COUNTABLE MEETS IN COHERENT SPACES WITH APPLICATIONS TO THE CYCLIC SPECTRUM

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Abstract. This paper reviews the basic properties of coherent spaces, characterizes them, and proves a theorem about countable meets of open sets. A number of examples of coherent spaces are given, including the set of all congruences (equipped with the Scott topology) of a model of a theory based on a set of partial operations. We also use an unpublished theorem of Makkai's to give an alternate proof of our main theorem. Finally, we apply these results to the Boolean cyclic spectrum and give some relevant examples.

1. Introduction

A **frame** is a complete lattice in which finite infs distribute over arbitrary sups. We denote the empty inf by $\top$ and the empty sup by $\bot$, which are the top and bottom elements, respectively, of the lattice. A map of frames preserves finite infs and arbitrary sups. The motivating example of a frame is the open set lattice of a topological space. Moreover, continuous maps induce frame homomorphisms. The result is a contravariant functor from the category $\mathcal{Top}$ of topological spaces to the category $\mathcal{Frm}$ of frames. A closed subset D of a topological space is called **indecomposable** if it is not possible to write it as union of two proper closed subsets. A space is called **sober** if every indecomposable closed set is the closure of a unique point. On sober spaces this functor is full and faithful.

If we let $\mathcal{Loc}$ denote the category of **locales**, which is simply $\mathcal{Frm}^{\mathrm{op}}$, the opposite of the category of frames, this results in a covariant functor $\mathcal{Top} \longrightarrow \mathcal{Loc}$.

In Section 2, we review basic properties of coherent spaces and prove a characterization theorem which is similar to known results. Section 3 shows several ways in which coherent spaces arise. A notable example concerns models of a first order theory described by operations and partial operations. We show, for example, that the set of subobjects as well as the set of congruences of a model, when equipped with a certain topology, called the Scott topology, give coherent spaces. In Section 4 we state and prove our main theorem that shows that if X is a coherent space and $\{U_i\}$ a countable family of open

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subsets of X then the intersection $\bigcap U_i$ (in the lattice of subspaces of X) coincides with their localic intersection $\bigwedge U_i$ (in the lattice of sublocales of X). Section 5 discusses the connection with an unpublished theorem of Makkai's. In Section 6, we apply our results to the Boolean cyclic spectrum and thus extend the work in [Kennison, 2002, Kennison, 2006, Kennison, 2009]. Section 7 gives examples.

1.1. Remark. In dealing with locales, it is standard to use "sublocale" to mean regular subobject. This means that sublocales correspond to regular quotients of frames. Since $\mathcal{F}rm$ is equational, there is a one-one correspondence between regular quotients and equivalence relations that are also models of the theory. Such equivalence relations are called **congruences**. Thus if F is a frame there is a one-one correspondence between congruences on F and sublocales of the locale L corresponding to F.

2. Basic definitions and preliminary results

The results in this section can all be found, with somewhat different proofs, in [Johnstone, 1982].

2.1. congruences and nuclei. A **nucleus** j on a frame F is a function $j : F \longrightarrow F$ such that

Nuc-1. j is expansive: $u \leq j(u)$ for all $u \in F$;

Nuc-2. j preserves finite inf;

Nuc-3. j is idempotent.

2.2. Theorem. *There is a one-one correspondence between nuclei and congruences on a frame.*

Proof. Let F be a frame and j be a nucleus on F. Define a relation $\mathbf{E}$ by $u \,\mathbf{E}\, v$ if $j(u) = j(v)$. Since this clearly defines an equivalence relation, it is sufficient to show it is closed under the frame operations. If $u_1 \,\mathbf{E}\, v_1$ and $u_2 \,\mathbf{E}\, v_2$, it follows immediately from Nuc-2, that $(u_1 \wedge u_2) \,\mathbf{E}\, (v_1 \wedge v_2)$. One might expect that showing that $\mathbf{E}$ is closed under arbitrary sup would require that j preserve arbitrary sup, which it does not do in general. From Nuc-2, it is clear that j is order preserving. Suppose we have two families $\{u_\alpha\}$ and $\{v_\alpha\}$ such that $u_\alpha \,\mathbf{E}\, v_\alpha$ for all α. Then $u_\alpha \leq j(u_\alpha) = j(v_\alpha) \leq j(\bigvee v_\alpha)$. Thus $\bigvee u_\alpha \leq j(\bigvee v_\alpha)$ so that $j(\bigvee u_\alpha) \leq j^2(\bigvee u_\alpha) = j(\bigvee v_\alpha)$ and the opposite inequality follows by symmetry. Thus $\mathbf{E}$ is a congruence.

Now suppose that $\mathbf{E}$ is a congruence on F. Define j by $j(u) = \sup\{v \mid u \,\mathbf{E}\, v\}$. It is clear that j is expansive. Since $\mathbf{E}$ is closed under arbitrary sup, it is clear that $u \,\mathbf{E}\, j(u)$ from which it follows that $j(u) \,\mathbf{E}\, j^2(u)$ so that $j^2(u) \leq j(u)$. We next show that j preserves the order. For if $u \leq v$, then $v = (u \vee v) \,\mathbf{E}\, (j(u) \vee v)$ so that $j(u) \leq j(u) \vee v \leq j(v)$. It follows that $j(u_1 \wedge u_2) \leq j(u_1) \wedge j(u_2)$. Now suppose $u \leq j(u_1) \wedge j(u_2)$. Then $(u \vee u_1) \,\mathbf{E}\, u_1$ and $(u \vee u_2) \,\mathbf{E}\, u_2$ from which it follows that $(u \vee (u_1 \wedge u_2)) \,\mathbf{E}\, (u_1 \wedge u_2)$ from which it is clear that $u \leq j(u_1 \wedge u_2)$. It is easy to verify that these processes are inverse to each other. ∎

2.3. Coherent spaces. A topological space is said to be **coherent** if it is compact, sober, if the compact open sets are a base for the topology and the intersection of two compact open sets is compact.

If X is a topological space and $\mathcal{M}$ is a subbase for the topology on X, let $\mathcal{N}$ be the set of complements of sets in $\mathcal{M}$. We call the topology generated by $\mathcal{M} \cup \mathcal{N}$ the **s-topology** (for **strong topology**) and call a subset of X **s-open, s-closed,** or **s-compact**, respectively, if it is open, closed or compact, respectively, in the s-topology. It is clear that open and closed sets are s-open and s-closed, respectively, while an s-compact set is compact.

2.4. Theorem. *A topological space X is coherent if and only if it has a subbase $\mathcal{M}$ such that the topology generated by $\mathcal{M}$ and the complements of the sets in $\mathcal{M}$ is compact Hausdorff.*

Proof. We start by replacing $\mathcal{M}$ by its closure under finite joins and meets. This expanded set satisfies the hypotheses if the original one does. If $\mathcal{N}$ is the set of complements of sets in the expanded $\mathcal{M}$, the set $\mathcal{M} \cup \mathcal{N}$ is still a subbase for the s-topology.

The forward implication is based on the proof of [Hochster, 1969, Theorem 1]. The set $\mathcal{M} \cup \mathcal{N}$ is a subbase for the closed set lattice in the s-topology (which means the set of its complements, also $\mathcal{M} \cup \mathcal{N}$, is a subbase for the open set lattice in that topology). By dualizing [Kelley, 1955, Theorem 4.6], it will suffice to show that for any $\mathcal{M}_0 \subseteq \mathcal{M}$ and any $\mathcal{N}_0 \subseteq \mathcal{N}$, if $\mathcal{M}_0 \cup \mathcal{N}_0$ has the finite intersection property (FIP), then it has a non-empty meet. We will do this using a series of claims.

We can assume that $\mathcal{M}_0$ is closed under finite meets and that $\mathcal{N}_0$ is maximal. The first is trivial, while the second follows readily from the fact that the join of any chain of families with the FIP has that property, since the FIP is determined by the finite subfamilies.

The meet $D = \bigcap_{N \in \mathcal{N}_0} N \neq \emptyset$ and meets every $M \in \mathcal{M}_0$ so that the family $\mathcal{M}_0 \cup \{D\}$ has the FIP. Fix $M \in \mathcal{M}_0$. The family $\{M \cap N \mid N \in \mathcal{N}_0\}$ certainly has the FIP and is a family of closed subsets of the compact space M.

D is indecomposable. Suppose $D = D_1 \cup D_2$ with D_1 and D_2 closed subsets of D. At least one of $\mathcal{M}_0 \cup \{D_1\}$ and $\mathcal{M}_0 \cup \{D_2\}$ has the FIP. Suppose that $\mathcal{M}_0 \cup \{D_1\}$ has the FIP. Since D_1 is closed, it is an intersection of sets in $\mathcal{N}$. These sets can be added to $\mathcal{N}_0$ without destroying the FIP in $\mathcal{M}_0 \cup \mathcal{N}_0$ and, by maximality, must already belong to $\mathcal{N}_0$. But this implies that $D_1 \supseteq \bigcap_{N \in \mathcal{N}_0} N$ and hence $D = D_1$.

