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THE POINT OF THE EMPTY SET

by Michael BARR

The point, of course, is that there isn't any point there. This manifests itself in various ways. For example, see [Barr,a], in which the lack of splitting to a map $\phi \to X$, $X \neq \phi$, plays an essential role. In [Barr, b] the empty set came in in a different way, as a non-trivial subobject of the terminal object. There it made the difference between being able to embed an abelian category into a category of (abelian) M-sets while, in the general case of an exact category, having to allow M to have several objects (i.e. be a category).

In this paper we study two more examples in which the empty set complicates an otherwise straightforward situation. The first concerns tripleableness of a functor category (over the base, or somewhat more generally over a category over which the base is tripleable). This result was first announced by Beck, who also introduced the notion of a pure functor, thus being the first to recognize the distinguished role played by empty set. The second example concerns many-sorted algebras. It is frequently supposed that many-sorted algebras are special cases of 1-sorted or ordinary algebras. Again the empty set rears its ugly head to complicate the situation. One way around this problem is to adopt the point of view of Grätzer (see, e.g. [Grätzer], p.8) and simply legislate out of existence any empty algebra. This certainly works, but at the cost of both completness and cocompletness (see, e.g., [Grätzer], § 9, Lemma 1 or § 24, Corollary 1 to Lemma 1). And it is not clear that even Grätzer would want to obviate the possibility of having many-sorted algebras in which some components are empty and others not. But this is exactly where the problem appears and must be faced.

The first three sections deal with preliminary material, including an exposition of and the main application of the VTT. The next two sections consider the two situations discussed above.

1. Definitions

A functor $U : \underline{B} \to \underline{A}$ is said *to create limits* if for any diagram $D : \underline{I} \to \underline{B}$ for which a limit $\varphi : A \to UD$ exists there is an object $B \in \underline{B}$ and a map $f : B \to D$ which are unique up to a unique isomorphism satisfying $Uf \cong \varphi$; moreover $f : B \to D$ is a limit of D.

A pair of maps $B_1 \overset{d^1}{\underset{d^0}{\rightrightarrows}} B_0$ in $\underline{B}$ is called a *split coequalizer pair* if there is an object $B \in \underline{B}$ and maps $d : B_0 \to B$, $s_0 : B \to B_0$ and $s_1 : B_0 \to B_1$ such that

$$d d^0 = d d^1 , \; d^0 s_1 = B_0 \quad \text{and} \quad d^1 s_1 = s_0 d .$$

It easily follows from these equations that the diagram

$$B_1 \overset{d^0}{\underset{d^1}{\rightrightarrows}} B_0 \xrightarrow{\; d \;} B$$

is, in fact, a coequalizer under those circumstances.

The split coequalizer equations imply that

$$d^0 s_1 = B_0 \quad \text{and} \quad d^1 s_1 d^1 = d^1 s_1 d^0 .$$

If we assume that idempotents split in $\underline{B}$, the existence of an $s_1 : B_0 \to B_1$ satisfying those equations is equivalent to d^0, d^1 being a split coequalizer pair. For then B may be defined as the object which splits the idempotent $d^1 s_1$.

If $U : \underline{B} \to \underline{A}$ is a functor, we say that $B_1 \overset{d^0}{\underset{d^1}{\rightrightarrows}} B_0$ is a *U-split-pair* if $U B_1 \overset{U d^0}{\underset{U d^1}{\rightrightarrows}} U B_0$ is a split pair.

The functor $U : \underline{B} \to \underline{A}$ is said to satisfy the VTT provided that it creates limits and every U-split pair is a split pair; it is said to satisfy the CTT if it creates limits and preserves all coequalizers; it is said to satisfy the PTT if it creates limits and if the coequalizers of all U-split pairs exist and are preserved by U.

