1(U,A) \longrightarrow H^1(U,A'')$$

is an exact sequence of abelian groups. Thus we need only show that the connecting homomorphism $\delta\colon H^0(A'') \longrightarrow H^1(U,A')$ is additive. That is, given

$$\begin{array}{ccc} X_1 & \longrightarrow & 1 \\ {\scriptstyle q_1}\downarrow & & \downarrow{\scriptstyle d_1} \\ A & \longrightarrow & A'' \end{array} \qquad\qquad \begin{array}{ccc} X_2 & \longrightarrow & 1 \\ {\scriptstyle q_2}\downarrow & & \downarrow \\ A & \longrightarrow & A'' \end{array}$$

pullback squares, we must show that there is a pullback square

$$\begin{array}{ccc} X_1 \otimes_A X_2 & \longrightarrow & 1 \\ \downarrow & & \downarrow{\scriptstyle d_1 + d_2} \\ A & \longrightarrow & A'' \end{array}$$

As in the proof of (1.10), it is sufficient merely to exhibit a commutative square of that sort. Consider the diagram

$$\begin{array}{ccccc} X_1 \times A' \times X_2 & \rightrightarrows & X_1 \times X_2 & \longrightarrow & X_1 \otimes_{A'} X_2 \\ & & \downarrow{\scriptstyle q_1 \times q_2} & & \\ & & A \times A & & \\ & & \downarrow{\scriptstyle m} & & \\ & & A & & \end{array}$$

where m is the addition. By applying the metatheorem we see that the vertical map coequalizes the given maps and induces $X_1 \otimes_{A'} X_2 \longrightarrow A$. Another application of the embedding (or a simple direct argument based on the facts that m induces the addition in $(-,A)$ and that $A \longrightarrow A''$ is a homomorphism) shows that

$$\begin{array}{ccc} X_1 \otimes X_2 & \longrightarrow & 1 \\ \downarrow & & \downarrow{\scriptstyle d_1 + d_2} \\ A & \longrightarrow & A'' \end{array}$$

commutes.

4. Extensions.

(4.1) Consider an exact category $\underline{X}$ and a fixed object X. Then $\underline{Y} = (\underline{X}, X)$ is also exact by I.(5.4). This category also has a terminal object, $X \longrightarrow X$, by the identity map. A map $Y \twoheadrightarrow X$ will be called an extension of X. If G is a group of $\underline{Y}$, we say that G is an X-group. A principal G-object is a $Y \twoheadrightarrow X$ on which G operates principally. It is in particular an extension and will be called a singular extension with kernel G. $G \twoheadrightarrow X$ itself will be called the split extension with kernel G. Note that the unit law shows up in this case as a map $X \longrightarrow G$ which splits $G \twoheadrightarrow X$, so that this really is a split epimorphism. In particular, a U-split extension is one which really splits when U is applied.

(4.2) Suppose $\underline{X}$ is the category $\underline{Gp}$ of groups and $X \in \underline{X}$ is a fixed group. Then an X-group G is a $G \twoheadrightarrow X$ whose group law considered as a map $G \times_X G \longrightarrow G$ is a homomorphism of groups. Since $G \twoheadrightarrow X$ is split, G is a semi-direct product $X \times M$ where M is the kernel of $G \longrightarrow X$. $G \times_X G$ is $X \times M \times M$ and it is a moment's calculation to see that M must be abelian and that G operates on M as a G-module.

(4.3) If

$$0 \longrightarrow M \longrightarrow G \longrightarrow X \longrightarrow 1$$

and

$$0 \longrightarrow M \longrightarrow Y \longrightarrow X \longrightarrow 1$$

are (still in the category of groups) two singular extensions of X with kernel M, the upper being split, then we can form the pullbacks

$$\begin{array}{ll}
(1) & 0 \longrightarrow M \longrightarrow G \longrightarrow X \longrightarrow 1 \\
 & \qquad\quad \| \qquad\quad \uparrow p_1 \qquad\qquad \uparrow \\
(2) & 0 \longrightarrow M \longrightarrow G \times_X Y \xrightarrow{a} Y \longrightarrow 1
\end{array}$$

$$\begin{tikzcd}
(3) & 0 \arrow[r] & M \arrow[r] \arrow[d, equal] & Y \times_X Y \arrow[r, "p_1"] \arrow[d, "p_2"] & Y \arrow[r] \arrow[d] & 1 \\
(4) & 0 \arrow[r] & M \arrow[r] & Y \arrow[r] & X \arrow[r] & 1.
\end{tikzcd}$$

Both sequences (2) and (3) split, the first because (1) is split and the second by the diagonal $Y \longrightarrow Y \times_X Y$. It is a familiar fact in extension theory (and reappears as IV.(3.6) in this formulation) that any two split sequences are equivalent, which means that

$G \times_X Y \xrightarrow{(a,p_2)} Y \times_X Y$ is an equivalence. It can be seen directly (e.g. use the metatheorem) that a determines an action, evidently principal, of G on Y. Note,of course, that fibred product over X is precisely cartesian product in $\underline{Y}$.

