WHAT IS THE CENTER?

by

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Category theory was invented to define "natural". Despite this, certain very natural object constructions are not functorial in any obvious way[1]. Examples of these are completions of all kinds, injective envelope constructions and the construction of the center of a group. All except the last-named have categorical interpretations; we wish to provide one for the center. In doing so, we were motivated by considerations of obstruction theory in cohomology. The solution we derive seems right for that.

In 1. we give the basic definitions. The rest of the paper is concerned with existence: finding conditions under which every object of some category $\underline{X}$ has a center. In 2. general conditions are given, and in 3. these are applied to equational categories.

1. Basic Definition.

Let X be a group. The center $Z \subset X$ is easily seen to be the largest subgroup of X such that there exists a group homomorphism

$$Z \times X \longrightarrow X$$

whose restriction to Z is the inclusion and whose restriction to

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[1]Actually Robert Paré, a student at McGill, has shown how all these may be made "functorial" if the mapping functions are replaced by relations.

X is the identity. Clearly, this property characterizes Z. Of course, a similar definition in an abstract category cannot make sense unless the category is pointed. Otherwise, it does not make sense to speak of restricting a map on a product to its coordinates. Accordingly, we have:

<u>Definition 1.1</u>: Let $\underline{X}$ be a pointed category with finite products, and let $X \in \underline{X}$. A subobject $Z \subset X$ is called <u>central</u> in X if there is a morphism $Z \times X \longrightarrow X$ whose restriction to Z is the inclusion, and whose restriction to X is the identity. Z is called the center of X if it is central and includes every central subobject of X.

Of course, this definition leaves the question of existence of a center wide open.

<u>2. The Main Theorem</u>.

<u>Definition 2.1</u>: A category $\underline{X}$ is called a Z-category if the following conditions are satisfied:

Z.1. $\underline{X}$ is pointed.

Z.2. $\underline{X}$ has finite projective limits.

Z.3. The "coordinate axes" $X_1 \longrightarrow X_1 \times X_2 \longleftarrow X_2$ are collectively epi for any $X_1, X_2 \in \underline{X}$.

Z.4. Any morphism $f: X \longrightarrow Y$ of $\underline{X}$ factors as $X \longrightarrow Y_0 \longrightarrow Y$ where $X \longrightarrow Y_0$ is a coequalizer (necessarily of its kernel pair) and $Y_0 \longrightarrow Y$ is monic.

Z.5. If $X \in \underline{X}$ and $\{X_i\}$ is a directed family of subobjects of X, then $\operatorname{colim} X_i$ exists and is a subobject of X.

Z.6. For any $X' \in \underline{X}$ the functor $X' \times -$ commutes with those inductive limits assumed in Z.4. and Z.5. This means that if $f\colon X \twoheadrightarrow Y$ is a morphism which factors as $X \twoheadrightarrow Y_0 \rightarrowtail Y$ as above, then $X' \times X \twoheadrightarrow X' \times Y_0$ is still a coequalizer (and $X' \times Y_0 \rightarrowtail X' \times Y$ remains a monic). Similarly, if $\{X_i\}$ is a collection of subobjects of X, then $\operatorname{colim}(X' \times X_i) \longrightarrow X' \times \operatorname{colim} X_i$ by the natural map is an isomorphism.

This appears to be quite a restrictive set of hypotheses. However, many algebraic categories of interest to us satisfy them. We shall discuss this in 3.

If $X_1, \ldots, X_m, Y_1, \ldots, Y_n \in \underline{X}$ and $f\colon X_1 \times \ldots \times X_m \longrightarrow Y_1 \times \ldots \times Y_n$ is a morphism, then f has a matrix

$$\| f \| = \left\| \begin{matrix} f_{11} & & f_{1n} \\ & & \\ & & \\ f_{m1} & & f_{mn} \end{matrix} \right\|$$

where f_{ij} is the composition

$$X_i \longrightarrow X_1 \times \ldots \times X_m \longrightarrow Y_1 \times \ldots \times Y_n \longrightarrow Y_j \ .$$

The correspondence $f \longmapsto \| f \|$ is not an isomorphism as it is in an additive category, but Z.3. together with the usual properties of products insures that this correspondence is injective. If

$$X \xrightarrow{f} X_1 \times \dots \times X_n \xrightarrow{g} X'$$

have matrices $\| f_1, \dots, f_n \|$ and $\left\| \begin{matrix} g_1 \\ \vdots \\ g_n \end{matrix} \right\|$, we will let

$g_1 f_1 + \dots + g_n f_n$ denote gf. The "+" does not necessarily have any real significance except that it now permits composition of maps between products to be represented by ordinary matrix multiplication. The details are familiar and will be omitted. We will frequently write down a matrix to denote a morphism, understanding, of course, that not every matrix stands for a morphism. However, a matrix with at most one non-zero map in each row always represents a morphism. For example, $\| 0, \dots, 0, f_i, 0, \dots, 0 \| :$ $X_1 \times \dots \times X_n \longrightarrow X$ represents $X_1 \times \dots \times X_n \xrightarrow{\text{proj.}} X_i \xrightarrow{f_i} X$.