The point of these definitions is that the PTT conditions are not invariant under composition of functors (those of the VTT and the CTT are). The obvious theorem is that when $U = U_1 U_2 U_3$ then U satisfies the PTT provided that U_1 satisfies the VTT, U_2 satisfies the PTT and U_3 satisfies the CTT. Of course, Beck's theorem states that U is tripleable if and only if U has a left adjoint and satisfies the PTT.

A category $\underline{A}$ is said to satisfy the *weak axiom of choice* (WAC) if it has a terminal object 1 and for each X, the terminal map $X \to 1$ factors as $X \to S \to 1$ where S is a subobject of 1 and $X \to S$ is a split epimorphism. The object S is called the support of X and we will write $S = supp\, X$.

For example, the category of sets, or any discrete power of it as well as any pointed category, all satisfy this WAC. Note that the category of abelian groups, being pointed, satisfies it while it does not satisfy the AC.

If $\underline{A}$ satisfies the WAC we define, after Jon Beck [unpublished] a functor $D : \underline{X} \to \underline{A}$ to be *pure* if every DX, $X \in \underline{X}$, has the same support. If $U : \underline{B} \to \underline{A}$ is a functor, then $D : \underline{X} \to \underline{B}$ is called *U-pure* if UD is pure. We define $\underline{P}(\underline{X}, \underline{A})$ and $U\text{-}\underline{P}(\underline{X}, \underline{B})$ to be the full subcategories of the functor categories $(\underline{X}, \underline{A})$ and $(\underline{X}, \underline{B})$ respectively, consisting of the pure and U-pure functors respectively.

We note that if U is itself pure, then $U\text{-}\underline{P}(\underline{X}, \underline{B}) = (\underline{X}, \underline{B})$.

If $\underline{A}$ satisfies the WAC and the sub-objects of 1 form a complete lattice, we define, for a functor $D : \underline{X} \to \underline{A}$,

$$supp\, D = \bigcap \{ supp\, DX \mid X \in \underline{X} \}.$$

2. The VTT

THEOREM 1. *Let $U : \underline{B} \to \underline{A}$ be the inclusion of a full subcategory. Then if U creates limits, it satisfies the VTT.*

PROOF. Trivial.

In particular, the inclusion of any full coreflexive subcategory satisfies the VTT.

THEOREM 2. *Let I be a small discrete category and $\underline{A}$ be complete and satisfy the WAC. Then the functor $\Pi : \underline{P}(I, \underline{A}) \to \underline{A}$, which assigns to* *each I-indexed family its product, satisfies the VTT.*

PROOF. A parallel pair is an I-indexed family $A_i' \underset{d_i^1}{\overset{d_i^0}{\rightrightarrows}} A_i$ with $supp\, A_i = supp\, A_i'$ the same for all i, and it is Π-split provided that $A' \underset{d^1}{\overset{d^0}{\rightrightarrows}} A$ is a split pair where

$$A = \Pi A_i, \quad A' = \Pi A_i', \quad d^0 = \Pi d_i^0 \text{ and } d^1 = \Pi d_i^1 .$$

Let $s: A \to A'$ such that $d^0 s = A$ and $d^1 s d^1 = d^1 s d^0$. Let $S = supp\, A_i$ and $t_i : A_i \to S$ (respectively $t_i' : A_i' \to S$) be the unique maps of A_i (respectively A_i') to S. For each i let $u_i : S \to A_i$ be a map so that $t_i u_i = S$. Note that since S is a subobject of 1, any diagram terminating in S commutes. Let $p_i : A \to A_i$ and $p_i' : A \to A_i'$ denote the product projections. Then there is a map $u : S \to A$ determined by $p_i u = u_i$. By replacing, if necessary, u by $d^1 s u$, we may suppose that $u = d^1 s u$. (This replacement will affect the u_i but they play no further role.) If we define $u' : S \to A'$ by $u' = s u$, then $d^0 u' = d^0 s u$ and $d^1 u' = d^1 s u = u$. Next define $v_i : A_i \to A$ by letting $p_i v_i = A_i$ while for $j \neq i$, $p_j v_i = p_j u t_i$. We similarly define $v_i' : A_i' \to A'$ by

$$p_i' v_i' = A_i' \text{ and for } j \neq i, \; p_j' v_i' = p_j' u' t_i' .$$

At this point we need a computation.

