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ET GEOMETRIE DIFFERENTIELLE

A CLOSED CATEGORY OF REFLEXIVE TOPOLOGICAL ABELIAN GROUPS

by Michael BARR *

The category $\mathfrak{L}$ of locally compact abelian groups has a structure which is fairly unusual as categories go. There is a duality functor on $\mathfrak{L}$, denoted $L \mapsto L^*$, which determines an equivalence between $\mathfrak{L}$ and $\mathfrak{L}^{op}$. This category is not complete or cocomplete, either in the usual external sense of having limits and colimits or the internal one of being closed monoidal. In part, it resembles the categories of finite dimensional vector spaces over a field as well as the category of finite dimensional Banach spaces. The purpose of this paper is to adapt to $\mathfrak{L}$ the constructions used in [B_1, B_2, B_3 and B_4] to find an extension of $\mathfrak{L}$ which is still self dual but which is now complete and cocomplete and which, moreover, is closed and monoidal.

The first results in this direction are those of Kaplan [K_1, K_2], who extended the Pontrjagin duality theory to those groups which are the product of locally compact groups and, of course, the duals of such groups. As well, he extended the theory to inverse limits of sequences of locally compact groups. However not all inverse limits are reflexive (see [Mi]) so we will have to restrict categories somewhat to obtain the desired results.

The results of the two papers of Kaplan are absolutely crucial in this paper.

If the category $\mathfrak{L}$ is the analogue of the finite dimensional spaces, then products and sums of spaces on $\mathfrak{L}$ are the analogues of the (linearly) compact and the discrete spaces respectively. (It could be argued that closed subspaces of the former and Hausdorff quotients of the latter are better analogues but I have not been able to make them perform well. There are results known (see [Mo] and [HMP]) which suggest that this is not as serious

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a restriction as it seems.)

The following conventions and notations are used without further mention. All groups are additive abelian Hausdorff topological groups. If G and H are groups and $U \subset G$, $V \subset H$, $N(U,V)$ denotes the subset of $Hom(G,H)$ of all these $f\colon G \to H$ for which $f(U) \subset V$. We let R, Z, T denote the reals, integers and circle group, respectively. If $\eta \in R$, $0 < \eta < \frac{1}{2}$ we let $[-\eta, \eta]$ denote either the usual interval in R or the image of that interval in $R/Z \approx T$. We let $N(U,\eta)$ denote also $N(U,[-\eta,\eta])$. If G is a group, we let G^* denote the character group. When G is locally compact, G^* will always be assumed to have the compact open topology. The topology in other cases is described in Section 3.

All maps between groups are assumed additive without further mention and all maps between topological spaces or groups are assumed continuous unless explicitly qualified by a phrase such as «not necessarily continuous». Except in the proof of (1.6), a product is always assumed to have the product, or Tychonoff, topology and a sum to have the asterisk topology (see (1.2)).

1. THE CATEGORIES $\mathfrak{C}$ AND $\mathfrak{D}$.

(1.1) Definition of $\frac{1}{n}U$ and $(g:U)$.

Let G be a group and U be a subset of G which contains 0. We let

$$\frac{1}{n}U = \{\, g \in G \mid mg \in U,\ m = 0, 1, \dots, n \,\}.$$

It is clear from continuity of addition that when U is a neighborhood of 0, so is $\frac{1}{n}U$. Warning: it is not in general true that $n(\frac{1}{n}U) \supset U$.

If $g \in U$, let

$$(g:U) = inf\{\, \tfrac{1}{n} \mid g \in \tfrac{1}{n}U \,\}.$$

(1.2) Definition of asterisk sum.