The generic point x of D is in every $M \in \mathcal{M}_0$. For if $x \notin M \in \mathcal{M}_0$, then $D - M$ would be a proper closed subset of D that contained x.

This completes the proof of the forward implication. For the converse, we begin by noting that the sets in $\mathcal{M}$ are s-closed and hence s-compact so they are also compact. By assumption, $\mathcal{M}$ is a base for the topology and we easily see that every compact open set belongs to $\mathcal{M}$.

Since X is s-compact, it is also compact. In view of the above definition of coherent spaces, it suffices to show that X is sober. It is clear that X is T_0 for if $x, y \in X$ are such that for every $M \in \mathcal{M}$ we have $x \in M$ if and only if $y \in M$, then the same is true for the family consisting of all sets in $\mathcal{M} \cup \mathcal{N}$. Since this family forms a base for the s-topology, which is Hausdorff, it follows that $x = y$. We denote by $\bar{x}$, the closure of $\{x\}$. Let A be a closed, indecomposable subset. We have to find a point p, necessarily in A, such that A is $\overline{p}$. Since X is T_0, such a point is unique if it exists. Assume that no such point p exists. Then for every $a \in A$, we can choose a point $\varphi(a) \in A$ such that $\varphi(a) \notin \overline{a}$. Since $\varphi(a) \notin \overline{a}$, there exists a basic neighbourhood, $M_a \in \mathcal{M}$, of $\varphi(a)$ which misses a. Then a is in $\neg M_a$, the complement of M_a. Since $\neg M_a$ is s-open and A is s-closed, hence s-compact, there is a finite subset $F \subseteq A$ such that A is covered by $\{\neg M_a \mid a \in F\}$. Assume that F is chosen as small as possible. The set F cannot consists of a single element since $\varphi(a) \notin \neg M_a$. But if $F = F_1 \cup F_2$ is the union of two non-empty subsets, then $A \subseteq (\bigcup_{a \in F_1} \neg M_a) \cup (\bigcup_{a \in F_2} \neg M_a)$. But these are closed sets and A is indecomposable, so it must be contained in the one or the other factor. This contradicts the assumption that F was chosen as small as possible. ∎

The s-topology on a coherent space is usually referred to as the **patch topology** and we will adopt this terminology. We will sometimes call the original topology the **w-topology**.

2.5. Proposition. *Suppose X is a coherent space with base $\mathcal{M}$ of compact open sets. Suppose $\{M_\alpha\}$ is a family of sets from $\mathcal{M}$ and U is an open subset of X. If $\bigcap M_\alpha \subseteq U$, then for some finite set, say $\alpha_1, \ldots, \alpha_m$ of indices, we have that $\bigcap_{i=1}^m M_{\alpha_i} \subseteq U$.*

Proof. The sets M_α are s-closed in a compact space. The set U is open, hence s-open and therefore each $M_\alpha - U$ is closed. If $\bigcap M_\alpha \subseteq U$, then $\bigcap (M_\alpha - U) = \emptyset$, whence a finite intersection of them is empty. ∎

3. Examples of Coherent Spaces

This section shows that coherent spaces arise in many ways. Often the proof that a given space is coherent is omitted because it easily follows from the definition or from Theorem 2.4.

3.1. Notation. Whenever X is a given coherent space, $\mathcal{M}$ will denote the base of all compact open subsets and $\mathcal{N}$ will denote the family of all sets whose complements are in $\mathcal{M}$. When constructing a coherent space, $\mathcal{M}$ will denote a family satisfying the conditions of Theorem 2.5 (and, after closing $\mathcal{M}$ up under finite joins and meets, $\mathcal{N}$ will denote the family of all sets whose complements are in $\mathcal{M}$.)

3.2. Example. *Any s-closed subspace of a coherent space is coherent.*

3.3. Example. *Let X be coherent and let $\mathcal{M}$ and $\mathcal{N}$ be as above. Then X with the topology generated by $\mathcal{N}$ is coherent. We call the topology generated by $\mathcal{N}$ the* **dual** *of the original topology, generated by $\mathcal{M}$.*

3.4. Definition. *Let S be any set and let 2^S be the family of all subsets of S. For each $a \in S$ let $M(a) = \{A \subseteq S \mid a \in A\}$ and $\mathcal{M} = \{M(a) \mid a \in S\}$. Then:*

1. *the w-topology on 2^S is the one generated by the subbase $\mathcal{M}$;*

2. *the s-topology on 2^S is generated by $\mathcal{M}$ together with $\mathcal{N}$, the family of all complements of members of $\mathcal{M}$;*

3. *if $\mathcal{F} \subseteq 2^S$ then the w-topology (respectively the s-topology) on $\mathcal{F}$ is the relative topology on $\mathcal{F}$ obtained from the w-topology (resp. the s-topology) on 2^S.*

3.5. Example. *The space 2^S, with the w-topology, is coherent.*

In general, if $\mathcal{F} \subseteq 2^S$ is an s-closed subset, then $\mathcal{F}$ with the w-topology, is also coherent.

Proof. The w-topology on 2^S is generated by $\mathcal{M}$ as defined above. In view of Theorem 2.5, it suffices to observe that the s-topology, generated by $\mathcal{M} \cup \mathcal{N}$ is the product topology, obtained by regarding 2^S as a product of S copies of 2 (where 2 is the discrete space with two points). The assertion about $\mathcal{F}$ follows from 3.2. ∎

3.6. Notation. If $\mathcal{U}$ is an ultrafilter on 2^S, then $A_{\mathcal{U}} \subseteq S$ denotes the subspace for which $a \in A_{\mathcal{U}}$ if and only if $M(a) \in \mathcal{U}$.

3.7. Example. *Let $\mathcal{F} \subseteq 2^S$ be given and suppose that $A_{\mathcal{U}} \in \mathcal{F}$ whenever $\mathcal{U}$ is an ultrafilter on 2^S with $\mathcal{F} \in \mathcal{U}$. Then $\mathcal{F}$ with the w-topology is coherent.*

Proof. It suffices to observe that $A_{\mathcal{U}}$ is the limit of $\mathcal{U}$ in the s-topology on 2^S. So the given condition implies that $\mathcal{F}$ is an s-closed subset of 2^S. ∎

3.8. Definition. *Let $\mathbf{T}$ be a first order theory. We will say it is generated by finitary partial operations if there is a family $\Omega = \{\Omega_0, \Omega_1, \ldots, \Omega_n, \ldots\}$ of sets such that an algebra S for $\mathbf{T}$ is given by a partial function $\omega S : S^n \dashrightarrow S$ for each $n \in \mathbf{N}$ and each $\omega \in \Omega_n$. These partial operations may be subject to equations and Horn clauses, but they play no role in the construction.*

3.9. Example. *Let S be a $\mathbf{T}$-algebra. Let $\mathcal{F}$ be the family of all subsets of $F \subseteq S$ which satisfy (finitary) first order conditions built up from equality, the operations of T, the conditions $x \in F$ and closed under binary infs, sups and negation. Then $\mathcal{F}$, with the w-topology, is coherent.*

Such families would include $\mathbf{T}$-subalgebras and, $\mathbf{T}$-congruences, in case $S = R \times R$ where R is a model of a $\mathbf{T}$-algebra.

Proof. Such a family $\mathcal{F}$ satisfies the condition that whenever $\mathcal{U}$ is an ultrafilter on 2^S and $\mathcal{F} \in \mathcal{U}$, then $A_{\mathcal{U}} \in \mathcal{F}$. ∎

3.10. Example. *If R is a ring, then the set $\mathcal{P}$ of all prime ideals of R, with the w-toplogy, is coherent. The dual of this space is the set $\mathcal{P}$ with the Zariski topology.*

3.11. Remark. It is shown in [Hochster] that every coherent space X is homeomorphic to a space of the form $\mathcal{P}$ (as in the above example, with the w-topology) and also homeomorphic to a space $\mathcal{P}$ with the Zariski topology.

4. Countable meets

Recall from 1.1 that a sublocale is a regular subobject in the category of locales and that there is a one-one correspondence between nuclei and congruences on a frame. If the frame is $\mathcal{O}(X)$, the lattice of open sets of a space X, and if $A \subseteq X$ is a subspace, then the nucleus j_A defined by $j_A(U) = \bigcup\{V \in \mathcal{O}(X) \mid V \cap A \subseteq U\}$ corresponds to the subframe $\mathcal{O}(A)$.