(4.4) Considering the same diagram, we see that $(a,p_2): G \times_X Y \longrightarrow Y \times_X Y$ gives that $G \times_X Y$ and $Y \times_X Y$ are extensions of Y with the same kernel M, which implies that G and Y are extensions of X with the same kernel M, the first being split. Hence we have shown:

(4.5) <u>Theorem</u>. Let X be a group, M an X-module, G the split extension of X with kernel M. Then singular extensions of X with kernel M are equivalent to principal G-objects in $(\underline{Gp},X)$. Equivalent extensions correspond to isomorphic objects of $\underline{PLO}(G)$.

Proof. We have shown everything but the last, but that is obvious.

(4.6) <u>Proposition</u>. Let M,X,G be as above. Then

$$\mathrm{Der}(X,G) \cong (\underline{Gp},X)(X \longrightarrow X, G \longrightarrow X).$$

Proof. Note that the last is $\underline{Y}(1,G) = H^0(G)$. The proof is easy and also well-known. See the remark in the middle of p.255 of [4].

(4.7) Thus we have identified $H^0(G)$ with $H^0(X,M) = \mathrm{Der}(X,M)$ and

$H^1(G)$ with $H^1(X,M)$, the usual group of singular extensions of (2.2) corresponds, as far as it goes, with the usual one. It is also evident that the identical analysis would work for any of the standard equational categories: associative, commutative, Lie, Jordan rings or algebras, etc. In each of those categories, as well as any equational category in which there is a group law among the operations, each group object must be abelian.

(4.8) In all these categories of algebras we might consider a relative cohomology, relative to some suitable functor. In the common examples this functor is algebraic, i.e. induced by a map of triples, and hence exact. The most common is the underlying functor from a category of K-algebras of some type to K-modules. In that case the relative cohomology classifies, in dimension one, those singular extensions which are split as K-modules. The Hochschild cohomology of associative algebras is of this form, while the corresponding absolute cohomology was given by Shukla. See [BB] for some of the details and further references.

(4.9) The Baer sum of singular extensions is defined in the following way. Given

$$\begin{tikzcd} & & Y_1 \arrow[dr] & & & \\ 0 \arrow[r] & M \arrow[ur] \arrow[dr] & & X \arrow[r] & 1 & , \\ & & Y_2 \arrow[ur] & & & \end{tikzcd}$$

two extensions with the same kernel, we first form $Y_1 \times_X Y_2$ and then observe that there are two embeddings $M \rightrightarrows Y_1 \times_X Y_2$. When these are rendered equal (or coequalized), the result is the Baer sum. We may indicate the process as

$$M \rightrightarrows Y_1 \times_X Y_2 \longrightarrow Y_1 \divideontimes Y_2,$$

where $Y_1 \divideontimes Y_2$ is the Baer sum. In our generality, the embeddings $M \longrightarrow Y_i$ are replaced by actions $G \times Y_i \longrightarrow Y_i$, $i = 1,2$. The fibred product $Y_1 \times_X Y_2$ is simply the product in the category $(\underline{Gp}, X)$. Thus it seems more or less likely and is trivial to prove that the above sequence corresponds to our definition of the product in $H^1(G)$ (G commutative) given by the following diagram being a coequalizer:

$$Y_1 \times G \times Y_2 \rightrightarrows Y_1 \times Y_2 \longrightarrow Y_i \otimes_G Y_2.$$

This proves:

(4.10) Theorem: The equivalence between $H'(G)$ and $H^1(X,M)$ given by (3.5) takes the tensor product multiplication in the first to the Baer sum in the second. Analogous results hold in the relative case.

Appendix: Giraud's theorem.

(A.1) After the completion of the five preceding chapters, I received from Ira Wolf a sketch of his proof of the Giraud theorem characterizing toposes. As I read it I realized that exact categories made a very convenient setting for the proof. This appendix presents a proof given along these lines. The proof is actually much closer to the one published by Verdier [Ve] than to Wolf's. It differs from the former in that it treats the question entirely in terms of Grothendieck topologies (in the sense of Artin) and that it involves neither a change of universe nor any essential use of an illegitimate category.