Let $\{G_\omega\}$, $\omega \in \Omega$ be a family of groups. Let $G = \Sigma G_\omega$ denote the ordinary algebraic direct sum. If U_ω is a 0-neighborhood in G_ω, for all $\omega \in \Omega$, we let $\Gamma U_\omega = \Gamma_\omega U_\omega$ denote the subset of ΣG_ω of all sums Σg_ω

for which

$$g_\omega \epsilon U_\omega \quad \text{and} \quad \Sigma(g_\omega : U_\omega) < 1.$$

The sets ΓU_ω so defined determine a neighborhood system at O for a group topology on G, called the asterisk topology. See [K_1] for details although the definition is slightly different. This is not the coproduct in the category of topological abelian groups ([BHM], p. 21, Remark 1) but it will turn out to be the coproduct in the categories we are constructing. *Whenever we form a sum in this paper it is understood to bear the asterisk topology.* If I is an index set, we denote by $I.G$ the direct sum of I copies of G.

(1.3) Definition of $\mathfrak{C}_0$, $\mathfrak{D}_0$, $\mathfrak{C}$, $\mathfrak{D}$.

These are four full subcategories of all groups. The groups in $\mathfrak{C}$ are those of the form $C \times R^I \times Z^J$ where C is compact and I and J are arbitrary index sets. $\mathfrak{C}_0 = \mathfrak{C} \cap \mathfrak{L}$ consists of those groups of the above form for which I and J are finite. The groups in $\mathfrak{D}$ are those of the form

$$D \oplus I.R \oplus J.T \quad \text{with } D \text{ discrete.}$$

Again $\mathfrak{D}_0 = \mathfrak{D} \cap \mathfrak{L}$ consists of those for which I and J are finite. It is evident that $\mathfrak{C}_0$ and $\mathfrak{D}_0$ are dual via the Pontrjagin duality and it follows from the duality theorem of [K_1] that so are $\mathfrak{C}$ and $\mathfrak{D}$.

(1.4) PROPOSITION. *Let $L \epsilon \mathfrak{L}$ and U a neighborhood of O in L. Then there is a $D \epsilon \mathfrak{D}_0$, a neighborhood V of O in D and a map $f: L \to D$ such that $U \supset f^{-1}(V)$.*

PROOF. There is a compact set $X \subset L^*$ and an

$$\eta \epsilon (0, 1/2) \quad \text{such that} \quad U \supset N(X, \eta).$$

(we identify L and L^{**}). Since L^* is locally compact, we may enlarge X to be a compact neighborhood of O. When that is done, the subgroup C generated by X has a non-empty interior, hence is open, hence locally compact. It follows from [HR], (9.8), that $C \epsilon \mathfrak{C}_0$. The inclusion $C \to L^*$ gives $f: L \to C^*$ and $U \supset f^{-1}(N(X, \eta))$.

(1.5) We know from [K_1] that $I.R$ is the dual of R^I with the compact open topology. A basis for compact sets in R^I consists of sets of the form:

$\Pi[r_i, -r_i]$. The set $N(\Pi[-r_i, r_i], 1/4)$ can be described as the set of

$$\Sigma s_i \in I.R \quad \text{such that} \quad \Sigma\, |r_i s_i| \leqslant 1/4.$$

This is equivalent to taking as neighborhood base at O all sets of the form

$$\{\, \Sigma s_i \in I.R \mid \Sigma |r_i s_i| \leqslant 1/4 \,\},$$

where $(r_i) \in R^I$ is arbitrary.

Exactly the same arguments work for $I.T$. Given any sequence of integers $\{n_i\}$, $i \in I$, we consider

$$\{\, \Sigma t_i \in I.T \mid \Sigma m_i t_i \in [-\tfrac{1}{4}, \tfrac{1}{4}],\ -n_i \leqslant m_i \leqslant n_i \}.$$

This will happen iff, choosing s_i as the absolutely least representative of t_i, $\Sigma |n_i s_i| \leqslant \frac{1}{4}$.