PROPOSITION 1. *For* $k = 0, 1$ *and for all* i, $d^k v_i' = v_i d_i^k$.

PROOF. We compute using the product projections. First,

$$p_i d^k v_i' = d_i^k p_i' v_i' = d_i^k = p_i v_i d_i^k .$$

For $j \neq i$,

$$p_j d^k v_i' = d_j^k p_j' v_i' = d_j^k p_j' u' t_i' = p_j d^k u' t_i' = p_j u t_i' = p_j u t_i d_i^k = p_j v_i d_i^k .$$

Now we return to the proof of Theorem 2. Define $s_i : A_i \to A_i'$ by $s_i = p_i' s v_i$. Then

$$d_i^0 s_i = d_i^0 p_i' s v_i = p_i d^0 s v_i = p_i v_i = A_i .$$

Also

$$d_i^1 s_i d_i^1 = d_i^1 p_i' s v_i d_i^1 = p_i d^1 s d^1 v_i' = p_i d^1 s d^0 v_i'$$

$$= d_i^1 \, p_i' \, s \, v_i \, d_i^0 = d_i^1 \, s_i \, d_i^0 .$$

Since we have assumed that $\underline{A}$ is complete, idempotents split, and by the remark after the definition of split coequalizer pairs, it follows that $A_i' \rightrightarrows A_i$ is a split coequalizer pair for every i. The functor Π has an adjoint which associates to an object $A \in \underline{A}$ the constant family $\{ A_i = A$ for all $i \}$. It then follows that Π preserves all inverse limits. To see that Π creates them, it is sufficient to show that $\underline{P}(I, \underline{A})$ has inverse limits and that Π reflects isomorphisms.

PROPOSITION 2. *If $A \in \underline{A}$ and S is a subobject of 1, then $S \cong supp\, A$ if and only if $(S, A) \neq \phi$ and $(A, S) \neq \phi$.*

PROOF. The direct implication is guaranteed by our assumption that supports split. If $S \to A$ is a map, then $A \to S$ is a split epic, since S has only one endomorphism. But it factors $A \to supp\, A \to S$, and the second factor is a monomorphism and simultaneously a split epic and consequently an isomorphism.

PROPOSITION 3. *Let $\{ A_j \mid j \in J \}$ be a family of objects of $\underline{A}$. Let $S_i = supp\, A_i$. Then $supp\, \Pi A_i = \Pi S_i = \bigcap S_i$.*

PROOF. The first equality is clear, since a product of split epimorphisms is one again. The second is trivial for subobjects of 1.

COROLLARY 1. *Let $D : \underline{X} \to \underline{A}$ be a functor. Let $S = supp\, D$. Then $S \times D$ (the functor whose value at $X \in \underline{X}$ is $S \times DX$) is pure. Moreover if $E : \underline{X} \to \underline{A}$ is pure, then $(E, D) \cong (E, S \times D)$.*

PROOF. The first part is clear, since $supp(S \times DX) = S \times supp\, DX = S$. For the second, note that $(E, S \times D) \cong (E, S) \times (E, D)$, where S is denoting the constant functor on S. Now there is at most one natural transformation $E \to S$, so that $(E, S \times D)$ is either ϕ or (E, D) according as (E, S) is or isn't ϕ. But if $(E, S) = \phi$, then $T \nleq S$ where $T = supp\, E$. Since $S = supp\, D$ and E has constant support, it follows that for some $X \in \underline{X}$, $T \nleq supp\, DX$. But then $(EX, DX) = \phi$ also, which implies that $(E, D) = \phi = (E, D) \times (E, S)$. In either case $(E, D) \cong (E, D \times S)$.