4.1. Theorem. *Suppose that X is coherent and suppose $\{U_i\}$ is a sequence of open sets. Then their spatial intersection, $\bigcap U_i$, coincides with their localic intersection, $\bigwedge U_i$.*

Proof. Let $\mathcal{M}$ be the base of compact open sets. Suppose $A = \bigcap U_n$ and that $U_n = \bigcup_{\sigma\in\Sigma_n} M_{n,\sigma}$ with each $M_{n,\sigma} \in \mathcal{M}$. Let $L = \bigwedge U_n$ and denote by j_n, J_A, and j_L, resp. the nuclei corresponding to U_n, A, and L. By definition, $j_L = \bigvee j_n$, the sup taken in the lattice of nuclei. Since $A \subseteq U_n$ for all n, we see that $j_n \leq j_A$ whence $j_L \leq j_A$.

By a **choice function**, we mean a map $\xi : \mathbf{N} \longrightarrow \bigcup \Sigma_n$ such that $\xi(n) \in \Sigma_n$ for all $n > 0$. If ξ is a choice function, then from $M_{n,\xi(n)} \subseteq U_n$, it follows that $\bigcap_{n=1}^{\infty} M_{n,\xi(n)} \subseteq A$.

If we suppose that $L \subsetneqq A$, then $j_L \lneqq j_A$. Thus there is an open set V such that $j_L(V) \subsetneqq j_A(V)$ and hence there is an $M_0 \in \mathcal{M}$ with $M_0 \subseteq j_A(V)$ while $M_0 \not\subseteq j_L(V)$. This last implies that for all $n > 0$, $M_0 \not\subseteq j_n(V)$ which, we will show, leads to a contradiction.

4.2. Lemma. *Suppose that $M \in \mathcal{M}$ with $M \not\subseteq j_L(V)$. Then for each $n > 0$, there is a $\sigma \in \Sigma_n$ such that $M \cap M_{n,\sigma} \not\subseteq j_L(V)$.*

Since $M \not\subseteq j_L(V) = j_L^2(V)$ and $j_L = \bigvee j_n$, we see that $M \not\subseteq j_n(j_L(V))$ and hence $M \cap U_n \not\subseteq j_L(V)$. But $U_n = \bigcup_{\sigma\in\Sigma_n} M_{n,\sigma}$ so there must be some $\sigma \in \Sigma_n$ with $M \cap M_{n,\sigma} \not\subseteq j_L(V)$. ∎

We will use this lemma to construct a choice function ξ such that $M_0 \cap A_\xi \not\subseteq j_L(U)$. Assuming this can be done, it follows that $M_0 \cap A = M_0 \cap \bigcup_{\xi\in\Xi} A_\xi \not\subseteq U$ from which we conclude that $M_0 \not\subseteq j_A(U)$ in contradiction to our supposition.

In this proof we use the standard notation := to mean "defined as".

By the lemma, there is a $\xi(1) \in \Sigma_1$ such that $M_1 := M_0 \cap M_{1,\xi(1)} \not\subseteq j_L(U)$. Since $\mathcal{M}$ is closed under finite meets, it follows that $M_1 \in \mathcal{M}$. Another application of the lemma allows us to find a $\xi(2) \in \Sigma_2$ such that $M_2 := M_1 \cap M_{2,\xi(2)} \not\subseteq j_L(U)$. Since no term in the descending chain

$$M_0 \supseteq M_1 \supseteq \cdots \supseteq M_n \supseteq \cdots$$

is included in $j_L(U)$, it follows from Proposition 2.5 that $\bigcap M_n \not\subseteq j_L(U)$. Since $M_n \subseteq M_{n,\xi(n)}$ it also follows that $M_0 \cap A_\xi = M_0 \cap \bigcap_{n\in\mathbf{N}} M_{n,\xi(n)} \not\subseteq U$ and hence that $M_0 \cap A \not\subseteq U$ which means that $M_0 \not\subseteq j_A(U)$, contrary to our assumption. ∎

5. Connections with a Theorem of Makkai's

Theorem 4.1 can be derived from an unpublished theorem of Michael Makkai's that extends a famous result of [Rasiowa & Sikorski, 1950], which can also be found in [Rasiowa & Sikorski, 1968, p. 88]. In order to discuss the connection, we need to recall a few basic concepts. If L is a locale, then a **point** of L is defined to be a localic map $p : 1 \longrightarrow L$ where 1 stands for the one-point topological space (regarded as a locale). In other words, a point of L is a frame homomorphism $p : L \longrightarrow \{\bot, \top\}$. For example, if $L = \mathcal{O}(X)$ is a spatial locale, then every element $x \in X$ determines a point $\widehat{x}$ for which $\widehat{x}(U) = \top$ if and only if $x \in U$.

5.1. Enough points We say that the locale L **has enough points** if whenever $x \neq y$ there is a point $p : L \longrightarrow \{\bot, \top\}$ for which $p(u) \neq p(v)$. Recall that a locale is **spatial** if and only if it is isomorphic to the locale of all open subsets of a topological space. The following straightforward result is well-known:

5.2. Proposition. *A locale is spatial if and only if it has enough points.*

Proof. If L is isomorphic to $\mathcal{O}(X)$ where X is a topological space, then it clearly has enough points of the form $\widehat{x}$ for $x \in X$.

Conversely, assume that L has enough points. Let $\mathrm{pt}(L)$ be the set of all points of L and for each $u \in L$ define an open subset $\widehat{u} \subseteq \mathrm{pt}(L)$ by $\widehat{u} = \{p \in X \mid p(u) = \top\}$. It readily follows that L it is isomorphic to $\mathcal{O}(\mathrm{pt}(L))$. ∎

In the next proposition, we use the obvious fact that a T_0 space is sober if and only every indecomposable closed set has at least one generic point.

5.3. Proposition. *The set theoretic meet of any family of sober subspaces of a* T_0 *topological space is sober.*

Proof. Let X be a T_0 space and let $\{Y_\alpha\}$ be a family of sober subspaces of X. Let $p : 1 \longrightarrow Y$ be any point of $Y = \bigcap Y_\alpha$. Then, for each α, there is a corresponding point $p_\alpha : 1 \longrightarrow Y_\alpha$ given by $p_\alpha = i_\alpha p$ where i_α is the inclusion $Y \subseteq Y_\alpha$. Since Y_α is sober, the point p_α is represented by an element $x_\alpha \in Y_\alpha$. By factoring through the inclusion of $Y_\alpha \longrightarrow X$ we get a point of X which is represented by x_α. Since X is T_0, the elements x_α must all coincide, and must therefore be in Y. ∎

5.4. Proposition. *Let* $\{Y_\alpha\}$ *be a family of sober subspaces of a* T_0 *topological space* X*. Then,* $Y := \bigcap Y_\alpha$*, the intersection of the family in the lattice of all subspaces, coincides with* $L := \bigwedge Y_\alpha$*, the intersection in the lattice of all sublocales of* X*, if and only if the sublocale* $\bigwedge Y_\alpha$ *has enough points.*

Proof. If the two intersections coincide, then L is spatial so it must have enough points. Conversely, assume that L has enough points. Since L is contained in each spatial sublocale Y_α, we see that every point of L is a point of each Y_α which, by sobriety, corresponds to an element of Y_α and hence to an element of Y. Let $\mathbf{E}_{Y_\alpha}$, $\mathbf{E}_Y$ and $\mathbf{E}_L$ denote the congruences on $\mathcal{O}(X)$ determined by Y_α, Y, and L respectively. Since $\mathbf{E}_L$ is the sup in the lattice of congruences of the $\mathbf{E}_{Y_\alpha}$ and $\mathbf{E}_{Y_\alpha} \subseteq \mathbf{E}_Y$ for all α, it is immediate that $\mathbf{E}_L \subseteq \mathbf{E}_Y$. Thus we have a surjection $\mathcal{O}(X)/\,\mathbf{E}_L \twoheadrightarrow \mathcal{O}(X)/\,\mathbf{E}_Y$. But since the set P of points of L coincide with the set $|Y|$ we have that the bottom row of

$$\begin{tikzcd}
\mathcal{O}(X)/\,\mathbf{E}_L \arrow[r, two heads] \arrow[d, tail] & \mathcal{O}(X)/\,\mathbf{E}_Y \arrow[d, tail] \\
2^P \arrow[r, "\cong"'] & 2^{|Y|}
\end{tikzcd}$$

is an isomorphism from which it is evident that the top map is also an isomorphism. ∎

There are lots of sober subspaces in view of the above proposition and:

5.5. Proposition.

1. *Every closed subset of a sober space is sober.*

2. *Every open subset of a sober space is sober.*

Proof.

1. Straightforward, by looking at closed indecomposable subsets.

2. Let X be a sober space and let $U \subseteq X$ be an open subset. Let $A_0 \neq \emptyset$ be a (relatively) closed indecomposable subset of U. Let A be the closure of A_0 in X. We first claim that A is indecomposable in X. Suppose $A = B \cup C$. Let $B_0 = B \cap U$ and $C_0 = C \cap U$. Then $A_0 = B_0 \cup C_0$. Since A_0 is indecomposable in U we see that either $A_0 = B_0$ or $A_0 = C_0$. Say $A_0 = B_0$. Then the closure of B_0 is contained in B but the closure of B_0 is the closure of A_0, which is A so $A = B$.