(A.2) The following terminology will be used throughout.
Let $\underline{C}$ be a category, C an object, $F: \underline{C}^{op} \longrightarrow \underline{S}$ a functor. A family of maps to C, $\{C_i \longrightarrow C\}$, is called a sieve (or a sieve on C). A sieve is called an F-sieve if every $C_i \times_C C_j$ exists and

$$FC \longrightarrow \Pi FC_i \rightrightarrows \Pi F(C_i \times_C C_j)$$

is an equalizer. It is called a universal F-sieve if for $C' \longrightarrow C$, every $C' \times_C C_i$ exists and $\{C' \times_C C_i \longrightarrow C'\}$ is an F-sieve. It is evident that if it is a universal F-sieve, then $\{C' \times_C C_i \longrightarrow C'\}$ will be universal also. If C" is an object of $\underline{C}$, a sieve is called a (universal)C"-sieve if it is a (universal) (-,C")-sieve. It is called a regular epimorphic sieve if it is a C"-sieve for every object C" of $\underline{C}$ (this is an evident generalization of $\twoheadrightarrow$) and a universal regular epimorphic sieve if it is a universal C"-sieve for every C" of $\underline{C}$. These last two notions will be abbreviated r.e.s. and u.r.e.s. respectively.

(A.3) <u>Proposition</u>. Let $\{C_i \longrightarrow C\}$, and for each i, $\{C_{ij} \longrightarrow C_i\}$ be universal F-sieves. Then $\{C_{ij} \longrightarrow C\}$ is one also.

Proof. It is sufficient to show it is an F-sieve, since pullback commutes with composition. In order to do this we need the following lemma.

(A.4) Lemma. Let the diagram

$$\begin{tikzcd}
X_o \arrow[r, "d"] \arrow[dr, "f"'] & Y_o \arrow[r, shift left, "d^o"] \arrow[r, shift right, "d^1"'] \arrow[d, "e"] & Z_o \arrow[dd, tail, "g"] \\
& Y_1 \arrow[d, shift right, "e^o"'] \arrow[d, shift left, "e^1"] \arrow[dr, shift left, "f^1"] \arrow[dr, shift right, "f^o"'] & \\
& Y_2 & Z_2 \arrow[l, "h"]
\end{tikzcd}$$

commute (that is, with d^o, e^o, f^o and with d^1, e^1, f^1), g be a monomorphism and e be the equalizer of e^o and e^1. Then d is the equalizer of d^o and d^1 if and only if f is the equalizer of f^o and f^1.

Proof. Chase the diagram.

(A.5) Now we return to the proof of (1.3). Apply the lemma with

$X_o = FC$, $Y_o = \prod_i FC_i$,

$Z_o = \prod_{i,k} F(C_i \times_C C_k)$, $Y_1 = \prod_{i,j} FC_{ij}$,

$Y_2 = \prod_{i,j,\ell} F(C_{ij} \times_{C_i} C_{i\ell})$, $Z_2 = \prod_{i,j,k,\ell} F(C_{ij} \times_C C_{k\ell})$.

The maps e and d are equalizers by assumption and we need only define h and show g is a monomorphism. The former is easily done by product projections. As for the latter, we define $Z_1 = \prod_{i,j,k} F(C_{ij} \times_C C_k)$. Now $\{C_{ij} \longrightarrow C_i\}$ is a universal F-sieve, so that by pulling back along the projection $\{C_i \times_C C_k \longrightarrow C_i\}$ we find that $\{C_{ij} \times_C C_k \longrightarrow C_i \times_C C_k\}$ is an F-sieve. This implies at least that $F(C_i \times_C C_k) \rightarrowtail \prod_j F(C_{ij} \times_C C_k)$ or that

$$\prod_{i,k} F(C_i \times_C C_k) \rightarrowtail \prod_{i,j,k} F(C_{ij} \times_C C_k),$$

which is $Z_o \rightarrowtail Z_1$. Similarly, $\{C_{kl} \longrightarrow C_k\}$ is a universal F-sieve, and by pulling it back along $C_{ij} \times_C C_k \longrightarrow C_k$ we see that $\{C_{ij} \times_C C_{kl} \longrightarrow C_{ij} \times_C C_k\}$ is an F-sieve too. Thus

$$F(C_{ij} \times_C C_k) \rightarrowtail \prod_{\ell} F(C_{ij} \times_C C_{k\ell}),$$

and by taking products over i,j,k we find $Z_1 \rightarrowtail Z_2$.