(1.6) PROPOSITION. *The groups in $\mathfrak{C}$ and $\mathfrak{D}$ are complete in their natural uniformity.*

PROOF. For the groups in $\mathfrak{L}$ and hence in $\mathfrak{C}$ this is trivial. As for $\mathfrak{D}$, it is also trivial for discrete groups so we need only consider $I.R$ and $I.T$. For $s \in R$ let $<s> = |s|$ and for $s \in T$ let $<s>$ denote the absolute value of the absolutely least representative of s in R. Let D denote R or T. Then, topologize D^I by taking as a basic neighborhood of O all sets of the form

$$(*) \qquad \{\, (s_i) \mid \Sigma n_i < s_i > \, < 1/4 \,\}$$

where (n_i) is a sequence of non-negative integers. Thus topologized, D^I is a topological group and $I.D$ is a subgroup. I claim the neighborhood (*) above is a closed set in the Tychonoff topology. In fact, if

$$\Sigma\, n_i < s_i > \; > 1/4,$$

there is already a finite set of indices, say $i = 1, \dots, k$ and an $\eta > 0$ such that

$$n_1 < s_1 > + \dots + n_k < s_k > \; > \tfrac{1}{4} + \eta .$$

Consider the neighborhood in the product defined by the finite set of conditions

$$<t_i> < \eta / k n_i, \quad i = 1, \ldots, k.$$

This determines a Tychonoff neighborhood and if $s_i' = s_i + t_i$,

$$<s_i'> > <s_i> - \eta / k n_i, \quad n_i <s_i'> \geqslant n_i <s_i> - \eta / k$$

and finally

$$n_1 <s_1'> + \ldots + n_k <s_k'> > 1/4.$$

It now follows from [S], 1.6 (which works as well for topological groups as for vector spaces) that D^I, so topologized, is complete. The result is proved by showing that $I.D$ is a closed subgroup. But if $(s_i) \epsilon D$ and infinitely many $s_i \neq 0$, choose for each such i, $n_i > 1/s_i$ and n_i arbitrarily for all other i. If

$$\Sigma n_i <t_i> < 1/4 \quad \text{and} \quad s_i' = s_i + t_i,$$

we have $n_i <t_i> < 1/4$, so

$$<t_i / s_i> < 1/4, \quad <t_i> < 1/4 <s_i>$$

and hence $<s_i'> \geqslant \frac{3}{4} <s_i>$ and certainly $s_i' \neq 0$ either for that set of indices.

2. THE CATEGORY $\mathfrak{G}$.

(2.1) Definition of $\mathfrak{G}$.

It was mentioned in the introduction that the direct sums of locally compact groups would play the role that the discrete spaces did in [B_1]. It will actually turn out that the groups in $\mathfrak{D}$ suffice. For the purposes of definition, however, we define $\mathfrak{G}$ to be the full category of those groups which are subgroups of products of sums of locally compact groups. Since a finite product of sums is a finite sum of sums which is just a sum, this definition reduces to the following: $G \epsilon \mathfrak{G}$ provided that for every neighborhood of O, $U \subset G$, there is a family $\{L_\omega\}$, $\omega \epsilon \Omega$ of locally compact groups, a map $f: G \to \Sigma L_\omega$ and a neighborhood of O, $V \subset \Sigma L_\omega$ such that $U \supset f^{-1}(V)$.

(2.2) PROPOSITION. *In the definition above we may assume without loss of generality that each* $L_\omega \epsilon \mathfrak{D}_0$, *whence* $\Sigma L_\omega \epsilon \mathfrak{D}$.

PROOF. The set V contains a set of the form ΓV_ω where V_ω is a neighborhood of O in L_ω. From (1.4) it follows that for all ω there is a map

$$e_\omega : L_\omega \to D_\omega \quad \text{with} \quad D_\omega \in \mathfrak{D}$$

and a neighborhood of O,

$$W_\omega \subset D_\omega \quad \text{with} \quad V_\omega \supset e_\omega^{-1}(W_\omega).$$

The rest is easy.

(2.3) If $G \in \mathfrak{G}$, $D \in \mathfrak{D}$, a map $G \to D$ is called a *$\mathfrak{D}$-representation of G*. A family $\{G \to D_\omega\}$, $\omega \in \Omega$, is called a *$\mathfrak{D}$-envelopment of G* if the induced map $G \to \Pi D_\omega$ embeds G as a subspace.