   Now, since A is indecomposable in X there exists an element $x \in X$ such that A is the closure of $\{x\}$. It suffices to show that $x \in U$. Let $u \in A_0$. Then $u \in A$ and therefore is in the closure of $\{x\}$. Thus x is in every neighbourhood of u. Since U is a neighborhood of u we must have $x \in U$. ∎

5.6. Proposition. *Let X be a topological space and let $\{A_\alpha\}$ be a family of closed subsets of X. Then $\bigcap A_\alpha$, the intersection in the lattice of all subspaces of X, coincides with $\bigwedge A_\alpha$, the intersection in the lattice of all sublocales of X.*

Proof. For each α, let $W_\alpha \subseteq X$ be the complement of A_α. Then each W_α is obviously open. Let j_α be the nucleus of A_α, viewed as a sublocale of $\mathcal{O}(X)$. It is readily shown that j_α is given by $j_\alpha(U) = U \vee W_\alpha$ for all $U \in \mathcal{O}(X)$. Now let j be the sup of $\{j_\alpha\}$ in the lattice of all nuclei on $\mathcal{O}(X)$. It is readily shown that $j(U) = U \vee (\bigvee W_\alpha)$ but this is the nucleus for $\bigcap A_\alpha$, viewed as a sublocale of $\mathcal{O}(X)$. ■

5.7. Makkai's Theorem. The theorem of Rasiowa and Sikorski mentioned at the beginning of this section can be paraphrased as follows.

5.8. Theorem. *Let A be a Boolean algebra and Q be a countable family of subsets of A. Let B be the Boolean algebra freely generated by A together with one element forced to be a sup for each set in Q. Then there are enough 2-valued Boolean representations of B that preserve all the sups from Q to separate the points of A.*

Had the conclusion been that there were enough such "points" to separate the points of B, this would have given a different proof of our Theorem 4.1 in the special case of a Stone space. However, in a so-far unpublished work, Makkai has strengthened the Rasiowa-Sikorski theorem in two ways: the theorem is generalized to meet semi-lattices and the conclusion has been strengthened in the way required to give an alternate proof of our theorem in the general case.

5.9. Theorem. [Makkai, unpublished] *Assume that P is a meet-semi-lattice with a coverage system generated by $Y_1 \cup Y_2$ where Y_1 is a countable set of covers and Y_2 is a set of finite covers. Then the locale generated by these data (see [Johnstone, 1982, pp. 57–59]) has enough points.*

We now sketch how this result can be used to give an alternate proof of 4.1.

Proof. We work in the meet-semilattice $\mathcal{M}$ of all compact open subsets of X. For each i, we write $U_i = \bigcup \Sigma_i$ where $\Sigma_i \subseteq \mathcal{M}$. We let Y_1 be the countable set of covers given by saying that Σ_i is a cover of the top element of $\mathcal{O}(X)$. We let Y_2 be all covers of the form (M, C) where $M \in \mathcal{M}$ and $C \subseteq \mathcal{M}$ is a finite subset for which $\bigcup C = M$. It is readily shown that the locale generated by the meet-semilattice $\mathcal{M}$ with the coverage system generated by $Y_1 \cup Y_2$ is the sublocale $\bigwedge U_i$. By Makkai's result, it follows that $\bigwedge U_i$ has enough points, and the proof of this proposition then follows from 5.4. ■

6. Applications to the Boolean cyclic spectrum

In this section, we use the result about countable meets of open subsets of a coherent space to obtain results about the cyclic spectrum of a Boolean flow. Here we assume that (B, τ) is a Boolean flow, that W is the coherent space of all flow ideals of (B, τ), that Q is the canonical sheaf over W and that Q_{cyc} is the cyclic spectrum, obtained by forcing Q to become a cyclic flow. The cyclic spectrum, Q_{cyc} is a sheaf over $\mathcal{O}(W)_{\text{cyc}}$, a sublocale of the locale $\mathcal{O}(W)$ of all open subsets of W and we let j_{cyc} denote the associated nucleus. (For details, see the previous papers, see [Kennison, 2002, Kennison, 2006]. Here we have used Q for what was previously denoted B^0, Q_{cyc} is used for what was previously B^*, and $\mathcal{O}(W)_{\text{cyc}}$ for what was previously L_{cyc}.)

From here on, we will assume that (B, τ) is a given Boolean flow and we will use the above notation of W, Q, $\mathcal{O}(W)$, $\mathcal{O}(W)_{\text{cyc}}$, Q_{cyc}, j_{cyc}. We let $\Gamma(Q_{\text{cyc}})$ denote the set of global sections over the cyclic spectrum. Using Theorem 4.1, we will:

1. show that the cyclic spectrum of a countable flow is always spatial;
2. give an explicit description of the nucleus j_{cyc};
3. show that the cyclic spectrum always has the Lindelöf property;
4. describe $\Gamma(Q_{\text{cyc}})$, the set of global sections over the cyclic spectrum.

The cyclic spectrum of a countable flow is spatial.

6.1. Proposition. *If B is countable then the locale $\mathcal{O}(W)_{\text{cyc}}$ is spatial.*

Proof. The base of the cyclic spectrum, $\mathcal{O}(W)_{\text{cyc}}$, can be defined as the largest sublocale of $\mathcal{O}(W)$ for which $b \in B$ becomes cyclic, meaning that, for each such b, the basic open sets $\{N(b - \tau^k b)\}$ cover $\mathcal{O}(W)_{\text{cyc}}$. See [Kennison, 2006]. It follows that, if we let $\text{cyc}(b) = \bigcup\{N(b - \tau^k b) \mid k > 0\}$, then $\mathcal{O}(W)_{\text{cyc}}$ is the localic meet of $\{\text{cyc}(b) \mid b \in B\}$. If B is countable, then this meet is spatial by Theorem 4.1. ∎

For technical reasons, we want to generalize the above result. We need the following definition.

6.2. Definition. *Let $C \subseteq B$ be a countable subset. Let W_C be the largest sublocale of $\mathcal{O}(W)$ which makes every $c \in C$ cyclic. That is, W_C is the localic meet of $\{\text{cyc}(c) \mid c \in C\}$. Furthermore, we say that a flow ideal $I \in W$ is C-cyclic if for every $c \in C$, there exists $k > 0$ such that $I \in N(c - \tau^k c)$.*

6.3. Proposition. *The sublocale W_C is spatial for every countable $C \subseteq B$. The sublocale $W_C \subseteq W$ can be identified with the subspace of all C-cyclic flow ideals.*

Proof. The proof of the previous proposition clearly applies here. ∎

6.4. Remark. We will routinely identify W_C with the subspace of all C-cyclic flow ideals.

Description of the nucleus j_{cyc} and the Lindelöf property. Our next result gives a fairly technical, but quite useful, characterization of j_{cyc}. We first need some definitions and notation.

6.5. Definition. *An open set $U \in \mathcal{O}(W)$ is* **countably basic** *if we can write U as a countable union of basic open subsets of the form $N(b)$ for $b \in B$.*

6.6. Theorem. *Let (B, τ) be a Boolean flow and let $b \in B$ and $U \in \mathcal{O}(W)$ be given. Then $N(b) \subseteq j_{\text{cyc}}(U)$ if and only if there exists a countable subset $C \subseteq B$ and a countably basic open set $U_0 \subseteq U$ such that $N(b) \cap W_C \subseteq U_0$.*

PROOF. We define $J(U)$ as the union of all $N(b)$ for which there exists a countable subset $C \subseteq B$ and a countably basic open set $U_0 \subseteq U$ such that $N(b) \cap W_C \subseteq U_0$. We claim that that J is a nucleus. The only non-trivial step is proving that J is idempotent. By examining the nucleus j_C for the subspace $W_C \subseteq W$, it readily follows that $N(b) \cap W_C \subseteq U_0$ if and only if $N(b) \subseteq j_C(U_0)$. Assume $N(b) \subseteq J(J(U))$. Then there exists a countable subset $C \subseteq B$ and a countably basic set $V \subseteq J(U)$ such that $N(b) \subseteq j_C(V)$. Write $V = \bigcup N(d_n)$ where $N(d_n) \subseteq J(U)$ for all $n \in \mathbf{N}$. Then for each n, there exists a countably basic $V_n \subseteq U$ and a countable subset $C(n) \subseteq B$ with $N(d_n) \subseteq j_{C(n)}(V_n)$. Let $U_0 = \bigcup V_n$ and $D = C \cup \bigcup C(n)$. It suffices to show that $N(b) \subseteq j_D(U_0)$. But $j_D \geq j_C$ and $j_D \geq j_{C(n)}$ for all n. So $V \subseteq j_D(U_0)$ and $j_C(V) \subseteq j_D(j_D(U_0)) = j_D(U_0)$ and the claim follows.

