FIXED POINTS IN CARTESIAN CLOSED CATEGORIES

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Abstract

The purpose of this paper is to begin the study of domain theory in a context that is also appropriate for semantic models of other aspects of computation, that is in cartesian closed categories with a natural numbers object. I show that if D is an internally ω-complete partial order with bottom in such a category, then the usual construction of least fixed point of an ω-continuous endomorphism can be internalized as an arrow from the object of ω-continuous endomorphisms of D (suitably defined) to D itself.

1 Introduction

Since the appearance of Reynolds' paper, Polymorphism is not set-theoretic, [1984] at least, it has been apparent that any semantic model rich enough to include polymorphism would have to take place in some category other than that of standard ZFC set theory. One possibility is modest sets [Rosolini, to appear] and [Carboni, Freyd & Ščedrov, 1988], but there are many other possibilities. It thus becomes necessary to make sense of standard computer scientific notions in more general categories.

The purpose of this paper is to look at one aspect of domain theory in the setting of a cartesian closed category with finite limits and a natural numbers object. We give a definition of an ω-complete partial order in such a category and show how to define the object of increasing sequences and an object of sup-preserving morphisms. We then show that in this setting an ω-complete partial order with bottom has a fixed point operator in

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the sense of a mapping from the object of sup-preserving endomorphisms to the object itself that realizes the least fixed point.

The suggestion to use natural numbers objects to construct a fixed point combinator was also made at the Boulder meeting by Phil Mulry [1988].

Many of these constructions would become simpler were we to suppose that the category has countable limits. On the other hand, the category of modest sets does not have countable limits. It would seem plausible that any model of polymorphism lacks countable sums. In any case, existence of countable limits does not seem computationally reasonable. For a countable sum, say, of copies of 1 would require an arrow to exist subject to an infinite set of unrelated conditions. Denote the natural numbers object by $\mathbf{N}$. For an object X of a category with countable sums, the object $X^{\mathbf{N}}$ is the set of *all* (including totally uncomputable) countable sequences of elements of X. On the other hand, in a category in which countable sums don't exist, it may (and in the category of modest sets does) consist of only the recursive sequences.

In the proofs below, we make use, usually without explicit mention of the possibility of reasoning in a category just as though we had sets with elements. Of course, this can be done only with certain kinds of arguments. One way of justifying this is by using the equivalence between cartesian closed categories and typed λ-calculuses. See [Lambek & Scott, 1986] for details. Another semantics for this is to interpret an element $x \in A$ as a morphism to A with unspecified domain. If, say, we construct from this an element we call $f(x) \in B$ and make no assumption about properties, then what we have really done is construct a natural transformation $\mathrm{Hom}(-, A) \longrightarrow \mathrm{Hom}(-, B)$. The Yoneda lemma ([Barr and Wells, 1985], Section 1.5) asserts that there is one-one correspondence between such natural transformations and morphisms $A \longrightarrow B$. Thus there is a morphism $f: A \longrightarrow B$ so that $f(x) = f \circ x$. The uniqueness implies that if we can show that $f(x) = g(x)$, then $f = g$.

Unless it is explicitly mentioned otherwise we will be dealing with a category $\mathscr{C}$ which is cartesian closed and has finite limits and a natural numbers object $\mathbf{N}$. We let $0: 1 \longrightarrow \mathbf{N}$ and $s: \mathbf{N} \longrightarrow \mathbf{N}$ denote the zero and successor morphisms, respectively. We will denote the exponential objects mostly by $[A \longrightarrow B]$, but will sometimes write B^A, when that is convenient.

2 Partially ordered objects

To give a partial order on a set X one can describe a subset of $\leq \subseteq X \times X$, namely $\{(x, y) \| x \leq y\}$. Then if two functions $f, g: Y \longrightarrow X$ are given, $f(y) \leq g(y)$ (that is, $f \leq g$ in the pointwise order on functions) if and only if the pair $\langle f, g \rangle: Y \longrightarrow X \times X$ factors through $\leq$. This can be generalized in an arbitrary category, so long as it has finite products.

Let C be an object of $\mathscr{C}$. A partial order on C is a subobject $\leq\ \subseteq C \times C$ such that for any other object A, the relation on $\mathrm{Hom}(A, C)$ such that $f \leq g$ if and only if $(f, g)\colon A \longrightarrow C \times C$ factors through $\leq$, is a partial order. In terms of elements, this reduces to the familiar requirements that $c \leq c$, that $c_1 \leq c_2$ and $c_2 \leq c_3$ imply $c_1 \leq c_3$ and that $c_1 \leq c_2$ and $c_2 \leq c_1$ imply $c_1 = c_2$. These conditions can also be stated in terms of commutative diagrams involving finite limits.