(2.4) It is clear that a product of groups in $\mathfrak{G}$ is still in $\mathfrak{G}$ and so that is the categorical product. If $\{G_\omega\}$, $\omega \in \Omega$, is a family of groups in $\mathfrak{G}$, the asterisk sum ΣG_ω lies in $\mathfrak{G}$. In fact, let U_ω be a neighborhood of O in each G_ω. Choose for each ω a $\mathfrak{D}$-representation $f_\omega : G_\omega \to D_\omega$ and a neighborhood of O,

$$V_\omega \subset D_\omega \quad \text{such that} \quad f_\omega^{-1}(V_\omega) \subset U_\omega.$$

Then from the easily proved

$$(g_\omega : U_\omega) < (f_\omega(g_\omega) : V_\omega),$$

It follows that $(\Sigma f_\omega)(\Gamma U_\omega) \subset \Gamma V_\omega$. Since an asterisk sum of asterisk sums is again an asterisk sum (trivial), it follows that $\Sigma G_\omega \in \mathfrak{G}$.

(2.5) Let

$$D = D_1 \oplus I.R \oplus J.T$$

in $\mathfrak{D}$ with D_1 discrete. A basic neighborhood of O is of the form

$$U = \Gamma[-\eta_i, \eta_i] + \Gamma[-\delta_j, \delta_j] \quad \text{where } \eta_i > 0 \text{ and } 0 < \delta_j \leqslant \tfrac{1}{4}.$$

Here we are thinking of $[-\eta_i, \eta_i]$ in the ith copy of R and $[-\delta_j, \delta_j]$ in the jth copy of T. For such a U it is evident that

$$\frac{1}{n} U = \Gamma[-\eta_i/n, \eta_i/n] + \Gamma[-\delta_j/n, \delta_j/n].$$

If $n_1, \dots, n_k$ is a finite sequence of integers such that $\Sigma\, 1/n_k < 1$, we have that $\Sigma\, 1/n_k\, U \subset U$. Now suppose that $\{G_\omega\}$, $\omega \in \Omega$, is a family of groups of $\mathfrak{G}$, $G = \Sigma G_\omega$. If $f_\omega : G_\omega \to D$ is given for each $\omega \in \Omega$, there is of course induced an additive function $f : \Sigma G_\omega \to D$. To show it is continuous, let U a neighborhood as above. Let $U_\omega = f_\omega^{-1}(U)$. Then the remarks above make it clear that $f(\Gamma U_\omega) \subset U$.

(2.6) Now suppose that $H \in \mathfrak{G}$ is arbitrary. Given $f_\omega : G_\omega \to H$ we still get an additive map $f : \Sigma G_\omega \to H$. To see that it is continuous, follow it by any $\mathfrak{D}$-representation of H and apply the above. Since it is clear that $G_\omega \to \Sigma G_\omega$ is continuous, we can now conclude :

PROPOSITION. *The asterisk sum is the coproduct in* $\mathfrak{G}$.

(2.7) It should be clear to the reader that the above argument is exactly the sort of thing you do with locally convex spaces. In fact, the Γ operator was named with this analogy in mind.

3. DUALITY.

(3.1) For $G \in \mathfrak{G}$, there are enough $\mathfrak{D}$-representations to separate points. There are enough characters on any $D \in \mathfrak{D}$ to separate points and hence every $G \in \mathfrak{G}$ has enough characters. Now we wish to define a topology on the dual G^* with the property that the characters on G^* are determined precisely by the elements of G. The compact open topology is too fine, as the following example shows. Let G be the group of integers topologized 2-adically. Since G is dense in the 2-adic integers $Z_{(2)}$, any character on G has a unique extension to $Z_{(2)}$. Hence G^* is algebraically isomorphic to $Z_{(2)}^* = Z_{2^\infty}$. The set

$$X = \{0, 1, 2, 4, 8, \dots\}$$

consists of a convergent sequence together with its limit and is hence compact. It is easy to see that $N(X, 1/4) = 0$ and so G^* would be discrete whence $G^{**} = Z_{(2)}$.