(A.6) <u>Proposition</u>. If $\{C_i \longrightarrow C\}$, and for each $\{C_{ij} \longrightarrow C_i\}$ are u.r.e.s, then so is $\{C_{ij} \longrightarrow C\}$.

(A.7) From the previous proposition it is clear that the class of all u.r.e.s. in a category $\underline{C}$ forms a topology, called the canonical topology. Any topology less fine than the canonical topology is called a standard topology.

(A.8) Another consequence of this proposition is that the usual assumption in a Grothendieck topology that the composition of covers is a cover (I.(4.1).b) is unnecessary. In fact, it is an easy corollary that given an arbitrary collection of sieves, the sheaves for the coarsest topology it generates are exactly those F for which every one of the given sieves is a universal F-sieve.

(A.9) <u>Proposition</u>. Let $\underline{C}$ have pullbacks. Then a topology on $\underline{C}$ is a standard topology if and only if every representable functor is a sheaf.

The proof is very easy and is omitted.

(A.10) Let $\underline{E}$ be a category. $\underline{E}$ is called a topos if

a) $\underline{E}$ has finite limits.

b) $\underline{E}$ has disjoint universal sums.

c) $\underline{E}$ is exact.

d) $\underline{E}$ has a set of generators.

The precise meanings of these follow. a) is clear. b) means that for every family $\{E_i\}$ of objects there is a sum $\coprod E_i$; that the square

$$\begin{tikzcd} \delta_{ij}E_i \arrow[r] \arrow[d] & E_i \arrow[d] \\ E_j \arrow[r] & \coprod E_i \end{tikzcd}$$

is a pullback where

$$\delta_{ij}\ E_i = \begin{cases} E_i & \text{if } i = j \\ 0, & \text{the initial object, when } i \neq j; \end{cases}$$

and that given $E_i \longrightarrow E \longleftarrow E'$, $E' \times_E \coprod E_i \cong \coprod(E' \times_E E_i)$ by the natural map. By interpreting this condition when $i \in \emptyset$, we see that $E' \times_E 0 = 0$ for any $E' \longrightarrow E$ and if $E' \longrightarrow 0$, that $E' \cong E' \times_0 0 \cong 0$. This implies that 0 is empty and will henceforth be denoted by $\emptyset$. c) is used in the sense of this paper and d) in the sense of II.(1.3); that is, there is a set Γ of objects such that for any $E \rightarrowtail E'$ not an isomorphism there is a $G \in \Gamma$ and a map $G \longrightarrow E'$ which does not factor through E.

(A.11) Theorem (Giraud). Let $\underline{E}$ be a category. Then the following are equivalent.

a) There is a small category $\underline{C}$ with finite limits such that $\underline{E} =$ $= \mathfrak{F}(\underline{C}^{op}, \underline{S})$ for the canonical topology on $\underline{C}$.

b) There is a small category $\underline{C}$ such that $\underline{E} = \mathfrak{F}(\underline{C}^{op}, \underline{S})$, sheaves for some topology on $\underline{C}$.

c) There is a small category $\underline{C}$ and a full embedding $I: \underline{E} \longrightarrow (\underline{C}^{op}, \underline{S})$ which has an exact left adjoint.

d) $\underline{E}$ is a topos.

e) $\underline{E} = \mathfrak{F}(\underline{E}^{op},\underline{S})$,(canonical topology) and has a set of generators.

(A.12) It is obvious that a) $\Longrightarrow$ b). That b) $\Longrightarrow$ c) is found in [Ar] and since the setting of exact categories in no way improves his proof, we omit it. The only thing to note in this connection is that if $P \rightarrowtail F$ where $P,F\colon \underline{C}^{op} \longrightarrow \underline{S}$ and F is a sheaf (in some topology), then the sheaf P^* associated to P is the subfunctor of F gotten by adding to PC every point in $FC \cap \Pi FC_i$ where $\{C_i \longrightarrow C\}$ is a cover in the topology. This obviously works even when $\underline{C}$ is large and the associated sheaf functor may not exist. The P^* so constructed can easily be seen to have the required universal mapping property: $(P^*,F) \cong (P,F)$ when F is a sheaf.

(A.13) Proposition. Condition c) $\Longrightarrow$ condition d).