COROLLARY 2. *The inclusion $\underline{P}(\underline{X}, \underline{A}) \to (\underline{X}, \underline{A})$ has a right adjoint given by $D \mapsto (supp\, D) \times D$.*

3. Generalities on Reflective Subcategories

Let $\underline{X}$ be a category. By a reflective subcategory (warning: [Mac Lane] has interchanged the meanings of reflective and coreflective) is meant a full subcategory whose inclusion functor has a right adjoint. Up to categorical equivalence this is the same thing as an adjoint pair

$$\underline{X}_0 \underset{\check{I}}{\overset{I}{\rightleftarrows}} \underline{X}$$

for which the front adjunction $\eta : \underline{X}_0 \to \check{I} I$ is an isomorphism. It will simplify (without materially changing) the discussion below if we suppose that $\underline{X}_0$ is always replete: that is, if $X \cong X_0 \in \underline{X}_0$, then $X \in \underline{X}_0$ also. We let $\varepsilon : I\check{I} \to \underline{X}$ denote the back adjunction.

PROPOSITION 4. *A necessary and sufficient condition that X be in $\underline{X}_0$ is that there exist a map $x : X \to I\check{I}X$ with $\varepsilon X . x = X$.*

PROOF. The necessity of this condition is obvious. We must show that any such x must be an isomorphism. The other composite

$$I\check{I}X \xrightarrow{\varepsilon X} X \xrightarrow{x} I\check{I}X$$

is a map between objects of $\underline{X}_0$ and is the identity if and only if $\check{I}$ of it is. But $\check{I}\varepsilon X . \eta \check{I} X = X$, together with $\eta \check{I} X$ an isomorphism, implies that $\check{I}\varepsilon X = (\eta \check{I} X)^{-1}$. On the other hand, $\check{I}\varepsilon X . \check{I} x = \check{I} X$ and $\check{I} \varepsilon X$ an isomorphism implies that $\check{I} x = (\check{I}\varepsilon X)^{-1} = \eta \check{I} X$, from which the desired relation follows.

THEOREM 3. *Let $\underline{X}_0 \underset{\check{I}}{\overset{I}{\rightleftarrows}} \underline{X}$ and $\underline{Y}_0 \underset{\check{J}}{\overset{J}{\rightleftarrows}} \underline{Y}$ be reflective subcategories and $U : \underline{X} \to \underline{Y}$, $U_0 : \underline{X}_0 \to \underline{Y}_0$ be functors such that $J U_0 \cong U I$. If U has a right adjoint $\check{U}$, then $\check{I}\check{U}J = \check{U}_0$ is right adjoint to U_0. If U has a left adjoint $\hat{U}$ and if, in addition, $\check{J} U \cong U_0 \check{I}$, then $\hat{U} J$ factors as $I \hat{U}_0$ and $\hat{U}_0$ is left adjoint to U_0. If U is cotripleable, so is U_0. If U is tripleable, then subject to the same condition $\check{J} U \cong U_0 \check{I}$, so is U_0.*

PROOF. The first assertion follows from the computation, where $X_0 \in \underline{X}_0$, $Y_0 \in \underline{Y}_0$,

$$(U_0 X_0, Y_0) \cong (J U_0 X_0, J Y_0) \cong (U I X_0, J Y_0) \cong$$

$$\cong (I X_0, \check{U} J Y_0) \cong (X_0, \check{I}\check{U} J Y_0),$$

with all isomorphisms natural. To prove the second, let $\alpha : UI \to J U_0$ and $\beta : U_0 \check{I} \to \check{J} U$ be the given isomorphisms. We let

$$\eta_I : \underline{X}_0 \xrightarrow{\cong} \check{I} I, \quad \eta_J : \underline{Y}_0 \xrightarrow{\cong} \check{J} J \quad \text{and} \quad \eta_U : \underline{Y} \longrightarrow U \hat{U}$$