(3.2) On the other hand, we wish to have the compact open topology on the groups known to have a good duality theory. By Kaplan's Theorem $[K_1]$, this includes at least the cartesian products of locally compact groups. This imposes certain constraints on the topology on G^*. If $f: \Pi L_\omega \to G$ is a map, L_ω locally compact, if $X \subset \Pi L_\omega$ is compact, and if $Y = f(X)$, then it is necessary, in order that $G^* \to (\Pi L_\omega)^*$ be continuous, that $N(Y, \eta)$ be a neighborhood of O for all $\eta \in (0, 1/2)$. In that case we can make the following simplifications. Let X_ω be the image of X in L_ω. Enlarge X_ω to be a compact neighborhood of O and then suppose that L'_ω is the subgroup generated by X_ω. Then

$$L'_\omega \in \mathfrak{C}_0, \quad \text{so} \quad \Pi L'_\omega \in \mathfrak{C}.$$

The set $X' = \Pi X_\omega$ contains X, and if Y' is its image in G, we must have uniform convergence on Y' as well. Hence we define a compact set $Y \subset G$ to be *strongly compact* if there is a $C \in \mathfrak{C}$, a map $f: C \to G$ and compact set

$$X \subset C \quad \text{with} \quad f(X) \supset Y.$$

It is clear from the above discussion that any compact set in a group which is a product of locally compact groups is strongly compact. It is also clear that any map takes a strongly compact set to another one.

(3.3) Now we define the topology on G^* to be that of uniform convergence on strongly compact sets. It is clear that this topology is functorial and agrees with the compact open topology on products of locally compact groups.

(3.4) PROPOSITION. *Suppose $C \in \mathfrak{C}$ and $D \in \mathfrak{D}$. Then the image of any map $C \to D$ lies in $\mathfrak{C}_0 \cap \mathfrak{D}_0$.*

PROOF. Let

$$C = C_1 \times R^I \times Z^J \quad \text{and} \quad D = D_1 \oplus I.R \oplus J.T$$

with C_1 compact and D_1 discrete. It is easily seen that there is a neighborhood of O in D which contains no subgroups. The inverse image of that set contains a set of the form

$$M_1 \times \Pi[-r_i, r_i] \times \Pi[-s_j, s_j]$$

where M_1 is a O-neighborhood in C_1 and $r_i = \infty$ and $s_j = \frac{1}{2}$ except for finitely many indices, say except for

$$i \in I_0, \quad j \in J_0 .$$

The subgroup $R^{I-I_0} \times Z^{J-J_0}$ is annihilated by the map which means it factors through $C_I \times R^{I_0} \times Z^{J_0}$. Dually its image is contained in

$$D_I \oplus K_0 . R \oplus L_0 . T$$

for some finite subsets $K_0 \subset K$, $L_0 \subset L$. The image is then that of a map from one locally compact group to another and is thus locally compact. Since $C_I \times R^{I_0} \times Z^{J_0}$ is generated by a compact subset, so is the image. By [HR], (9.8), the image belongs to $\mathfrak{C}_0$. Dually, it belongs to $\mathfrak{D}_0$.

(3.5) PROPOSITION. *Let $D \in \mathfrak{D}$ and $H \subset D$ be a subgroup. Then the image of $D^* \to H^*$ is dense.*

PROOF. It follows from (3.4) that H^* is topologized as a subgroup of a product ΠC^*_ω, where C_ω runs over all the subgroups of H which belong to $\mathfrak{C}_0 \cap \mathfrak{D}_0$. Write

$$D = D_I \oplus I . R \oplus J . T .$$

It follows as above that whenever $C_\omega \subset H$ there are finite sets

$$I_0 \subset I, \ J_0 \subset J \quad \text{such that} \quad C_\omega \subset D_I \oplus I_0 . R \oplus J_0 . T = D_0 .$$

Then $D_0^* \to C^*_\omega$ is onto since both are in $\mathfrak{L}$ while it is obvious that $D^* \to D_0^*$ is onto. Thus every composite

$$D^* \to H^* \subset \Pi C^*_\omega \to C^*_\omega$$

is onto. This argument works unchanged for a finite number of indices so that $D^* \to \Pi C^*_\omega$, over any finite set of indices, is onto, from which it follows that $D^* \to \Pi C^*_\omega$ is dense and hence so is $D^* \to H^*$.