The nucleus J makes every $b \in B$ (by letting $C = \{b\}$ and $U_0 = \bigcup N(b - \tau^n b)$ covers W_C and is countably basic. It follows that $J \geq j_{\rm cyc}$ and the opposite inclusion, $J \leq j_{\rm cyc}$ is obvious. ■

6.7. DEFINITION. *A locale L has the* **Lindelöf property** *if whenever $F \subseteq L$ covers L (that is whenever $\bigvee F = \top$) then F has a countable subset $F_0 \subseteq F$ which also covers L.*

6.8. PROPOSITION. *The locale $\mathcal{O}(W)_{\rm cyc}$ has the Lindelöf property.*

PROOF. It suffices to show that any cover of $\mathcal{O}(W)_{\rm cyc}$ by basic opens $N(b)$ has a countable subcover. Suppose that $U = \bigcup N(b_\alpha)$ and that U covers $\mathcal{O}(W)_{\rm cyc}$. Then $j_{\rm cyc}(U) = \top = \mathbf{N}(0)$ so, by the above theorem, there is a countably basic $U_0 \subseteq U$ with $j_{\rm cyc}(U_0) = \top$. Let $U_0 = \bigcup N(c_n)$. Then for each n we have $N(c_n) \subseteq \bigcup N(b_\alpha)$ which readily implies that there exists α with $N(c_n) \subseteq N(b_\alpha)$ and so only a countable set of the $N(b_\alpha)$ is needed to cover U_0 and hence to cover $\mathcal{O}(W)_{\rm cyc}$. ■

DESCRIPTION OF THE GLOBAL SECTIONS OVER THE CYCLIC SPECTRUM. It remains to describe the Boolean flow $\Gamma(Q_{\rm cyc})$ of all global sections over the cyclic spectrum of B. We first do this when B is countable then show how to extend that result to arbitrary B.

6.9. NOTATION. *Let (B, τ) be a Boolean flow. By Stone duality, we can suppose that $B = \operatorname{clop}(X)$, the algebra of clopen sets of the Stone space X. Another use of Stone duality shows that there is a unique continuous map $t : X \longrightarrow X$ such that $\tau(b) = t^{-1}(b)$ for all $b \in B$. See [Kennison, 2002] for more details.*

6.10. DEFINITION. *Let X be as above, let $b \in B$ be a clopen subset of X and let $x \in X$ be given. We say that x is* **k-cyclic with respect to** *b if, for all $n \geq 0$, we have $t^n(x) \in b$ if and only if $t^{n+k}(x) \in b$.*

We say that x is **cyclic** *with respect to b if x is k-cyclic with respect to b for some $k > 0$ (in this case, k is a* **period** *of x).*

Further, $x \in X$ is **cyclic** *if, for all $b \in B$, x is cyclic with respect to b. We let $X_{\rm cyc}$ denote the subspace of all cyclic elements of X.*

6.11. EXAMPLE. *Suppose (B, τ) is a cyclic Boolean flow, meaning that for every $b \in B$ there exists $k > 0$ such that $b = \tau^k b$. Then $X = X_{\rm cyc}$.*

6.12. Definition. *Let $I \subseteq B$ be a flow ideal. Then I corresponds to the flow quotient B/I which, by Stone duality, corresponds to a closed subflow $A(I) \subseteq X$.*

If $b \in B$, we let $\langle b \rangle$ denote the flow ideal generated by b. By abuse of language, we use $A(b)$ to denote $A(\langle b \rangle)$.

6.13. Lemma.

1. *Let $I \subseteq B$ be a flow ideal. Then $A(I) = \bigcap\{\neg b \mid b \in I\}$. Also, $b \in I$ if and only if $b \cap A(I) = \emptyset$.*

2. *Let $I, J \subseteq B$ be flow ideals. Then $I \subseteq J$ if and only if $A(J) \subseteq A(I)$.*

3. *Let $b \in B$ be given and regard b as a clopen subset of X. Then $x \in A(b)$ if and only if $t^n(x) \notin b$ for all $n \geq 0$.*

Proof.

1. First, we show that if $A \subseteq X$ is a closed subflow, then the corresponding flow ideal is $\{b \mid b \cap A = \emptyset\}$. Let $i : A \longrightarrow X$ be the inclusion map. Then $i^{-1} : \mathrm{clop}(X) \longrightarrow \mathrm{clop}(A)$ is the corresponding quotient of $B = \mathrm{clop}(X)$. Obviously $i^{-1}(b) = 0$ if and only if $A \cap b = \emptyset$.

   It follows that if A is the closed subflow that corresponds to the flow ideal I, then $A(I) \subseteq \bigcap\{\neg b \mid b \in I\}$. It is readily checked that $\bigcap\{\neg b \mid b \in I\}$ is topologically closed and closed under the action of t (as I is closed under the action of τ). Suppose $d \cap \bigcap\{\neg b \mid b \in I\} = \emptyset$. We must show that $d \in I$. It follows that d is covered by the elements of I and, by compactness, by a finite subset of I. Since I is closed under finite unions, there exists $b \in I$ with $d \leq b$ and this implies that $d \in I$.

2. Straightforward, in view of the first paragraph, above.

3. Clearly $A(b)$ is the largest closed subflow of X which is disjoint from b. A straightforward check shows that the given description of $A(b)$ has this property. ∎

6.14. Lemma. *Let $b \in B$ and $x \in X$ be given. Then x is k-cyclic with respect to b, if and only if $x \in A(b - \tau^k b)$.*

Proof. Straightforward. ∎

6.15. Lemma. *Let $b \in B$ and $k > 0$ be given. Then for every non-zero multiple m of k, we have $\langle b - \tau^m b \rangle \subseteq \langle b - \tau^k b \rangle$.*

Proof. It clearly suffices to show that if $I \subseteq B$ is a flow ideal and $(b - \tau^k b) \in I$ then $(b - \tau^m b) \in I$. But suppose $(b - \tau^k b) \in I$. By applying τ^k, we see that $(\tau^k b - \tau^{2k} b) \in I$. Adding $(b - \tau^k b)$ to it gives us $(b - \tau^{2k} b) \in I$ and the result follows by an easy induction. ∎

6.16. Proposition. *Let $I \subseteq B$ be a flow ideal. Then I is a cyclic flow ideal if and only if $A(I) \subseteq X_{\mathrm{cyc}}$.*

Proof. Assume $I \in W_{\mathrm{cyc}}$ and let $x \in A(I)$ be given. To prove that $x \in X_{\mathrm{cyc}}$, suppose $b \in B$. Since I is a cyclic flow ideal, there exists $k > 0$ such that $(b - \tau^k b) \in I$. It readily follows that $A(I) \subseteq A(b - \tau^k b)$ and, in view of lemma 6.14, we see that x is k-cyclic with respect to b. Since b is an arbitrary member of B, we see that $x \in X_{\mathrm{cyc}}$.

Conversely, assume $A(I) \subseteq X_{\mathrm{cyc}}$ and that $b \in B$ is given. We observe that $\{\neg(b-\tau^k b) \mid k > 0\}$ covers X_{cyc}, which easily follows using Lemma 6.14. Since $A(I) \subseteq X_{\mathrm{cyc}}$ it is covered by a finite set $\{\neg(b - \tau^{k(i)} b)\}$. Let m be a common multiple of the set $\{k(i)\}$, then it is readily shown that $A(I) \subseteq \neg(b - \tau^m b)$ which shows that $(b - \tau^m b) \in I$. ∎

6.17. Proposition. *Assume that B is countable. Let $d \in B$ be given. Let $\widehat{d}$ denote the corresponding constant section in $\Gamma(Q)$ and let $\widehat{d}_{\mathrm{cyc}}$ denote the restriction of $\widehat{d}$ to the subspace W_{cyc}. Then $\widehat{d}_{\mathrm{cyc}} = 0$ if and only if $d \cap X_{\mathrm{cyc}} = \emptyset$.*

Proof. Recall that d is a clopen subset of X. Assume that $d \cap X_{\mathrm{cyc}} = \emptyset$. Let $I \in W_{\mathrm{cyc}}$ be a given cyclic flow ideal. As shown above, $A(I) \subseteq X_{\mathrm{cyc}}$ so $d \cap A(I) = \emptyset$ and therefore $d \in I$. Since $d \in I$ for all $I \in W_{\mathrm{cyc}}$, it follows that $\widehat{d}_{\mathrm{cyc}}$, the restriction of $\widehat{d}$ to W_{cyc} is 0.