denote the front adjunctions, while

$$\varepsilon_I : I \check{I} \to \underline{X}, \quad \varepsilon_J : J \check{J} \to \underline{Y} \quad \text{and} \quad \varepsilon_U : \hat{U} U \to \underline{X}$$

denote the back adjunctions. As above, $\check{I}\varepsilon_I$ is an isomorphism and naturality of β implies the commutativity of

$$\begin{tikzcd}
U_0 \check{I} I \check{I} \arrow[r, "\beta I \check{I}"] \arrow[d, "U_0 \check{I} \varepsilon_I"'] & \check{J} U I \check{I} \arrow[d, "\check{J} U \varepsilon_I"] \\
U_0 \check{I} \arrow[r, "\beta"] & \check{J} U
\end{tikzcd}$$

from which we infer that $\check{J} U \varepsilon_I$ is an isomorphism. But then the composite

$$U_0 \check{I} \xrightarrow{\eta_J U_0 \check{I}} \check{J} J U_0 \check{I} \xrightarrow{\check{J} \alpha^{-1} \check{I}} \check{J} U I \check{I} \xrightarrow{\check{J} U \varepsilon_I} \check{J} U$$

is also an isomorphism. Hence, without loss of generality, we may suppose that $\beta = \check{J} U \varepsilon_I . \check{J} \alpha^{-1} \check{I} . \eta_J U_0 \check{I}$. Now we let

$$\gamma = J\beta . \alpha \check{I} : U I \check{I} \to J \check{J} U.$$

First we claim that $\varepsilon_J U . \gamma = U \varepsilon_I$. In fact,

$$\begin{aligned}
\varepsilon_J U . \gamma &= \varepsilon_J U . J \check{J} U \varepsilon_I . J \check{J} \alpha^{-1} \check{I} . J \eta_J U_0 \check{I} . \alpha \check{I} \\
&= U \varepsilon_I . \varepsilon_J U I \check{I} . J \check{J} \alpha^{-1} \check{I} . J \eta_J U_0 \check{I} . \alpha \check{I} \\
&= U \varepsilon_I . \alpha^{-1} \check{I} . \varepsilon_J J U_0 \check{I} . J \eta_J U_0 \check{I} . \alpha \check{I} \\
&= U \varepsilon_I . \alpha^{-1} \check{I} . \alpha \check{I} \\
&= U \varepsilon_I .
\end{aligned}$$

Here we used the fact, standard for adjoints, that $\varepsilon_J J . J \eta_J = J$. Now define $\xi : \hat{U} J \to I \check{I} \hat{U} J$ by the composite

$$\hat{U}J \xrightarrow{\hat{U}J\eta_J} \hat{U}J\check{J}J \xrightarrow{\hat{U}J\check{J}\eta_U J} \hat{U}J\check{J}U\hat{U}J \xrightarrow{\hat{U}\gamma^{-1}\hat{U}J} \hat{U}UI\check{I}\hat{U}J \xrightarrow{\varepsilon_U I\check{I}\hat{U}J} I\check{I}\hat{U}J.$$

We compute

$$\begin{aligned}
\varepsilon_I \hat{U}J \,.\, \xi &= \varepsilon_I \hat{U}J \,.\, \varepsilon_U I\check{I}\hat{U}J \,.\, \hat{U}\gamma^{-1}\hat{U}J \,.\, \hat{U}J\check{J}\eta_U J \,.\, \hat{U}J\eta_J \\
&= \varepsilon_U \hat{U}J \,.\, \hat{U}U\varepsilon_I \hat{U}J \,.\, \hat{U}\gamma^{-1}\hat{U}J \,.\, \hat{U}J\check{J}\eta_U J \,.\, \hat{U}J\eta_J \\
&= \varepsilon_U \hat{U}J \,.\, \hat{U}\varepsilon_J U\hat{U}J \,.\, \hat{U}J\check{J}\eta_U J \,.\, \hat{U}J\eta_J \\
&= \varepsilon_U \hat{U}J \,.\, \hat{U}\eta_U J \,.\, \hat{U}\varepsilon_J J \,.\, \hat{U}J\eta_J \\
&= \hat{U}J.
\end{aligned}$$