We are now ready to give the main result of this paper.

<u>Theorem 2.2</u>. Let $\underline{X}$ be a Z-category. Then every object of $\underline{X}$ has a center.

<u>Proof</u>. Let $X \in \underline{X}$ and $\underline{Z} = \{Z_i\}$ be the class of central subobjects of X. We must show that $\underline{Z}$ contains a largest element. First, we show it is directed. If $Z_1, Z_2 \in \underline{Z}$ and $\alpha_i : Z_i \longrightarrow X$, $i = 1, 2$ is the inclusion, then there is map with matrix

$$\| \alpha_i, X \| : Z_i \times X \longrightarrow X .$$

(Of course we can always write down that matrix; Z_i is central if and only if that matrix represents a map.) Now $\|\alpha_1, \alpha_2\|$: $Z_1 \times Z_2 \longrightarrow X$ is a morphism since it can be factored, e.g., $\|\alpha_1, X\| \begin{Vmatrix} Z_1 & 0 \\ 0 & \alpha_2 \end{Vmatrix}$. Let $P \overset{d^0}{\underset{d^1}{\rightrightarrows}} Z_1 \times Z_2$ and $Z_1 \times Z_2 \xrightarrow{d} Z_1 Z_2$ be the kernel pair of $\|\alpha_1, \alpha_2\|$ and the coequalizer of d^0, d^1 respectively. Suppose that $d^i = \begin{Vmatrix} \partial_{1i} \\ \partial_{2i} \end{Vmatrix}$, $i = 0, 1$, and $d = \|\beta_1, \beta_2\|$. By Z.4. the induced map α: $Z_1 Z_2 \longrightarrow X$ is a subobject, and, of course, $\alpha\beta_i = \alpha_i$, $i = 1, 2$. Also $\beta_1 \partial_{10} + \beta_2 \partial_{20}$ and $\beta_1 \partial_{11} + \beta_2 \partial_{21}$ are defined and equal which implies that $\alpha_1 \partial_{10} + \alpha_2 \partial_{20}$ and $\alpha_1 \partial_{11} + \alpha_2 \partial_{21}$ are also defined and equal. Now by Z.6.

$$P \times X \overset{d^0 \times X}{\underset{d^1 \times X}{\rightrightarrows}} Z_1 \times Z_2 \times X \xrightarrow{d \times X} Z_1 Z_2 \times X$$

is also a coequalizer. The map with matrix $\|\alpha_1, X\|(Z_1 \times \|\alpha_2, X\|)$ $= \|\alpha_1, X\| \begin{Vmatrix} Z_1 & 0 & 0 \\ 0 & \alpha_2 & X \end{Vmatrix} = \|\alpha_1, \alpha_2, X\|$ coequalizes $d^0 \times X$ and $d^1 \times X$. In fact $\|\alpha_1, \alpha_2, X\| \begin{Vmatrix} \partial_{10} & 0 \\ \partial_{20} & 0 \\ 0 & X \end{Vmatrix} = \|\alpha_1, \partial_{10} + \alpha_2 \partial_{20}, X\|$

$= \|\alpha_1 \partial_{11} + \alpha_2 \partial_{21}, X\| = \|\alpha_1, \alpha_2, X\| \begin{Vmatrix} \partial_{11} & 0 \\ \partial_{21} & 0 \\ 0 & X \end{Vmatrix}$. Thus

there is induced a map $\|\gamma_1, \gamma_2\|$: $Z_1 Z_2 \times X \longrightarrow X$ with

$\|\gamma_1, \gamma_2\|\cdot(d \times X) = \|\alpha_1, \alpha_2, X\| = \|\alpha\beta_1, \alpha\beta_2, X\|$

$= \|\gamma_1, \gamma_2\| \begin{Vmatrix} \beta_1 & \beta_2 & 0 \\ 0 & 0 & X \end{Vmatrix} = \|\gamma_1\beta_1, \gamma_1\beta_2, \gamma_2\|$. Then

$\gamma_2 = X$ and $\|\alpha\beta_1, \alpha\beta_2\| = \|\gamma_1\beta_1, \gamma_1\beta_2\|$ or $\alpha d = \gamma_1 d$. Since d is a coequalizer, hence epi, it follows that $\alpha = \gamma_1$ and Z_1Z_2 is central. Of course $Z_i \subset Z_1Z_2$, i = 1, 2, since the inclusion map $Z_j \longrightarrow X$ factors through it.