Proof. Suppose $I\colon \underline{E} \longrightarrow (\underline{C}^{op},\underline{S})$ is a full embedding with left adjoint J. Then sums (as well as other colimits) are computed in $\underline{E}$ by $\coprod E_i = J \coprod IE_i$. We leave to the reader the easy task of showing that $(\underline{C}^{op},\underline{S})$ is itself a topos. In what follows we automatically identily the composite JI with the identity functor on $\underline{E}$. Then for a family $\{E_i\}$ of objects of $\underline{E}$.

$$\begin{tikzcd}
\delta_{ij} IE_i \arrow[r] \arrow[d] & IE_i \arrow[d] \\
IE_j \arrow[r] & \coprod IE_i
\end{tikzcd}$$

is a pullback. If we apply J and recall that J preserves initial objects, we get that

$$\begin{tikzcd}
\delta_{ij} E_i \arrow[r] \arrow[d] & E_i \arrow[d] \\
E_j \arrow[r] & \coprod E_i
\end{tikzcd}$$

is a pullback. Similarly, given $E_i \longrightarrow E$ and $E' \longrightarrow E$, we have

$$E' \times_E \coprod E_i \cong JIE' \times_{JIE} J(\coprod IE_i)$$

$$\cong J(IE' \times_{IE} \coprod IE_i) \cong J(\coprod IE' \times_{IE} IE_i)$$

$$\cong J(\coprod I(E' \times_E E_i)) \cong \coprod(E' \times_E E_i).$$

Thus $\underline{E}$ has universal disjoint sums. If

$$\begin{tikzcd} E'_o \arrow[r] \arrow[d] & E'_1 \arrow[d] \\ E_o \arrow[r, two heads] & E_1 \end{tikzcd}$$

is a pullback in $\underline{E}$, apply I and factor $IE_o \longrightarrow IE_1$ to get

$$\begin{tikzcd} IE'_o \arrow[r, two heads] \arrow[d] & F' \arrow[r, tail] \arrow[d] & IE'_1 \arrow[d] \\ IE_o \arrow[r, two heads] & F \arrow[r, tail] & IE_1 \end{tikzcd} .$$

F' is defined to make the right hand square a pullback, and since the whole square is a pullback, so is the left hand square, whence $IE'_o \twoheadrightarrow F'$ as shown. The functor J preserves both $\twoheadrightarrow$ and $\rightarrowtail$ (the latter because it preserves finite limits), so we can apply I to get

$$\begin{tikzcd} E'_o \arrow[r, two heads] \arrow[d] & JF' \arrow[r, tail] \arrow[d] & E'_1 \arrow[d] \\ E_o \arrow[r, two heads] & JF \arrow[r, tail] & E_1 \end{tikzcd} .$$

in which both squares are pullbacks. But since $E_o \twoheadrightarrow E_1$, it follows that $JF \twoheadrightarrow E_1$, whence $JF \xrightarrow{\sim} E_1$, and then $JF' \xrightarrow{\sim} E'_1$, which implies that $E'_o \twoheadrightarrow E'_1$. Thus the pullback of a regular epimorphism is also a regular epimorphism.

Suppose $E_1 \rightrightarrows E_o$ is an equivalence on E_o. It is clear from I(5.3) that a limit preserving functor preserves equivalence relations, so that there is an exact sequence

$$IE_1 \rightrightarrows IE_o \longrightarrow F$$

in $(\underline{C}^{op},\underline{S})$, and since J is exact,

$$E_1 \rightrightarrows E_o \longrightarrow JF$$

is an exact sequence as well. Thus $\underline{E}$ is exact.

Finally, if $E \rightarrowtail E'$ is not an isomorphism, it follows, since I is full and limit preserving, that $IE \rightarrowtail IE'$ and is not an isomorphism. This means there is a $C \in \underline{C}$ with $IEC \rightarrowtail IE'C$ not an isomorphism or, by the Yoneda lemma, a map $(-,C) \longrightarrow IE'$ which does not factor through IE. In view of adjointness, this is the same as a map $J(-,C) \longrightarrow E'$ which does not factor through E. Thus the objects $J(-,C)$, $C \in \underline{C}$ generate $\underline{E}$.

This completes the proof of (A.13).

(A.14) Now we turn our attention to showing d) $\Longrightarrow$ e). Until that is finished, $\underline{E}$ denotes a topos; $\mathfrak{F}(\underline{E}^{op},\underline{S})$, the category of sheaves in the canonical topology; and $R: \underline{E} \longrightarrow \mathfrak{F}(\underline{E}^{op},\underline{S})$, the embedding as representable functors.

(A.15) Proposition. R is exact.

Proof. The proof of I(4.3) is equally valid for any topology less fine than the canonical and finer than the regular epimorphism topology.