Hence for every $Y_0 \in \underline{Y}_0$, $\xi Y_0 : \hat{U}JY_0 \to I\check{I}\hat{U}JY_0$ is a map whose composite with $\varepsilon_I \hat{U}JY_0$ is $\hat{U}JY_0$, which shows that $\hat{U}JY_0 \in \underline{X}_0$; and we may write $\hat{U}JY_0 \cong I\hat{U}_0 Y_0$ where $\hat{U}_0 = \check{I}\hat{U}J$. Then we compute for $Y_0 \in \underline{Y}_0$, $X_0 \in \underline{X}_0$,

$$\begin{aligned}
(Y_0, U_0X_0) &\cong (JY_0, JU_0X_0) \cong (JY_0, UIX_0) \cong (\hat{U}JY_0, IX_0) \\
&\cong (I\hat{U}_0Y_0, IX_0) \cong (\hat{U}_0Y_0, X_0),
\end{aligned}$$

which gives the adjointness.

Now suppose that U is cotripleable. Evidently I reflects isomorphisms and then $UI = JU_0$ does, which, a fortiori, implies that U_0 does. Then suppose that $X_0 \to X_0' \rightrightarrows X_0''$ is a U_0-split equalizer diagram. Then it is a $JU_0 = UI$-split diagram as well, which implies, since U is cotripleable, that $IX_0 \to IX_0' \rightrightarrows IX_0''$ is an equalizer. Then for any $X \in \underline{X}$,

$$(X, IX_0) \to (X, IX_0') \rightrightarrows (X, IX_0'')$$

is an equalizer. If $X = IX_0'''$, then, since I is full and faithful, we see that

$$(X_0''', X_0) \to (X_0''', X_0') \rightrightarrows (X_0''', X_0'')$$

is an equalizer for all $X_0''' \in \underline{X}_0$, which means that $X_0 \to X_0 \rightrightarrows X_0''$ is an equalizer. The argument when U is tripleable is similar, since we never used the fact that it has a right adjoint, only that it is full and faithful.

4. Functor Categories

The main theorem on pure functors follows. It was first given

- more or less - by Beck with an *ad hoc* and somewhat mysterious proof.

THEOREM 4. *Let* $\underline{A}$ *satisfy the WAC and be complete and cocomplete. Let* $U:\underline{B}\to\underline{A}$ *be tripleable and* $\underline{C}$ *be a small category. Then* $U\text{-}\underline{P}(\underline{C},\underline{B})\to\underline{A}$ *by the composite*

$$U\text{-}\underline{P}(\underline{C},\underline{B}) \hookrightarrow (\underline{C},\underline{B}) \xrightarrow{(\underline{C},U)} (\underline{C},\underline{A}) \xrightarrow{\Pi} \underline{A}$$

is tripleable.

PROOF. First we establish the existence of an adjoint. Composition with F, the left adjoint of U, gives a left adjoint to the functor $(\underline{C},\underline{B})\to(\underline{C},\underline{A})$. The functor $(\underline{C},\underline{A})\to(|\underline{C}|,\underline{A})$ has, since $\underline{A}$ is cocomplete, a left adjoint given by the Kan extension. Here $|\underline{C}|$ is the discrete category which is the set of objects of $\underline{C}$. Finally, the product functor $(|\underline{C}|,\underline{A})\to\underline{A}$ has a left adjoint, the constant functor functor. The result then follows from Theorem 3 applied to the diagram

$$\begin{tikzcd}
U\text{-}\underline{P}(\underline{C},\underline{B}) \arrow[rr, shift left] \arrow[d] & & (\underline{C},\underline{B}) \arrow[ll, shift left] \arrow[d, shift left] \\
\underline{A} \arrow[rr, shift left] & & \underline{A} \arrow[ll, shift left] \arrow[u, shift left]
\end{tikzcd}$$