(3.6) Let $G^\Delta \subset G^{**}$ denote the subgroup of G^{**} of all characters on G^* represented (necessarily uniquely) by elements of G. Thus as sets G and G^Δ are isomorphic.

(3.7) PROPOSITION. *The natural group isomorphism $G^\Delta \to G$ is continuous.*

PROOF. For any neighborhood M of O in G there is a $D \in \mathfrak{D}$ and a map

$G \to D$ which forces M to be a neighborhood of 0. The map

$$G^{\Delta} \to G^{**} \to D^{**} \approx D$$

forces the image of M in G^{Δ} to be a 0-neighborhood as well.

(3.8) We say G *is quasi-reflexive* if $G^{\Delta} = G^{**}$, *prereflexive* if $G^{\Delta} \to G$ is an isomorphism and *reflexive* if both conditions hold.

(3.9) PROPOSITION. *If G is complete, so is G^{Δ}.*

PROOF. It suffices (cf. proof of (1.6)) to show that G^{Δ} has a neighborhood base at 0 consisting of sets closed in G. But for any $\phi \in G^*$,

$$\{ g \mid |\phi(g)| \leqslant \eta \}$$

is closed in G and so if $X \subset G^*$ is arbitrary,

$$\{ g \mid |\phi(g)| \leqslant \eta \ \ \forall \phi \in X \} = \bigcap_{\phi \in X} \{ g \mid |\phi(g)| \leqslant \eta \}$$

is closed in G. But as X ranges over the strongly compact sets and $\eta > 0$, these are a neighborhood base at 0.

(3.10) PROPOSITION. *Let G be complete. Then it is quasi-reflexive.*

PROOF. Let $\{ C_\omega \to G \}$ run over a family of maps, $C_\omega \in \mathfrak{C}$ such that G^* is a subgroup of ΠC_ω^*. Let $f: G^* \to T$ be a character. The set

$$f^{-1}(-1/4, 1/4) \supset G \cap \Pi M_\omega,$$

where M_ω is a neighborhood of 0 in C_ω^* and $M_\omega = C_\omega^*$ except for finitely many indices ω. Since $\mathfrak{C}$ is closed under finite products we may suppose $M_\omega = C_\omega^*$ except for a single index, say $\omega = \psi$. In that case the subgroup $H_0 \subset G^*$ consisting of the elements whose ψ-coordinate is 0 is taken by f into $(-1/4, 1/4)$ and hence to 0. The induced map $G^*/H_0 \to C_\psi^*$ is clearly 1-1 and we denote its image by H. The map f induces a not necessarily continuous map $e: H \to T$. But if $h \in H \cap M_\psi$ any preimage of h belongs to ΠM_ω and is sent by f into $(-1/4, 1/4)$. Thus

$$e(h) \in (-1/4, 1/4) \quad \text{and so} \quad e^{-1}(-1/4, 1/4) \supset M_\psi .$$

By [K$_2$], (2.1), e is then continuous. Let $E_0 \subset H^*$ be the image of the dual of $H \to C_\psi^*$. Then we have a commutative diagram

$$\begin{tikzcd} C_\psi \arrow[r] \arrow[d] & E_0 \arrow[r] \arrow[d] & H^* \arrow[d] \\ G & G^\Delta \arrow[l] \arrow[r] & G^{**} \end{tikzcd}$$

With $G^\Delta \to G$ a group isomorphism, $C_\psi \to E_0$ onto and $G^\Delta \to G^{**}$ 1-1, there is a diagonal fill-in $E_0 \to G^\Delta$ at the underlying group level. With $G^\Delta \subset G^{**}$, commutativity implies this map is continuous. Since G and hence G^Δ is complete, this extends to a map $H^* \to G^\Delta$. The two maps $H^* \to G^{**}$ agree on the dense subgroup E_0 and are hence equal. Thus the image of H^* in G^{**} lies in G^Δ. Since $e \in H^*$ mapped to an arbitrary element of G^{**}, it follows that $G^\Delta = G^{**}$.