Now since $\underline{Z}$ is a directed family of subobjects of X, it has a colimit Z which is also a subobject α: $Z \longrightarrow X$ by Z.5. If β_i: $Z_i \longrightarrow Z$ is the map to the colimit, then also the β_i are mono and $\alpha\beta_i = \alpha_i$. By Z.6., $Z \times X$ = colim $Z_i \times X$ and since for each i, $\|\alpha_i, X\|$: $Z_i \times X \longrightarrow X$ is a map, there is induced a map $\|\gamma, \gamma'\|$: $Z \times X \longrightarrow X$ such that for each i, $\|\gamma, \gamma'\|(\beta_i \times X)$ $= \|\alpha_i, X\|$. This gives $\|\gamma\beta_i, \gamma' X\| = \|\alpha_i, X\|$ or $\gamma' = X$, $\gamma\beta_i = \alpha_i$ for all i. Since also $\alpha\beta_i = \alpha_i$, the uniqueness of map extensions guarantees that $\gamma = \alpha$, so $\|\alpha, X\|$ is a map. Thus Z is central and clearly contains all central subobjects.

3. Equational Categories.

By an equational category, we mean a category $\underline{X}$ equipped with an algebraic functor U: $\underline{X} \longrightarrow$ Sets (i.e., one which is tripleable as soon as it has an adjoint). This means that if F is a functor with codomain $\underline{X}$ and S = lim UF, then there is a unique (up to isomorphism) $X \in \underline{X}$ with X = lim F and UX = lim UF. Also, if $X \subset Y \times Y$ is such that $UX \subset UY \times UY$ is an equivalence relation, then $X \rightrightarrows Y$ has a coequalizer $Y \longrightarrow Z$ and $UX \rightrightarrows UY \longrightarrow UZ$ is a coequalizer. If n is any set (possibly infinite), an n-ary operation is a natural transformation of $U^n \longrightarrow U$. U has a left adjoint F if and only if, for each set n, the class of natural transformations of $U^n \longrightarrow U$ is a proper set. (And then UFn = nat. trans. (U^n, U).) A nullary operation, also called a constant, is a natural transformation of $U^0 = 1 \longrightarrow U$. A natural transformation $U^n \longrightarrow U^m$ is called a projection if it is of the form U^f where $f\colon m \longrightarrow n$ is a function. We say that "all operations are finite" when we actually mean that any n-ary operation $U^n \longrightarrow U$ factors as $U^n \longrightarrow U^{n_0} \longrightarrow U$ where the first map is a projection and n_0 is a finite set. For more details of the theory of equational categories see [2].

<u>Theorem 3.1</u>. Let $\underline{X}$ be an equational category. Then:

1. $\underline{X}$ satisfies Z.1. if and only if there is exactly one nullary operation.
2. $\underline{X}$ satisfies Z.2.
3. $\underline{X}$ satisfies Z.3. (when $\underline{X}$ is pointed) if there is a binary operation "+" satisfying $x + 0 = 0 + x$ for $x \in\in \underline{X}$ where 0 is the base point. A tripleable category $\underline{X}$ satisfies Z.3$'$; $X * Y \longrightarrow X \times Y$ onto, if and only if there is such a "+". (Here $*$ is the coproduct.)
4. $\underline{X}$ satisfies Z.4.
5. $\underline{X}$ satisfies Z.5. if all operations are finite. If $\underline{X}$ is tripleable, the converse holds.

6. $\underline{X}$ satisfies Z.6.

Proof:

1. This is well-known. Permit me to observe, however, that it requires showing that if α is an n-ary operation, then $\alpha(0, \ldots, 0) = 0$. But if this were not an equation in the system, then $\alpha(0, \ldots, 0)$ would define a new nullary operation.

2. See, for example [2], p. 87.

3. It is well-known that in an equational category, there are coproducts which we denote by $*$. Then Z.3. is just the statement that the natural map $X_1 * X_2 \longrightarrow X_1 \times X_2$ is an epimorphism. If there is a binary operation + with $x + 0 = 0 + x = x$, then $(x_1, x_2) = (x_1, 0) + (0, x_2)$. Each of those is clearly in the image, so their sum is. Thus the natural map is onto, and Z.3' holds. Conversely, if Z.3' holds and $\underline{X}$ is tripleable, the natural map $F1 * F1 \longrightarrow F1 \times F1$ is onto and we can find an element $\xi \in UF2 = U(F1 * F1)$ whose image in $UF1 \times UF1$ is (η, η) where η is the generator of F1. UF2 = nat. trans. (U^2, U) and the natrual transformation corresponding to ξ is the desired one. The details are left to the reader (see [2]).