(A.16) Proposition. Let F be a sheaf. Then $F(\coprod E_i) = \Pi FE_i$ for any family of objects E_i of $\underline{E}$.

Proof. First observe that $\{E_i \longrightarrow \emptyset\}_{i\in\emptyset}$ is a cover. This is so since for any E",

$$= (\emptyset,E") \longrightarrow \prod_{i\in\emptyset} (E_i,E") \rightrightarrows \prod_{i,j\in\emptyset} E_i \times_{\emptyset} E_j,E")$$

is an equalizer, while there are no non-trivial $E' \longrightarrow \emptyset$ to pull back along. Replacing $(-,E")$ by any sheaf F, we see that $F\emptyset = 1$. Now let $E = \coprod E_i$. Since $E_i \times_E E_j = \delta_{ij} E_i$ we have, for any $E"$, that

$$(E,E") \longrightarrow \Pi(E_i,E") \rightrightarrows \Pi(E_i \times_E E_j,E")$$

is an equalizer (all maps being isomorphisms). Hence $\{E_i \longrightarrow E\}_{i\in\emptyset}$ is an r.e.s. and, using the universality of the sums, it is easily seen to be a u.r.e.s. Then for any sheaf F,

$$FE \longrightarrow \Pi FE_i \rightrightarrows \Pi F(E_i \times_E E_j)$$

is an equalizer. Since $E_i \times_E E_j = \delta_{ij} E_i$ and $F\emptyset = 1$, the third term is the same as the second, which implies that $FE = \Pi FE_i$.

(A.17) Proposition. R preserves sums.

Proof. For any F and any $\{E_i\}$, $(R \coprod E_i, F) = F(\coprod E_i) = \Pi FE_i = \Pi(RE_i,F) = (\coprod RE_i, F)$.

(A.18) Proposition. Every map of $\mathfrak{F}(\underline{E}^{op},\underline{S})$ factors as $\cdot \twoheadrightarrow \cdot \rightarrowtail \cdot$.

Proof. Let $F' \longrightarrow F$ be a map. Let P be the image as a functor. Then

$$F' \times_F F' \rightrightarrows F' \twoheadrightarrow P$$

is an exact sequence of functors and $F' \times_F F'$ is a sheaf. Since $P \rightarrowtail F$, P has an associated sheaf $P^* \rightarrowtail F$, which satisfies the universal mapping property that for $F"$ a sheaf, $(P,F") = (P^*,F")$. From this, we see that $F' \times_F F' \rightrightarrows F' \longrightarrow P^*$ is exact in $\mathfrak{F}(\underline{E}^{op},\underline{S})$ while $P^* \rightarrowtail F$ (see (A.12)).

(A.19) Proposition. A sieve $\{E_i \longrightarrow E\}$ is a cover in the canonical topology if and only if $\coprod E_i \twoheadrightarrow E$.

Proof. The "only if" is trivial. Suppose $\coprod E_i \twoheadrightarrow E$. Then

$$(\coprod E_i) \times_E (\coprod E_i) \rightrightarrows \coprod E_i \longrightarrow E$$

is exact. The kernel pair is

$$\coprod E_i \times_E \coprod E_i \cong \coprod (E_i \times_E \coprod E_j) = \coprod (E_i \times_E E_j),$$

so that

$$\coprod (E_i \times_E E_j) \rightrightarrows E_i \longrightarrow E$$

is exact, from which

$$(E,E') \longrightarrow \Pi(E_i,E') \rightrightarrows \Pi(E_i \times_E E_j, E')$$

is an equalizer for all E and $\{E_i \longrightarrow E\}$ is an r.e.s. The universality follows easily from that of sums.

(A.20) Proposition. The set of objects RG, with $G \in \Gamma$, is a set of generators for $\mathfrak{F}(\underline{E}^{op}, \underline{S})$.

Proof. Suppose $F \rightarrowtail F'$ is a monomorphism of sheaves such that $FG \xrightarrow{\sim} F'G$ for each $G \in \Gamma$. We will show that $F \xrightarrow{\sim} F'$. Let B be an object and find $\coprod G_i \twoheadrightarrow E$ with each $G_i \in \Gamma$. Then $\{G_i \longrightarrow E\}$ is a cover and hence we have the commutative diagram

$$\begin{tikzcd}
FE \arrow[r] \arrow[d, tail] & \Pi F G_i \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\cong"'] & \Pi F(G_i \times_E G_j) \arrow[d, tail] \\
F'E \arrow[r] & \Pi F' G_i \arrow[r, shift left] \arrow[r, shift right] & \Pi F'(G_i \times_E G_j),
\end{tikzcd}$$

whose rows are equalizers, and an easy diagram chase shows $FE \xrightarrow{\sim} F'E$.