(3.11) In Section 6, we will show how this argument can be extended to a wider class of groups. I do not know whether the result is true for all groups in $\mathfrak{G}$.

4. THE INTERNAL $Hom: \mathfrak{C}^{op} \times \mathfrak{D} \to \mathfrak{D}$.

We wish to introduce a topology on the sets $Hom(G,H)$ so as to make them into topological groups and give a closed category structure. As in [B_3] and [B_4] we begin by a preliminary topology which is then modified somewhat. As in those papers, the first thing to consider is

$$Hom(C,D), \text{ with } C \in \mathfrak{C},\ D \in \mathfrak{D}.$$

(4.1) We begin by observing that the decomposition of $C \in \mathfrak{C}$ as

$$C_1 \times R^I \times Z^J, \quad C_1 \text{ compact,}$$

is unique. In fact, C_1 consists of those elements whose orbit closure is compact. When C_1 is factored out, the identity component is R^I and the quotient is Z^J. The dual statement is true of the decomposition of a group in $\mathfrak{D}$ as $D_1 \oplus I.R \oplus J.T$.

(4.2) Now we consider

$$C = C_1 \times R^I \times Z^J \text{ and } D = D_1 \oplus K.R \oplus L.T.$$

To define the topology on $Hom(C,D)$ we will consider the nine cases separately. Both $Hom(C_1,D)$ and $Hom(C,D_1)$ are discrete in the compact open topology. Topologized thus, we denote them (C_1,D) and (C,D_1), respectively.

(4.3) Now suppose $C = R$ or T. Suppose $f: C^I \to K.D$. Let $U \subset K.D$ be a neighborhood of 0 which contains no non-zero subgroups. $f^{-1}(U)$ contains a set of the form $\prod_{i \in I} V_i$, where V_i is open in C for all i and $V_i = C$ with only finitely many exceptions, which we suppose are labeled $i = 1, \ldots,$ n. Then the subgroup of C^I of those elements with 0 in coordinates $1, \ldots$ $\ldots, n$ is carried into U by f. Since D contains no small subgroups it is in fact annihilated. Then if $I' = \{1, \ldots, n\}$, the map f factors through $C^{I'}$. This shows that $Hom(C^I, K.D)$ is algebraically isomorphic to the group $I.Hom(C, K.D)$. By duality,

$$Hom(C,K.D) \approx Hom((D^*)^K, C^*) \approx K.Hom(D^*, C^*) \approx K.Hom(C,D).$$

(4.4) Thus

$$Hom(C^I, K.D) \approx (I \times K).\,Hom(C,D).$$

Each of the four possible choices of C and D leads to one of the groups R or T as $Hom(C,D)$. When so topologized, we denote it (C,D). We would like now to define

$$\text{(i)} \qquad (C^I, K.D) = (I \times K).(C,D).$$

There is, however, a problem in that the representations as C^I are not unique. A different basis could be chosen for $K.D$ and dually for C^I. It is easy to see that the cardinalities are invariant but the actual representations are not. Similarly, it is not clear that this definition of the topology is functional, since homomorphisms do not preserve the decompositions. We will abuse notation and use (4.4.i) temporarily while recognizing that it is, *a priori*, a functor of four variables C, D, I, K, not C^I and $K.D$.

(4.5) PROPOSITION. *Let*

$$C = C_1 \times R^I \times Z^J \quad \textit{and} \quad D = D_1 \oplus K.R \oplus L.T$$

with C_1 *compact and* D_1 *discrete. If* (C, D) *is defined as indicated above, then it has the compact open topology whenever* $C \in \mathfrak{C}_0$ *or* $D \in \mathfrak{D}_0$.