4. See [2], p. 88 (called the First Isomorphism theorem).

5. This seems to be known, but as I have been unable to find a reference in the literature, I will include a proof. If $\underline{X}$ is finitary and $X, Y \in \underline{X}$, a set function $f: X \longrightarrow Y$ need only commute with finite

operations to be a morphism, since commuting with projections is automatic. Now if $\{X_i\}$ is a directed family of subobjects of X and if $\{f_i\}$, $f_i\colon X_i \longrightarrow Y$ is a family of maps on the direct system, let $X' = \bigcup X_i$ (set union). X' is a subobject, for if ω is an n-ary operation, n finite, and $x_{i_1}, \ldots, x_{i_n} \in X'$, I can already find X_i with each of $x_{i_1}, \ldots, x_{i_n}$ and hence $\omega(x_{i_1}, \ldots, x_{i_n})$ being elements of X_i. Similarly, the $\{f_i\}$ extends to a set map $f\colon X' \longrightarrow Y$ and $f\omega(x_{i_1}, \ldots, x_{i_n}) = f_\alpha \omega(x_{i_1}, \ldots, x_{i_n}) =$ $\omega(f_i(x_{i_1}), \ldots, f_i(x_{i_n})) = \omega(f(x_{i_1}), \ldots, f(x_{i_n}))$, since f extends f_i and f_i is a morphism. Conversely, if Z.5 holds and $\underline{X}$ is tripleable, we have $n = \operatorname{colim} n_0$ where n_0 ranges over the finite subsets of n. But a left adjoint F commutes with colimits, so $Fn = \operatorname{colim} Fn_0$, certainly $\{Fn_0\}$ is directed and their union is exactly the n-ary operations which are composites of projections and finitary operations. If Fn is just this union, then this union includes all the n-ary operations.

6. In an equational category a map is a coequalizer if and only if it is surjective. Let $f\colon X \longrightarrow Y$ be a map. The point set image Y_1 of f is also its categorical image and we have f factoring as

$X \xrightarrow{\text{onto}} Y_0 \xrightarrow{\text{1-1}} Y$. Also $X' \rtimes -$ preserves both properties of being 1-1 and onto and so $X' \rtimes f$ factors $X' \rtimes X \longrightarrow X' \rtimes Y' \longrightarrow X' \rtimes Y$ with the first being a coequalizer and the second being 1-1, and hence the image of $X' \rtimes f$. As for the second half, if $\{X_i\}$ is a directed family of subobjects, the colim X_i is just the set theoretic union (of course all operations are finite). $\{X' \rtimes X_i\}$ is still directed and $X' \rtimes -$ commutes with set union.

Thus we have proved,

<u>Theorem 3.2</u>. Let $\underline{X}$ be a pointed equational category with all operations finitary and in which there is a binary operation for which the base point is a 2-sided unit. Then every object of $\underline{X}$ has a center.

Let us examine this situation more closely. If Z is the center of X and if there is a map $\tau: Z \rtimes X \longrightarrow X$ with $\tau(z, 0) = z$ and $\tau(0, x) = x$ for $z \in Z$, $x \in X$, then τ must commute with all the operations. If Z.3' holds, then it must in particular commute with the distinguished binary operation, denoted by +, appearing in the statement of theorem 3.2. Thus:

$$\tau(z + z', x + x') = \tau(z, x) + \tau(z', x') .$$

If $z' = x = 0$, this says $\tau(z, x') = \tau(z, 0) + \tau(0, x') = z + x'$ so $\tau = +$. Then $(z + z') + (x + x') = (z + x) + (z' + x')$. Then if $z = x' = 0$, we get $z' + x = x + z'$. Finally, letting $z' = 0$, we have $z + (x + x') = (z + x) + x'$. Thus Z is a commutative associative monoid and the operation of Z on X is commutative and associative. A modification of this result to make Z into a group has long been known in universal algebra; see for example [1] pp. 799-800.

Note: It has recently come to the author's attention that S. A. Huq [commutator, nilpotency and solvability in categories, Quart. J. Math. Oxford (2), 19 (1968), 363-389] has considered closely related concepts (with arbitrary maps rather than subobjects) except that his axioms are strong enough to make central subobjects be abelian groups (not merely monoids) but lacking continuity axioms Z.5 and Z.6 he cannot prove that centers exist.

REFERENCES

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2. F.E.J. Linton, Some aspects of equational categories, Proceedings of the conference on categorical algebra, La Jolla 1965, 84-94.