Conversely, assume that $\widehat{d}_{\mathrm{cyc}} = 0$ and that $x \in d \cap X_{\mathrm{cyc}}$. We need to derive a contradiction. Since $x \in X_{\mathrm{cyc}}$, we can, for every $b \in B$, find a positive integer $k(b)$ such that x is $k(b)$-cyclic with respect to b. This implies that $t^n(x) \notin (b - \tau^{k(b)} b)$ for all $n \geq 0$. Let I be the set of all $c \in B$ such that $t^n(x) \notin c$ for all $n \geq 0$. Then I is readily seen to be a flow ideal of B and a cyclic flow ideal as $(b - \tau^{k(b)} b) \in I$ for all $b \in B$. Moreover, $x \in A(I)$ so $d \cap A(I) \neq \emptyset$ as $x \in d \cap A(I)$. So $d \notin I$ and this implies that $\widehat{d}(I) \neq 0$ which contradicts the assumption that $\widehat{d}_{\mathrm{cyc}} = 0$. ∎

6.18. Proposition. *Let $c, d \in B$ be given (and regard each element of B as a clopen subset of X). Then*

$$N(c) \cap W_{\mathrm{cyc}} \subseteq N(d) \text{ if and only if } A(c) \cap X_{\mathrm{cyc}} \subseteq A(d)$$

Proof. First, assume $A(c) \cap X_{\mathrm{cyc}} \subseteq A(d)$. Let $I \in N(c) \cap W_{\mathrm{cyc}}$ be given. We need to show that $d \in I$. Since $c \in I$, we have $\langle c \rangle \subseteq I$ so $A(I) \subseteq A(c)$. By Proposition 6.16, we have $A(I) \subseteq X_{\mathrm{cyc}}$, so, by our assumption, $A(I) \subseteq A(d)$. But then, by (2) Lemma 6.13, $\langle d \rangle \subseteq I$ and $d \in I$.

Conversely, assume $N(c) \cap W_{\mathrm{cyc}} \subseteq N(d)$. Let $x \in A(c) \cap X_{\mathrm{cyc}}$ be given. Since $x \in X_{\mathrm{cyc}}$, we can choose, for each $b \in B$, an integer $k(b) > 0$ such that x is $k(b)$-cyclic with respect to b. Let I be the flow ideal generated by c and $\{b - \tau^{k(b)} b \mid b \in B\}$. Then I is the smallest flow ideal containing c and each $b - \tau^{k(b)} b$ so $A(I)$ is the largest closed subflow contained in $A(c)$ and each $A(b - \tau^{k(b)} b)$ which means that $A(I) = A(c) \cap \bigcap_b A(b - \tau^{k(b)} b)$. By the choice of $k(b)$, we have $x \in \bigcap_b A(b - \tau^{k(b)} b)$ and we assumed that $x \in A(c)$ so $x \in A(I)$. Clearly $t^n(x) \in A(I)$ for all $n \geq 0$. But by our assumption that $N(c) \cap W_{\mathrm{cyc}} \subseteq N(d)$, we see that $d \in I$ so $d \cap A(I) = \emptyset$ so $t^n(x) \notin d$ (as $t^n(x) \in A(I)$) which implies that $x \in A(d)$. ∎

6.19. Corollary. *Let $c_1, c_2, d \in B$ be given. Then:*

$$N(c_1) \cap N(c_2) \cap W_{\text{cyc}} \subseteq N(d) \text{ if and only if } A(c_1) \cap A(c_2) \cap X_{\text{cyc}} \subseteq A(d)$$

Proof. This follows from the above proposition with $c = c_1 \vee c_2$. Note that $N(c_1 \vee c_2) = N(c_1) \cap N(c_2)$ and $A(c_1 \vee c_2) = A(c_1) \cap A(c_2)$. ■

6.20. Definition. *A subset $S \subseteq X_{\text{cyc}}$ is* **rectified** *by $b \in B$ if there exists $d \in B$ such that*

$$S \cap A(b) = d \cap A(b) \cap X_{\text{cyc}}$$

We let $\mathrm{Rect}(S)$ denote the set of all $b \in B$ which rectify S. We say that $S \subseteq X_{\text{cyc}}$ is **regular** *if*

$$W_{\text{cyc}} \subseteq \bigcup \{N(b) \mid b \in \mathrm{Rect}(S)\}.$$

6.21. Proposition.

1. *If $d \in B$, then $d \cap X_{\text{cyc}}$ is a regular subset of X_{cyc}.*

2. *The regular subsets of X_{cyc} are closed under complementation (within X_{cyc}) and under finite unions and intersections (which includes the empty subset and X_{cyc} itself).*

Proof.

1. First, it is clear that every $b \in B$ rectifies $d \cap X_{\text{cyc}}$ and when $b = \perp$, then $N(b)$ covers W.

2. Closure under complementation follows by verifying that b rectifies S if and only if b rectifies $X_{\text{cyc}} - S$. To prove closure under pairwise intersections, it suffices to verify that if $b \in \mathrm{Rect}(S)$ and $c \in \mathrm{Rect}(T)$, then $b \vee c \in \mathrm{Rect}(S \cap T)$. Note that if S is empty, then every $b \in B$ rectifies S. The rest of the proof follows by considering complements within X_{cyc}. ■

6.22. Notation. *Assume that B is countable. We let $\mathrm{Reg}(B)$ denote the Boolean algebra of all regular subsets of X_{cyc}. In view of (1) of the above proposition, there is a canonical Boolean homomorphism from B to $\mathrm{Reg}(B)$.*

6.23. Lemma. *If $d, e \in B$ are given, then $d \cap A(d - e) \subseteq e$.*

Proof. Assume the contrary, that there exists $x \in d \cap A(d - e)$ but with $x \notin e$. Then $x \in (d - e) \cap A(d - e)$ which is a contradiction. ■

6.24. THEOREM. *Assume that* B *is countable. The Boolean algebra* $\Gamma(Q_{\mathrm{cyc}})$ *is canonically isomorphic to* $\mathrm{Reg}(B)$. *Moreover, the isomorphism commutes with the map* $B \longrightarrow \mathrm{Reg}(B)$ *mentioned above and the map* $B \longrightarrow \Gamma(Q_{\mathrm{cyc}})$ *which sends* $d \in B$ *to* $\widehat{d}$.

PROOF. Let $\sigma \in \Gamma(Q_{\mathrm{cyc}})$ be given. Locally, σ agrees with constant sections of the form $\widehat{d}$ so we can find a family $\{(d_\alpha, b_\alpha)\}$ such that $\sigma = \widehat{d}_\alpha$ on $N(b_\alpha) \cap W_{\mathrm{cyc}}$. It follows that these sections are compatible, meaning that

$$N(b_\alpha) \cap N(b_\beta) \cap W_{\mathrm{cyc}} \subseteq N(d_\alpha - d_\beta)$$

Now define $S \subseteq X_{\mathrm{cyc}}$ as $\bigcup_\alpha (d_\alpha \cap A(b_\alpha) \cap X_{\mathrm{cyc}})$. We claim that each b_β rectifies S. We must show that $S \cap A(b_\beta) = d_\beta \cap A(b_\beta) \cap X_{\mathrm{cyc}}$. We have:

$$S \cap A(b_\beta) = \bigcup_\alpha (A(b_\beta) \cap d_\alpha \cap A(b_\alpha) \cap X_{\mathrm{cyc}})$$

We note that if $\beta = \alpha$ then $(A(b_\beta) \cap d_\alpha \cap A(b_\alpha) \cap X_{\mathrm{cyc}})$ reduces to $d_\beta \cap A(b_\beta) \cap X_{\mathrm{cyc}}$, so it suffices to show in general that $(A(b_\beta) \cap d_\alpha \cap A(b_\alpha) \cap X_{\mathrm{cyc}}) \subseteq d_\beta$. By Corollary 6.19, and the above condition that $N(b_\alpha) \cap N(b_\beta) \cap W_{\mathrm{cyc}} \subseteq N(d_\alpha - d_b e)$, we see that

$$A(b_\alpha) \cap A(b_\beta) \cap X_{\mathrm{cyc}} \subseteq A(d_\alpha - d_\beta).$$

So $(A(b_\beta) \cap d_\alpha \cap A(b_\alpha) \cap X_{\mathrm{cyc}}) \subseteq d_\alpha \cap A(d_\alpha - d_\beta)$. The claim now follows by the above lemma. The claim implies that S is regular, so we have associated the regular set S to the global section σ

Conversely, let $S \in \mathrm{Reg}(B)$ be given. Let $\{b_\alpha\}$ be a family of elements of B which rectify S and cover W_{cyc}. Then for each α there exists d_α such that

$$S \cap A(b_\alpha) = d_\alpha \cap A(b_\alpha) \cap X_{\mathrm{cyc}}.$$

Observe that for all α, β:

$$d_\alpha \cap A(b_\alpha) \cap A(b_\beta) \cap X_{\mathrm{cyc}} = d_\beta \cap A(b_\alpha) \cap A(b_\beta) \cap X_{\mathrm{cyc}}$$

as both are $S \cap A(b_\alpha) \cap A(b_\beta)$. Since $A(b_\alpha) \cap A(b_\beta) \cap X_{\mathrm{cyc}}$ is a subflow (closed under the action of t) the above result readily implies that

$$A(b_\alpha) \cap A(b_\beta) \cap X_{\mathrm{cyc}} \subseteq A(d_\alpha - d_\beta).$$

And by Corollary 6.19, this implies that

$$N(b_\alpha) \cap N(b_\beta) \cap W_{\mathrm{cyc}} \subseteq N(d_\alpha - d_\beta).$$

But this is precisely what we need to show that the local sections $\widehat{d}_\alpha$ on $N(b_\alpha)$ piece together to give us a global section σ.