(A.21) Proposition. For any sheaf F, there is a regular epimorphism $RE \twoheadrightarrow F$.

Proof. Since $\mathfrak{F}(\underline{E}^{op}, \underline{S})$ has $\cdot \twoheadrightarrow \cdot \rightarrowtail \cdot$ factorizations, we can repeat the argument of II(1.4) to see that

$$R\Big(\coprod_{G \in \Gamma} \coprod_{(RG,F)} G\Big) = \coprod \coprod RG \twoheadrightarrow F.$$

Proposition. Every sheaf is representable.

Proof. Consider the sequence

$$F' \rightrightarrows RE \longrightarrow F$$

where $RE \twoheadrightarrow F$ and F' is the kernel pair. Again we can find $RE' \twoheadrightarrow F'$.

Now we have $E' \longrightarrow E \times E$, which factors $E' \twoheadrightarrow E'' \rightarrowtail E \times E$, and since R is exact,

$$RE' \twoheadrightarrow RE'' \rightarrowtail R(E \times E),$$

and by the uniqueness of the factorization, $RE'' \cong F$. Then

$$RE'' \rightrightarrows RE \quad \text{is an}$$

equivalence relation and R is a full exact embedding, so that $E'' \rightrightarrows E$ is one too. Then there is an exact sequence

$$E'' \rightrightarrows E \longrightarrow E''',$$

and again, since R is exact, $RE''' \cong F$.

This completes the proof that d) $\Longrightarrow$ e).

(A.22) From now on $\underline{E}$ will be a category in which every sheaf for the canonical topology is representable. We suppose that $\underline{C}$ is a subcategory of $\underline{E}$ which is closed under subobjects and finite products and which contains a set of generators. Note that every sheaf's being representable implies that $\underline{E}$ has all limits. Our aim is to show that $\underline{E} \cong \mathfrak{F}(\underline{C}^{op},\underline{S})$ for the canonical topology on $\underline{C}$.

We say that a sieve $\{E_i \longrightarrow E\}$ is an extremal sieve if there is no subobject of E which factors each of the maps.

(A.23) Proposition. A sieve in $\underline{E}$ is extremal if and only if it is a cover in the canonical topology.

Proof. The "if" part is easy. For if $E' \rightarrowtail E$ were a subobject factoring all the $E_i \longrightarrow E$, then the fact that $(-,E')$ is a sheaf would provide an inverse to the inclusion $E' \rightarrowtail E$. To go the other way, suppose a sieve is extremal. Let $P\colon \underline{E}^{op} \longrightarrow \underline{S}$ be defined by

$PE_1 = \{f\colon E_1 \longrightarrow E \mid f$ factors through at least one $E_i \longrightarrow E\}$. Then $P \rightarrowtail (-,E)$, and by the remark (A.12) there is a sheaf $P^* \rightarrowtail (-,E)$ associated to P. If $P^* = (-,E')$, then $E' \rightarrowtail E$ factors every $E_i \longrightarrow E$, so $P^* = (-,E)$. Now in the category $(\underline{E}^{op},\underline{S})$,

$$\coprod(-,E_i) \times_P \coprod(-,E_i) \rightrightarrows \coprod(-,E_i) \longrightarrow P$$

is exact. Since $P \rightarrowtail (-,E)$, we have

$$\coprod(-,E_i) \times_P \coprod(-,E_i) \cong \coprod(-,E_i) \times_{(-,E)} \coprod(-,E_i)$$
$$\cong \coprod[(-,E_i) \times_{(-,E)} (-,E_j)] \cong \coprod(-,E_i \times_E E_j),$$

so that

$$\coprod(-,E_i \times_E E_j) \rightrightarrows \coprod(-,E_i) \longrightarrow P$$

is exact. Let E" be an arbitrary object. Then using the fact $(P,(-,E")) = (P^*,(-,E")) = (E,E")$ we hom this sequence into E" and have that

$$(E,E") \longrightarrow \Pi(E_i,E") \rightrightarrows \Pi(E_i \times_E E_j,E")$$

is an equalizer. Hence $\{E_i \longrightarrow E\}$ is an r.e.s. To show the universality, it is sufficient to show that for any $E' \longrightarrow E$, the sheaf associated to $P' = P \times_E (-,E')$ is $(-,E')$ itself. This is easily done by using the remark of (A.18) together with the usual proof that the associated sheaf functor is exact.*

(A.24) <u>Corollary</u>. The topology induced on $\underline{C}$ by the inclusion $\underline{C} \longrightarrow \underline{E}$ is the canonical topology.