We will usually not distinguish between a partially ordered object and its underlying object. For the most part, we will let the same symbol $\leq$ denote the partial order in any partially ordered object, but will sometimes write $\leq_D$ for clarity.

For objects D and E of a cartesian closed category, there is a way to internalize the arrow that takes $f\colon D \longrightarrow E$ to $f \times f\colon D \times D \longrightarrow E \times E$. Define $[D \longrightarrow E] \longrightarrow [D \times D \longrightarrow E \times E]$ as the exponential transpose of the composite

$$
\begin{aligned}
&[D \longrightarrow E] \times D \times D \xrightarrow{\text{diag}} [D \longrightarrow E] \times [D \longrightarrow E] \times D \times D \longrightarrow \\
&\qquad \xrightarrow{\langle p_1, p_3, p_2, p_4 \rangle} [D \longrightarrow E] \times D \times [D \longrightarrow E] \times D \xrightarrow{\text{eval} \times \text{eval}} E \times E
\end{aligned}
$$

If D and E are partially ordered objects in $\mathscr{C}$, we define the subobject $[D \longrightarrow E]_{\leq} \subseteq [D \longrightarrow E]$ of order preserving morphisms of D to E as the pullback

$$
\begin{tikzcd}
{[D \longrightarrow E]_{\leq}} \arrow[r, tail] \arrow[dd] & {[D \longrightarrow E]} \arrow[d] \\
 & {[D \times D \longrightarrow E \times E]} \arrow[d] \\
{[\leq_D \longrightarrow \leq_E]} \arrow[r, tail] & {[\leq_D \longrightarrow E \times E]}
\end{tikzcd}
$$

The upper left arrow in this diagram is the one described above. This pullback picks out those arrows from $D \longrightarrow E$ whose square, when restricted to the relation, come from a morphism between the relations. There is no difficulty in extending this definition to the object of morphisms between models of an arbitrary relation.

A global section $\bot\colon 1 \longrightarrow D$ will be called a **bottom** if for any object A and arrow $f\colon A \longrightarrow D$, $\bot \circ () \leq f$. Here we use $()$ to denote the unique arrow from A to 1.

3 The partial ordering on the natural numbers

This section is devoted to putting a partial order on the NNO and exploring some of its properties. We begin with the well-known fact that there is a commutative monoid structure $+$ on the NNO which satisfies the equations $n + 0 = n$ and $n + s(m) = s(n + m)$. It is also not hard to show that this operation gives a cancellation monoid with the additional property that $n + m = 0$ implies that $n = m = 0$.

We will not prove this, but just sketch how to do it. First define a morphism pred by $\mathrm{pred}(0) = 0$ and $\mathrm{pred}(sn) = n$. This is easily done using recursion. Next, define $m \dot{-} n$ by $m \dot{-} 0 = m$ and $m \dot{-} sn = \mathrm{pred}(m \dot{-} n)$. Then show that $(m + n) \dot{-} n = m$. From this, the above assertions are easy.

The cancellation implies that the map $p\colon \mathbf{N}\times\mathbf{N} \longrightarrow \mathbf{N}\times\mathbf{N}$ that takes (n, m) to $(n, n+m)$ is injective and hence is a binary relation we denote $\le$ on $\mathbf{N}$ such that $n \le n + m$. The associativity of the operation $+$ implies that this is transitive and the fact that $m + n = 0$ implies that $m = n = 0$ implies that it is a partial order.

It is not generally a total order, by the way. This fact is already evident in boolean-valued models of set theory. A natural number could be 0 on one component and 1 on another and a different one could be the reverse. Neither is greater than the other.

We say that an NNO is *stable* if for any objects A and B and any morphisms $f_0\colon A \longrightarrow B$ and $t\colon B \longrightarrow B$, there is a unique $f\colon A \times \mathbf{N} \longrightarrow B$ such that

$$\begin{tikzcd}
 & A \times \mathbf{N} \arrow[dd, "f"] & & A \times \mathbf{N} \arrow[ll, "A \times s"'] \arrow[dd, "f"] \\
A \arrow[ur, "A \times 0"] \arrow[dr, "f_0"'] & & & \\
 & B & & B \arrow[ll, "t"]
\end{tikzcd}$$

commutes. The object A is called an object of *parameters* and for that reason a stable NNO is often referred to as a *parametrized* NNO. However, it is not the NNO that is parametrized, but the definition.

In a cartesian closed category, any NNO is automatically stable. However the following is of independent interest, so it is worth stating under weaker conditions.