Here the lower arrows are, of course, the identity functor of $\underline{A}$. Next, we must show that the composite satisfies the PTT. To do this we make use of another chain, namely,

$$U\text{-}\underline{P}(\underline{C},\underline{B})\to U\text{-}\underline{P}(|\underline{C}|,\underline{B})\to U\text{-}\underline{P}(|\underline{C}|,\underline{A})\to\underline{A}.$$

Since $\underline{B}$ is complete, $(\underline{C},\underline{B})\to(|\underline{C}|,\underline{B})$ has a right adjoint, and, using Theorem 3, so does $U\text{-}\underline{P}(\underline{C},\underline{B})\to U\text{-}\underline{P}(|\underline{C}|,\underline{B})$. When $\underline{B}$ is cocomplete it has a left adjoint as well, but in any case it preserves all inverse limits, and by a simple modification of the proof of Theorem 3, so does

$$U\text{-}\underline{P}(\underline{C},\underline{B})\to U\text{-}\underline{P}(|\underline{C}|,\underline{B}).$$

The functor $(|\underline{C}|,\underline{B})\to(|\underline{C}|,\underline{A})$ has whatever properties $\underline{B}\to\underline{A}$ does and is, in particular, PTT. Then, by Theorem 3, the same is true for the functor $U\text{-}\underline{P}(|\underline{C}|,\underline{B})\to U\text{-}\underline{P}(|\underline{C}|,\underline{A})$. Finally, by Theorem 2, $U\text{-}\underline{P}(|\underline{C}|,\underline{A})\to\underline{A}$ satisfies the VTT. Putting it all together, we get the desired conclusion.

This theorem is usually applied under hypotheses which make $U\cdot \underline{P}(\underline{C},\underline{B})=(\underline{C},\underline{B})$. For example, if $\underline{A}=\underline{S}$, the category of sets, and $\underline{B}\to\underline{A}$ is tripleable, then $\underline{B}$ is a category of algebras for a theory (possibly with infinitary operations). If we suppose that there is a nullary operation in the theory, then every object of $\underline{B}$ has the same support and every functor is pure. On the other hand, regardless of the nature of $\underline{B}$, if $\underline{C}$ is strongly connected $-(C_1, C_2)\neq\phi$, $C_1, C_2 \in \underline{C}-$ then every functor is automatically pure. This is the case of the standard simplicial category Δ, and hence, whenever $\underline{B}\to\underline{A}$ is tripleable and $\underline{A}$ satisfies the other hypotheses of Theorem 4, we see that $simp\,\underline{B}=(\Delta^{op},\underline{B})$ is tripleable over $\underline{A}$. On the other hand, let $\underline{B}\to\underline{S}$ be tripleable but with no nullary operations. Then the empty set is an algebra. Now if we look at $simp^{+}\,\underline{B}$, the category of augmented simplicial objects, this is not tripleable over sets. There is the same adjoint pair but the algebras for the triple are the full subcategory consisting of the constantly empty simplicial object and those which are non-empty in all degrees, including -1. What gets omitted are those which are empty in non-negative degrees but augmented to a non-empty set.

5. Many-Sorted Algebras

An n-sorted algebra (where, for simplicity, we will imagine n to be finite) is a string $(S_1, \dots, S_n)$ of sets together with operations of the kind $S_1^{e_1}\times\dots\times S_n^{e_n}\to S_i$ and satisfying equations of the sort familiar in algebra. A category of 1-sorted, or ordinary, algebras can be described as a category of product preserving functors $\underline{Th}^{op}\to\underline{S}$ where there is given a coproduct preserving functor $\underline{S}\to\underline{Th}$, which is an isomorphism on objects. Similarly, we can describe n-sorted algebras by an n-fold theory, $\underline{S}^n\to\underline{Th}$.