Proof. Since $\underline{C}$ is closed under subobjects, a sieve $\{C_i \longrightarrow C\}$ is extremal in $\underline{C}$ if and only if it is in $\underline{E}$.

(A.25) This implies that there is a functor $I\colon \underline{E} \longrightarrow \mathcal{F}(\underline{C}^{op},\underline{S})$. This

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* I am indebted to H. Schubert for pointing out an error in my original proof of this proposition.

functor is faithful, since $\underline{C}$ contains a set of generators of $\underline{E}$. If we can find a $J\colon \mathfrak{F}(\underline{C}^{op},\underline{S}) \longrightarrow \underline{E}$ such that JI = identity, it follows that I is an equivalence. Let $F\colon \underline{C}^{op} \longrightarrow \underline{S}$ be a sheaf. We extend it to a functor $\bar{F}\colon \underline{E}^{op} \longrightarrow \underline{S}$ in what by (A.23) is the only possible way. For $E \in \underline{E}$, choose an extremal sieve

$$\{C_i \longrightarrow E\},\ C_i \in \underline{C},$$

which certainly exists, since $\underline{C}$ contains a set of generators. Now let $\bar{F}E$ be defined so that

$$\bar{F}E \longrightarrow \Pi FC_i \rightrightarrows \Pi F(C_i \times_E C_j)$$

is an equalizer. Note that $C_i \times_E C_j \subset C_i \times C_j$ and hence is an object of $\underline{C}$ for all i,j. There remain two problems: to show that $\bar{F}$ doesn't depend on the choice of an extremal sieve and that it is a sheaf. First we need:

(A.26) Lemma. Let the diagram

$$\begin{tikzcd}
 & Y_o \arrow[r, shift left, "d^o"] \arrow[r, shift right, "d^1"'] \arrow[d, dash] & Z_o \arrow[d, dash] \\
X_1 \arrow[r] \arrow[d, shift right, "e^o"'] \arrow[d, shift left, "e^1"] & Y_1 \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & Z_1 \arrow[d, shift left] \arrow[d, shift right] \\
X_2 \arrow[r] & Y_2 \arrow[r, shift left] \arrow[r, shift right] & Z_2
\end{tikzcd}$$

be commutative and the rows and columns be equalizers. Then the equalizer of d^o and d^1 is the same as that of e^o and e^1.

Proof. Chase the diagram.

(A.27) Proposition. $\bar{F}$ is well defined.

Proof. Let $\{C_i \longrightarrow E\}$ and $\{C'_k \longrightarrow E\}$ be two extremal sieves with

C_i, $C'_k \in \underline{C}$. Apply the above lemma with $Y_o = \Pi F C_i$, $Z_o = \Pi F(C_i \times_E C_j)$,

$X_1 = \Pi C'_k$, $X_2 = \Pi F(C'_k \times_E C'_\ell)$,

$Y_1 = \Pi F(C_i \times_E C'_k)$, $Z_1 = \Pi F(C_i \times_E C_j \times_E C'_k)$,

$Y_2 = \Pi F(C_i \times_E C'_k \times_E C'_\ell)$,

$Z_2 = \Pi F(C_i \times_E C_j \times_E C'_k \times_E C'_\ell)$. In all cases the products are taken over all available sets of indices.

(A.28) Proposition. $\bar{F}$ is a sheaf.

Proof. Let $\{E_i \longrightarrow E\}$ be an extremal sieve, and for each i, choose $\{C_{ij} \longrightarrow E_i\}$ an extremal sieve. Then $\{C_{ij} \longrightarrow E\}$ is an extremal sieve and can be used to define $\bar{F}E$. We now apply (A.4) with $X_o = \bar{F}E$, $Y_o = \Pi \bar{F} E_i$, $Z_o = \Pi \bar{F}(E_i \times_E E_\ell)$, $Y_1 = \Pi F(C_{ij})$, $Y_2 = \Pi F(C_{ij} \times_{E_i} C_{i\ell})$, $Z_2 = \Pi F(C_{ij} \times_E C_{k\ell})$. In applying the theorem in this direction, you do don't actually need g to be $\rightarrowtail$ if you know that $e^o.e = e^1.e$.

Thus $\bar{F}$ is a sheaf, and it is clear that $\bar{F}$ restricted to $\underline{C}$ is F. This completes the proof of Giraud's theorem.

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