3.1 Proposition. *Let D be a partially ordered object in a category with pullbacks and a stable NNO. Let $f\colon \mathbf{N} \longrightarrow D$ be a morphism such that $fn \le f(sn)$ for all n. Then f preserves the partial order.*

We begin with:

3.2 Lemma. *Let A be an object and $A_0 \subseteq A$ a subobject in a category with pullbacks and an NNO. Suppose $f\colon \mathbf{N} \longrightarrow A$ is a morphism and suppose that $f0$ factors through A_0 and that whenever fn factors through A_0 so does $f(sn)$. Then f factors through A_0.*

Proof. Form the pullback

$$\begin{tikzcd}
\mathbf{N}_0 \arrow[r, tail, "n"] \arrow[d, "f_0"'] & \mathbf{N} \arrow[d, "f"] \\
A_0 \arrow[r, tail, "a"'] & A
\end{tikzcd}$$

where $a: A_0 \rightarrowtail A$ is the inclusion. By hypothesis, the 0 map factors through A_0 and hence through $\mathbf{N}_0$. Since $f \circ n$ factors through $\mathbf{N}_0$, so does $f \circ n \circ s$ by a map we call g. Let s_0 be the fill-in in the diagram

$$\begin{tikzcd}
\mathbf{N}_0 \arrow[rr, "n"] \arrow[dr, "s_0"] \arrow[ddr, "g"'] & & \mathbf{N} \arrow[dr, "s"] & \\
& \mathbf{N}_0 \arrow[rr, "n"] \arrow[d, "f_0"] & & \mathbf{N} \arrow[d, "f"] \\
& A_0 \arrow[rr, "a"'] & & A
\end{tikzcd}$$

and let x be defined as the unique fill-in in the recursion diagram

$$\begin{tikzcd}
& \mathbf{N} \arrow[d, "x"] & \mathbf{N} \arrow[l, "s"'] \arrow[d, "x"] \\
1 \arrow[ur, "0"] \arrow[dr, "0"'] & & \\
& \mathbf{N}_0 & \mathbf{N}_0 \arrow[l, "s_0"]
\end{tikzcd}$$

Then $n \circ x = \mathrm{id}$ follows immediately from the uniqueness of recursively defined maps. Thus n is a split epi. It is also monic, being the pullback of a mono and is thus an isomorphism. This evidently means that f factors through A_0. □

Proof of the proposition. We apply the above in the category of objects over $\mathbf{N}$, but work in the original category. Let $f: \mathbf{N} \longrightarrow D$ satisfy the conditions of the proposition and let $g: \mathbf{N} \times \mathbf{N} \longrightarrow \mathbf{N} \times D \times D$ take (n, m) to $(n, fn, f(n+m))$. We want to complete the square (p is the inclusion of the order relation on $\mathbf{N}$):

$$\begin{tikzcd}
\mathbf{N}\times\mathbf{N} \arrow[r, tail, "p"] & \mathbf{N}\times\mathbf{N} \arrow[d, "g"] \\
\mathbf{N}\times\leq_D \arrow[r, tail] & \mathbf{N}\times D\times D
\end{tikzcd}$$

Then $(g\circ p)(n,0)=(n,fn,fn)$ certainly lies in $\mathbf{N}\times\leq$ and if $(g\circ p)(n,m)$ factors through $\mathbf{N}\times\leq$, then $(n,fn,f(n+m))$ factors through $\mathbf{N}\times\leq$, which means that $fn\leq f(n+m)$. Moreover, $f(n+m)\leq fs(n+m)$ by hypothesis, which is $f(n+sm)$. Thus $(g\circ p)(n,sm)=(n,fn,f(n+sm))$ also factors through $\mathbf{N}\times\leq$. From the lemma it follows that $g\circ p$ factors through $\mathbf{N}\times\leq$ which means that f is order preserving. □

3.3 Corollary. *If $f\colon\mathbf{N}\longrightarrow D$ has the property that $f(0)\leq f(s0)$ and that $f(n)\leq f(sn)$ implies that $f(sn)\leq f(ssn)$, then f is order preserving.*

Proof. Apply Lemma 3.2 to the map $g\colon\mathbf{N}\longrightarrow D\times D$ for which $g(n)=(f(n),fs(n))$ to conclude that $f(n)<fs(n)$ for all n and hence by Proposition 3.1 that f is order preserving. □

Finally, as an immediate corollary to Lemma 3.2, we have:

3.4 Corollary. *If $f,g\colon\mathbf{N}\longrightarrow D$ have the properties that $f(0)\leq g(0)$ and $f(n)\leq g(n)$ implies $f(sn)\leq g(sn)$, then $f\leq g$.* □

All the results of this section up to here have parametrized versions as well, which we will use without explicit mention.