The simplest category of 2-sorted algebras is the category $(\mathbf{1}+\mathbf{1}, \underline{S})=\underline{S}\times\underline{S}$ of pairs of sets. The underlying functor is product and at first glance it looks indeed to be tripleable. The operations are generated by a single binary operation which is associative and idempotent. The category of algebras for that triple (or that theory) is actually $\underline{P}(1+1, \underline{S})$, those pairs of sets of which either both are empty or both are non-empty. (What happens to tripleableness, by the way, is that the underlying functor does not reflect isomorphisms. The map $(\phi, \phi) \to (\phi, 1)$ is not an isomorphism; the induced $\phi \times \phi \to \phi \times 1$ is.)

A rather typical example of 2-sorted algebras is the category of all modules. Objects are pairs (R, M) where R is a ring and M is an R-module. A map $(\varphi, f) : (R, M) \to (R', M')$ consists of a ring homomorphism $f : R \to R'$, an abelian group homomorphism $f : M \to M'$ such that $f(rm) = \varphi(r) f(m)$ for $r \in R$, $m \in M$.

An interesting ω-sorted theory is the one whose algebras are the finite algebraic theories. A finitary algebraic theory is determined by a string $(\Omega_0, \Omega_1, \dots)$ where Ω_n is the set of n-ary operations. For each k and each $n_1, \dots, n_k$, there is a map $\Omega_k \times \Omega_{n_1} \times \dots \times \Omega_{n_k} \to \Omega_{n_1 + \dots + n_k}$ which assigns to $(\omega_0, \omega_1, \dots, \omega_k)$ the operation $\omega_0(\omega_1 \times \dots \times \omega_k)$. These are subject to associativity and unitary equations (the latter involving the projections) and give in obvious way an ω-sorted theory. The existence of the projections requires that $\Omega_i \neq \phi$ for all $i > 0$. However it is possible that $\Omega_0 = \phi$. This corresponds to theories without constants. Evidently it will not do to ignore theories without constants. Some of the most interesting theories lack constants. One way out would be to omit the theories with constants. A constant or nullary operation can always be replaced by a unary operation plus an equation which makes the unary operation constant. The only effect on the algebras would to make the empty set a model of every theory. This is what happens when you present the theory of groups by a single binary operation, $(x, y) \to xy^{-1}$, subject to some equations.

If $\underline{Th}$ is an n-sorted theory, define a *pure* $\underline{Th}$-algebra to be one all of whose components are empty or all non-empty. Let $\underline{P}(\underline{S}^{\underline{Th}})$ denote this category. It is, of course, the category of pure product preserving functors of $\underline{Th}^{op}$ to $\underline{S}$.

THEOREM 5. *For any n-sorted theory $\underline{Th}$ (where n may represent a finite or infinite cardinal), the category $\underline{P}(\underline{S}^{\underline{Th}})$ of pure Th-algebras is triple-*

able over $\underline{S}$ *by the product functor.*

PROOF. Let $|n|$ denote the discrete category with n objects. Then

$$\underline{P}(\underline{S}^{\underline{Th}}) \to \underline{P}(|n|, \underline{S})$$

is tripleable by the usual argument (see, e.g., [Mac Lane]). By Theorem 2, the functor $\underline{P}(|n|, \underline{S}) \to \underline{S}$ satisfies the VTT and putting them together, the result follows.

As with Theorem 4, the main interest of this result is likely to be when every algebra is pure. As in that case, this could result either because all components are required to be non-empty (like the theory of all modules) or by a kind of connectedness. We leave to the reader the easy exercise of constructing an example of the latter phenomenon.

References

M. BARR, (a) *Coalgebras in a category of algebras,* Lecture Notes in Math. 86 (1969), 1-12.

(b) *Exact categories,* Lecture Notes in Math. 236 (1971), 1-20.

G. GRÄTZER, *Universal Algebra,* D. Van Nostrand & Co., 1968.

S. MAC LANE, *Categories for the Working Mathematician,* Springer-Verlag 1971.

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