So, to each global section σ we have associated a regular set S and to each regular set S we have associated a global section σ. A routine check shows that this defines the desired isomorphism. ∎

Global sections of $\Gamma(Q_{\mathrm{cyc}})$ when B need not be countable. We recall the definition of W_C for each countable subset $C \subseteq B$. As noted above, W_C is a spatial locale for each such countable subset C. Let Q_C be the restriction of Q, the canonical sheaf over W, to the subspace W_C. The global sections $\Gamma(Q_C)$ can be determined by an approach strictly similar to the approach in the above theorem. That is, we can define $X_C \subseteq X$, as the set of all $x \in X$ which are cyclic with respect to every $c \in C$. We can then define a subset of X_C to be C-regular by an obvious modification of the definition of regular (in fact, just replace X_{cyc} by X_C). The argument used in the proof of 6.24 can then be used to show that $\Gamma(Q_C)$ is canonically isomorphic to the family of all C-regular subsets of X_C. Then the global sections over the cyclic spectrum, for arbitrary B, can be described using the following theorem.

6.25. Theorem. *$\Gamma(Q_{\mathrm{cyc}})$ is the colimit of $\Gamma(Q|W_C)$ where C varies over the filtered family of all countable subsets of B.*

Proof. We must prove that every global section in $\Gamma(Q_{\mathrm{cyc}})$ is the restriction of a global section in $\Gamma(Q|W_C)$ for some countable subset $C \subseteq B$. We must also show that two such global sections over W_C and W_D have the same restriction to $\mathcal{O}(W)_{\mathrm{cyc}}$ if and only if they have the same restriction to some W_E where $E \subseteq B$ is a countable subset with $C \cup D \subseteq E$.

Clearly, every global section $\sigma \in \Gamma(Q_{\mathrm{cyc}})$ is represented by a compatible family $\{(d_\alpha, b_\alpha)\}$ for which σ equals $\widehat{d}_\alpha$ on $N(b_\alpha)$. Since $\mathcal{O}(W)_{\mathrm{cyc}}$ is Lindelöf, we can assume that the family is countable and write it as $\{(b_n, d_n) \mid n \in \mathbf{N}\}$. The condition for the family being compatible is equivalent to a countable set of conditions of the form $N(b_\alpha) \cap N(b_\beta) \subseteq j_{\mathrm{cyc}}(N(\widehat{d}_\alpha - \widehat{d}_\beta)$. But by using Theorem 6.6, this condition holds if and only if it holds when we restrict to some W_C. It readily follows that $\{(d_\alpha, b_\alpha)\}$ will be a compatible family that defines a section in $\Gamma(Q_{\mathrm{cyc}})$ if and only if it is compatible enough to define a section in $\Gamma(Q|W_C)$ for some countable $C \subseteq B$. The remaining details are now straightforward. ∎

7. Examples

7.1. Example of a non-spatial cyclic spectrum. In constructing this example, it is notationally convenient to introduce, for each $n \in \mathbf{N}$, a symbol a_n and for each $f \in \mathbf{N}^{\mathbf{N}}$ a symbol h_f. We let

$$G = \{a_n \mid n \in \mathbf{N}\} \cup \{h_f \mid f \in \mathbf{N}^{\mathbf{N}}\}$$

We let (B, τ) be the free Boolean flow generated by G.

For each $n \in \mathbf{N}$ and $f \in \mathbf{N}^{\mathbf{N}}$ we let

$$U(n, f) = N(\tau^{f(n)} a_n - a_n) \cap N(\tau^n h_f - h_f)$$

We then claim that:

1. the family $\{U(n, f)\}$ covers W_{cyc};

2. the above family has no countable subcover;

3. the cyclic spectrum of this flow is not spatial.

PROOF.

1. Let $U \in W_{\text{cyc}}$ be given. Since I is cyclic, we can clearly define $f : \mathbf{N} \longrightarrow \mathbf{N}$ such that $\tau^{f(n)}a_n - a_n \in I$ for all $n \in \mathbf{N}$. But, there also must be an $n \in \mathbf{N}$ for which $\tau^n h_f - h_f \in I$ and it follows that $I \in U(n, f)$.

2. Assume there is a countable subcover. Then we can clearly find a sequence $(f_1, f_2, \ldots f_n, \ldots)$ of functions from $\mathbf{N}$ to $\mathbf{N}$ such that $\{U(n, f_i) \mid n, i \in \mathbf{N}\}$ covers W_{cyc}.

   Now define $u : \mathbf{N} \longrightarrow \mathbf{N}$ such that $u(n) > f_i(n)$ whenever $i \leq n$. Let $v : \mathbf{N}^{\mathbf{N}} \longrightarrow \mathbf{N}$ be any function for which $v(f_i) = i$ and $v(f) > 0$ for all f. Let I be the flow ideal generated by:

   $$\{\tau^{u(n)}a_n - a_n \mid n \in \mathbf{N}\} \cup \{\tau^{v(f)}h_f - h_f \mid f \in \mathbf{N}^{\mathbf{N}}\}$$

   Then I is obviously cyclic, so there exist $n, i \in \mathbf{N}$ with $I \in U(n, f_i)$.

   But this implies that $\tau^{u(n)}a_n - a_n \in I$ and so $u(n) < f_i(n)$ which implies that $i > n$. On the other hand, $\tau^n h_{f_i} - h_{f_i} \in I$ which implies that $n \geq v(f_i)$ so $i \leq n$ which is a contradiction.

3. The cyclic spectrum cannot be spatial because, as shown in [Kennison, 2006, Proposition 4.1] this implies that it is a sheaf over the space W_{cyc} and, by Proposition 6.8, that W_{cyc} is Lindelöf, which contradicts the above. ■

7.2. EXAMPLES OF REGULAR SETS.

7.3. DEFINITION. *Let (B, τ) be a Boolean flow. We say that $G \subseteq B$* **generates** B **as a flow** *if no proper subflow of B contains G,*

7.4. DEFINITION. *By a* k**-periodic subset of N** *we mean a subset $P \subseteq \mathbf{N}$ for which $n \in P$ if and only if $n+k \in P$. We further say that a subset is* **periodic** *if it is k-periodic for some $k > 0$.*

7.5. DEFINITION. *We say that $x \in X$ is* **periodic** *if there exists $n > 0$, such that $t^n x = x$. We let $\text{Per}(X)$ denote the set of all periodic elements of X. We note that $\text{Per}(X) \subseteq X_{\text{cyc}}$.*

7.6. EXAMPLE. Let $X = \{0, 1\}^{\mathbf{N}}$ and define $t : X \longrightarrow X$ as the shift map (so that $t(x_0, x_1, x_2, \ldots) = (x_1, x_2, \ldots)$). Let (B, τ) be the corresponding flow in Boolean algebras. Then $X_{\text{cyc}} = \text{Per}(X)$ is the set of all periodic sequences and every subset of X_{cyc} is regular.

Proof. $g = \pi_0^{-1}(1)$ generates B as a flow. Note that $x \in X$ is a periodic sequence if and only if x is cyclic with respect to g. It readily follows that $X_{\text{cyc}} = \text{Per}(X)$.

Let $b_n = \tau^n g - g$ and let $S \subseteq X_{\text{cyc}}$ be any subset. Then we claim that b_n rectifies S. It is readily shown that $A(b_n)$ is the set of all sequences in X which are n-periodic, which is a finite set. So every subset of $A(b_n)$ is relatively clopen and is clearly of the form $d \cap A(b_n)$. It easily follows that b_n rectifies any subset S. But the family of all $N(b_n)$ clearly covers W_{cyc} so S is regular. ∎

7.7. Remark. If (B, τ) is finitely generated (as a flow) then X_{cyc} always coincides with $\text{Per}(X)$ and every subset of X_{cyc} is regular, as the above argument generalizes.

The following proposition is useful in finding regular sets.

7.8. Proposition. *As usual, let (X, t) be a flow in Stone spaces and let (B, τ) be the corresponding flow in Boolean algebras. Let k-$\text{Cy}(c)$ be the set of all $x \in X_{\text{cyc}}$ which are k-cyclic with respect to c. Let $c \in B$ and the positive integer k be given. Then:*

1. *k-$\text{Cy}(c)$ is regular;*
2. *$S = X_{\text{cyc}} \cap \bigcap_{n \geq 0} \tau^n c$ is regular;*
3. *$S = X_{\text{cyc}} \cap \bigcap_{n \geq 0} \tau^n (\neg c)$ is regular.*

Proof.