We will denote the natural numbers object with this order on it by ω. If D is any partial order and if $\mathscr{C}$ is also a cartesian closed, we will denote the object $[\omega\longrightarrow D]_\leq$ of increasing sequences by D^ω.

3.5 Proposition. *If $t\colon D\longrightarrow D$, $f_0\colon A\longrightarrow D$ and $f\colon A\longrightarrow D^{\mathbf{N}}$ are such that t is order preserving and that*

$$\begin{tikzcd}
 & A\times\mathbf{N} \arrow[dd, "\widetilde{f}"] & A\times\mathbf{N} \arrow[l, "A\times s"'] \arrow[dd, "\widetilde{f}"] \\
A \arrow[ur, "A\times 0"] \arrow[dr, "f_0"'] & & \\
 & D & D \arrow[l, "t"]
\end{tikzcd}$$

commutes, then a necessary and sufficient condition that f factor through D^ω is that $\widetilde{f_0}\leq t\circ\widetilde{f_0}$.

Proof. The map f factors through D^ω if and only if $\widetilde{f}$ preserves order (where A is discrete and $A \times \mathbf{N}$ is given the product order). The condition $\widetilde{f}_0 \le t \circ \widetilde{f}_0 = \widetilde{f}(s0)$ is clearly necessary. If we suppose that $\widetilde{f}(n) \le \widetilde{f}(sn)$, we have, since t is order preserving, that $t \circ \widetilde{f}(n) \le t \circ \widetilde{f}(sn)$ or $\widetilde{f}(sn) \le \widetilde{f}(ssn)$. Thus it follows from Corollary 3.3 that $\widetilde{f}$ is order preserving and hence that f factors as claimed.

4 ω-complete partial orders

Let D be a partial order. We say that D is an ω**-complete partial order** (known as an ω-CPO) if there is an arrow $\bigvee : D^\omega \longrightarrow D$ such that for any arrow $f : A \longrightarrow D^\omega$ with transpose $\widetilde{f} : A \times \omega \longrightarrow D$, we have that $\widetilde{f} \le \bigvee \circ f \circ p_1$ and if $g : A \longrightarrow D$ is any arrow such that $\widetilde{f} \le g \circ p_1$, then $\bigvee \circ f \le g$. The relevant diagram is

$$A \times \omega \xrightarrow{p_1} A \xrightarrow{f} D^\omega \xrightarrow{\bigvee} D$$

4.1 Proposition. *Let D be an ω-CPO. Then $\bigvee \circ D^s = \bigvee$.*

Proof. Let $f : A \longrightarrow D^\omega$. Showing that $\bigvee \circ D^s \circ f = \bigvee \circ f$ is equivalent to showing that for any $g : A \longrightarrow D$, $\widetilde{f} \le g \circ p_1 : A \times \mathbf{N} \longrightarrow D$ if and only if $\widetilde{f} \circ (\mathrm{id} \times s) \le g$. In other words, we have to show that for all $a \in A$, $f(a, n) \le g(a)$ for all n if and only if $f(a, sn) \le g(a)$ for all $n \in \mathbf{N}$. But using Corollary 3.4, this is immediate. □

4.2 Proposition. *Let D be a partial order. Then there is a unique arrow, we will call $(-)^\omega : [D \longrightarrow D]_\le \longrightarrow [D^\omega \longrightarrow D^\omega]$ such that*

$$\begin{tikzcd}
{[D \longrightarrow D]_\le} \arrow[r, "(-)^\omega"] \arrow[d] & {[D^\omega \longrightarrow D^\omega]} \arrow[d, "{[D^\omega \longrightarrow \mathrm{Inc}]}"] \\
{[D^{\mathbf{N}} \longrightarrow D^{\mathbf{N}}]} \arrow[r, "{[\mathrm{Inc} \longrightarrow D^{\mathbf{N}}]}"'] & {[D^\omega \longrightarrow D^{\mathbf{N}}]}
\end{tikzcd}$$

commutes. In this diagram, Inc *is the inclusion map of D^ω into $D^{\mathbf{N}}$.*

Proof. The uniqueness comes from the fact that the right hand vertical arrow is monic. For existence, we will construct the transpose $[D \longrightarrow D]_\le \times D^\omega \longrightarrow D^\omega$, for which it is sufficient to construct its transpose $[D \longrightarrow D]_\le \times D^\omega \times \omega \longrightarrow D$. This takes (f, g, n) to $f(g(n))$ where f is a variable of type $[D \longrightarrow D]_\le$, g is a variable of type D^ω and n a variable of type ω. We have to show that this preserves order on the variable of type ω. But if $n \le m$, $g(n) \le g(m)$ since g preserves order and similarly $f(g(n)) \le f(g(m))$. □

Now we can internalize the definition of the set of ω-continuous endomorphisms of an ω-CPO D. We define the object $[D \longrightarrow D]_{\bigvee}$ as the equalizer of the two arrows

$$\begin{tikzcd} {[D \longrightarrow D]_{\leq}} \arrow[r, shift left, "(-)^{\omega}"] \arrow[r, shift right] & {[D^{\omega} \longrightarrow D^{\omega}]} \atop {[D \longrightarrow D]} \arrow[r, shift left, "{[D^{\omega} \longrightarrow \bigvee]}"] \arrow[r, shift right, "{[\bigvee \longrightarrow D]}"'] & {[D^{\omega} \longrightarrow D]} \end{tikzcd}$$

This is to be interpreted as the subobject of the object $[D \longrightarrow D]_{\leq}$ consisting of endomorphisms that commute with the $\bigvee$ operation.

We now come to the main theorem which internalizes the usual fixed point construction in ω-CPOs.

4.3 Theorem. *Let D be an ω-CPO with a bottom element $\perp$. Then there is an arrow* $\mathrm{fix}\colon [D \longrightarrow D]_{\bigvee} \longrightarrow D$ *such that* $f(\mathrm{fix}(f)) = \mathrm{fix}(f)$ *and if* $f(d) = d$, *then* $\mathrm{fix}(f) \leq d$.

Proof. Let $f\colon A \longrightarrow [D \longrightarrow D]_{\bigvee}$ be an element of $[D \longrightarrow D]_{\bigvee}$ defined on A. There is a morphism $\widetilde{g}\colon A \times \mathbf{N} \longrightarrow D$ given recursively by the equations $\widetilde{g}(a,0) = \perp$ and $\widetilde{g}(a,sn) = \widetilde{f}(a,\widetilde{g}(a,n))$. Formally, this is defined as the second coordinate of the map h in the diagram

$$\begin{tikzcd} & A \times \mathbf{N} \arrow[dd, "h"] & & A \times \mathbf{N} \arrow[ll, "A \times s"'] \arrow[dd, "h"] \\ A \arrow[ur, "A \times 0"] \arrow[dr, "h_0"'] & & & \\ & A \times D & & A \times D \arrow[ll, "t"] \end{tikzcd}$$

where $t(a,d) = (a,\widetilde{f}(a,d))$ and $h_0 a = (a,\perp)$. It is easy to see from the uniqueness of recursive definitions that the first coordinate of h is the projection on A. The fact that $\widetilde{f}$ preserves order implies that t does. The defining property of $\perp$ implies that $h_0 \leq t \circ h_0$ and hence it follows from Corollary 3.5 that $\widetilde{g}$ is increasing and that the associated morphism $g\colon A \longrightarrow D^{\mathbf{N}}$ factors through D^{ω}. Finally, we let $\mathrm{fix}(f) = \bigvee \circ g$. Except for the parameter A, this is the construction of the supremum of the sequence $\perp, f(\perp), f^2(\perp), \ldots$, just as in the standard proof.

Now we observe that the diagram

$$\begin{tikzcd} A \arrow[r, "\langle \mathrm{id}, g\rangle"] \arrow[d, "\langle \mathrm{id}, g\rangle"'] & A \times D^{\omega} \arrow[r, "\mathrm{id} \times \bigvee"] \arrow[d, "(\mathrm{id}, \widetilde{f}^{\omega})"] & A \times D \arrow[d, "(\mathrm{id}, \widetilde{f})"] \\ A \times D^{\omega} \arrow[r, "A \times D^{s}"'] & A \times D^{\omega} \arrow[r, "\mathrm{id} \times \bigvee"'] & A \times D \end{tikzcd}$$

commutes. The right hand square does because $\widetilde{f}$ commutes with $\bigvee$. As for the left hand square, it transposes to

$$\begin{tikzcd}
A \times \mathbf{N} \arrow[r, "A \times s"] \arrow[d, "\langle \pi_1, \widetilde{g} \rangle"'] & A \times \mathbf{N} \arrow[d, "\langle \pi_1, \widetilde{g} \rangle"] \\
A \times D \arrow[r, "(\mathrm{id}, \widetilde{f})"'] & A \times D
\end{tikzcd}$$

whose commutation is part of the recursive definition of $\widetilde{g}$. □

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