1. Let $b_n = \tau^n c - c$. It is readily shown that $A(b_n) = n$-$\text{Cy}(c)$. It follows that $S \cap A(b_n) = (k, n)$-$\text{Cy}(c)$, where $(k, n) = \gcd(k, n)$. A straightforward argument proves that the set (k, n)-$\text{Cy}(c)$ is relatively clopen in n-$\text{Cy}(c)$ (as we only have to restrict the values of $t^i x$ for $i = 0, 1, \ldots, n-1$). A standard argument, using the compactness of $A(b_n) = n$-$\text{Cy}(c)$, shows that there is a clopen set d of X such that (k, n)-$\text{Cy}(c) = d \cap A(b_n)$ and, from this, it follows that each b_n rectifies S. As noted in the previous proof, this shows that S is regular.
2. Here $S = X_{\text{cyc}} \cap k$-$\text{Cy}(X) \cap c$ is regular in view of Proposition 6.21.
3. Note that (2) implies (3) in view of the substitution of $\neg c$ for c. ∎

7.9. Example. Let $\Sigma_0 = \{\sigma_0, \sigma_1, \ldots, \sigma_n, \ldots\}$ be a sequence of "symbols" and give Σ_0 the discrete topology. Let $\Sigma = \Sigma_0 \cup \{\infty\}$ be its one-point compactification. Let $X = \Sigma^{\mathbf{N}}$ and define $t : X \longrightarrow X$ so that $t(x_0, x_1, x_2, \ldots) = (x_1, x_2, \ldots)$. Let $\pi_n : X \longrightarrow \Sigma$ denote the nth projection onto $\{0, 1\}$. For each $i \in \mathbf{N}$, let $g_i \in B$ be defined as $\pi_0^{-1}(\sigma_i)$. Let $G = \{g_i\}$. For this example, we claim that:

1. G generates B as a flow;
2. $\text{Per}(X)$ is a proper subset of X_{cyc};

3. there are regular subsets not of the form $b \cap X_{\text{cyc}}$ for $b \in B$ (so not every global section over the cyclic spectrum is of the form $\widehat{b}$ for $b \in B$);

4. not every subset of X_{cyc} is regular.

Before proving the above claims, we insert a useful definition and some lemmas.

7.10. Definition. *Let (B, τ) be a Boolean flow and let $G \subseteq B$ generate B as a flow. We say that $p \in B$ is G-**prescriptive** if there exists $\overline{g} = (g_1, \ldots, g_m) \in G^m$ and an m-tuple $\overline{k} = (k_1, \ldots, k_m)$ of positive integers such that*

$$p = p(\overline{g}, \overline{k}) = \bigvee_{1 \leq i \leq m} (\tau^{k_i} g_i - g_i)$$

The following lemma shows that, in a sense, a G-prescriptive element of B has the effect of prescribing a period to an m-tuple of elements of G.

7.11. Lemma. *Let $p = p(\overline{g}, \overline{k})$ be a G-prescriptive element of B. Then $A(p)$ is the set of all $x \in X$ which are k_i-cyclic with respect to g_i for $1 \leq i \leq m$.*

Proof. It is straightforward to show that x is k-cyclic with respect to g if and only if $t^n x$ is never in $\tau^k g - g$ (for any $n \in \mathbf{N}$). The proof then follows. ∎

Recall that $\langle c \rangle$ is the smallest flow ideal of B which contains c. Also $b \in \langle c \rangle$ if and only if b misses $A(c)$. Remember that for $b, c \in B$ we have that $c, A(c)$ and b are subsets of X while $N(c)$ and $N(b)$ are subsets of W.

7.12. Lemma. *Let $b, c \in B$ be given. Then the following are equivalent:*

1. *$N(c) \subseteq N(b)$;*

2. *$b \in \langle c \rangle$;*

3. *$A(c) \subseteq A(b)$.*

Proof. (1) $\Longleftrightarrow$ (2): If $N(c) \subseteq N(b)$, then $\langle c \rangle \in N(c) \subseteq N(b)$ so $b \in \langle c \rangle$. Conversely, assume $b \in \langle c \rangle$. If $I \in N(c)$ then $c \in I$ so $\langle c \rangle \subseteq I$ and $b \in \langle c \rangle \subseteq I$ so $I \in N(b)$.

(2)$\Longleftrightarrow$ (3): Assume $b \in \langle c \rangle$. Then $\langle b \rangle \subseteq \langle c \rangle$ which, by the duality between flow ideals of B and closed subflows of X, implies that $A(c) \subseteq A(b)$. Conversely, assume $A(c) \subseteq A(b)$. It follows that b misses $A(c)$ so $b \in \langle c \rangle$. ∎

7.13. Corollary. *Let $S \subseteq X_{\text{cyc}}$ be given and assume that b rectifies S. If $b \in \langle c \rangle$, then c rectifies S.*

Proof. By the above lemma, $A(c) \subseteq A(b)$ and the result easily follows. ∎

7.14. Proposition. *Let (B, τ) be a countable Boolean flow and let $G \subseteq B$ generate B as a flow. Let $S \subseteq X_{\text{cyc}}$ be given and let G-*Rect(S)* be the set of all G-prescriptive elements that rectify S. Then S is regular if and only if*

$$W_{\text{cyc}} \subseteq \bigcup \{N(p) \mid p \in G\text{-Rect}(S)\}$$

Proof. Assume that S is regular. Then

$$W_{\text{cyc}} \subseteq \bigcup \{N(b) \mid b \in \text{Rect}(S)\}.$$

Let $I \in W_{\text{cyc}}$ be given. Since I is a cyclic flow ideal, for each $g \in G$, we can choose $k(g) > 0$ such that $\tau^{k(g)}g - g \in I$. Let I_0 be the smallest flow ideal containing $\{\tau^{k(g)}g - g \mid g \in G\}$. Since G generates B as a flow, it readily follows that I_0 is cyclic so there exists $b \in \text{Rect}(S)$ such that $b \in I_0$. But for any element $b \in I_0$, there is a finite set $F \subseteq G$ such that b is in the smallest flow ideal containing $\tau^{k(g)}g - g$ for all $g \in F$. Write $F = \{g_1, \ldots, g_m\}$, let $\overline{g} = (g_1, \ldots, g_m)$ and $\overline{k} = (k_1, \ldots, k_m)$ where $k_i = k(g_i)$. Let $p = p(\overline{g}, \overline{k})$. Then by the choice of F, we see that $b \in \langle p \rangle$. By the above Lemma, we have $p \in G\text{-Rect}(S)$ and $p \in I_0 \subseteq I$. Since I is an arbitrary member of W_{cyc}, it follows that

$$W_{\text{cyc}} \subseteq \bigcup \{N(p) \mid p \in G\text{-Rect}(S)\}$$

The converse is trivial. ∎

Proof of Example 7.9

1. Note that the clopen set $\pi_n^{-1}(\sigma_i) = \tau^n(g_i)$. If $U \subseteq \Sigma$ is a clopen neighbourhood of ∞, then $F = \{i \in \mathbf{N} \mid \sigma_i \in \Sigma - U\}$ is finite and $\pi_n^{-1}(U) = \bigwedge_{i \in F} \tau^n(\neg g_i)$. The remaining details are now straightforward.

2. It is readily shown that $x \in X_{\text{cyc}}$ if and only if x is periodic in each $\sigma_i \in \Sigma_0$ separately. That is, if, for each i, the set $\{n \in \mathbf{N} \mid x_n = \sigma_i\}$ is a periodic subset of $\mathbf{N}$. The result easily follows.

3. Choose $g \in G$. The subset 2-Cy(g) is regular, in view of 7.8, but is clearly not of the form $b \cap X_{\text{cyc}}$ for any clopen subset $b \subseteq X$ (as clopen sets can only restrict x_n for finitely many n). It follows that the global section corresponding to 2-Cy(g) is not of the form $\widehat{b}$ for any $b \in B$.

4. Let S be the set of all $x \in X_{\text{cyc}}$ such that $x_n \neq \sigma_m$ for any *even* $m \in \mathbf{N}$. If S is regular then, by Proposition 7.8, there are enough G-prescriptive elements that rectify S. But suppose p is G-prescriptive and that $S \cap A(p) = d \cap A(p) \cap X_{\text{cyc}}$ for some $d \in B$. Choose g_k for an odd k such that g_k is not involved in any part of p or d. Let $x \in X$ be the sequence which is constantly σ_k. Then $x \in S \cap A(p)$ so $x \in d \cap A(p) \cap X_{\text{cyc}}$. But we could just as well have chosen k to be even, in which case x is still in $d \cap A(p) \cap X_{\text{cyc}}$, but x is not in S, which leads to a contradiction. ∎

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