PROOF. From the remarks in 4.2, it is clear that we need only consider the cases

$$(C^I, K.D) \text{ with } C = R \text{ or } Z \text{ and } D = R \text{ or } T.$$

Let $(C, D)_{c/o}$ denote the compact open topology. Now when K is finite,

$$(C, K.D)_{c/o} \approx K.(C, D)_{c/o} \quad\text{and}\quad (I \times K).(C, D) \approx K.(I.(C, D))$$

so it is sufficient to show that

$$(C^I, D)_{c/o} \approx I.(C, D).$$

This is proved in $[K_1]$, Section 4, when $D = T$, and the identical argument works when $D = R$. The case that I is finite is dual. That

$$(C^I, K.D)_{c/o} \approx ((D^*)^K, I.C^*)_{c/o}$$

is formal; each is topologized by uniform convergence on compact sets in $C^I \times (D^*)^K$. The analogous isomorphism for the other case is even easier.

(4.6) PROPOSITION. *Let* $C = R$ *or* Z, $C' = R$ *or* Z *and* $C^I \to C'$ *be any map. Then there is induced, for any* $D \in \mathfrak{D}$, *a map* $(C', D) \to (C^I, D)$. *Dually, for* $D = R$ *or* T, $D' = R$ *or* T, *and any* $D' \to K.D$, *there is induced, for any* $C \in \mathfrak{C}$, $(C, D') \to (C, K.D)$.

PROOF. Since $C' \in \mathfrak{D}_0$, it follows from (4.3) that any $C^I \to C'$, factors through C^J for some finite subset $J \subset I$. Then there is induced

$$(C^J, D) \approx J.(C, D) \to I.(C, D) \approx (C^I, D),$$

while we also have $(C', D) \to (C^J, D)$, since both have the compact open topology.

COROLLARY. *For any* $D \in \mathfrak{D}$, $C' \in \mathfrak{C}_0$, $C = R$ *or* Z, *and* $C^I \to C'$, *there is induced* $(C', D) \to (C^I, D)$. *Dually, for any* $C \in \mathfrak{C}$, $D' \in \mathfrak{D}_0$, $D = R$ *or* T, $D' \to K.D$, *there is induced* $(C, D') \to (C, K.D)$.

COROLLARY. *Let* $C \in \mathfrak{C}$, $C_0 \in \mathfrak{C}_0$, $D \in \mathfrak{D}$, $D_0 \in \mathfrak{D}_0$, $C \to C_0$, $D_0 \to D$. *Then there is induced* $(C_0, D_0) \to (C, D)$.

COROLLARY. *Let* $C, C' \in \mathfrak{C}$, $D, D' \in \mathfrak{D}$, *and maps* $C \to C'$ *and* $D' \to D$. *Then there is induced* $(C', D') \to (C, D)$.

PROOF. When C' is compact or D' is discrete, this is trivial. Thus it can be reduced to the case

$$C' = D_1^I,\ D' = J\,.\,D_1 \quad \text{where} \quad C_1 \text{ is } R \text{ or } Z \text{ and } D_1 \text{ is } R \text{ or } T\,.$$

For every $i \in I$, $j \in J$, the maps

$$C \to C' \xrightarrow{proj_i} C_1 \quad \text{and} \quad D_1 \xrightarrow{inj_j} D' \to D$$

induce $(C_1, D_1) \to (C, D)$. Since (C', D') is the coproduct of these (C_1, D_1) the result follows.

(4.7) PROPOSITION. (C, D) *has the strongest* $\mathfrak{G}$*-topology for which the above proposition and its corollaries are true.*

PROOF. Since it is the direct sum of spaces of the form

$$(C_0, D_0),\quad C_0 \in \mathfrak{C}_0,\ D_0 \in \mathfrak{D}_0,$$

it already has the finest topology for which those injections are continuous.

The point is that we have now freed the definition of (C, D) from the choice of the index sets. We could have defined (C, D) to have that finest topology, but that would have left the equivalent task of showing that it belongs to $\mathfrak{G}$. The results of this development can now be summarized.

(4.8) THEOREM. *Let* $C \in \mathfrak{C}$, $D \in \mathfrak{D}$. *Suppose* $\{C_\omega \to C\}$, $\omega \in \Omega$, *is indexed by all the* $\mathfrak{C}_0$*-subgroups of* C *and* $\{D \to D_\psi\}$, $\psi \in \Psi$, *is indexed by the* $\mathfrak{D}_0$*-quotients of* D. *Then each of the three colimits below is* (C, D):

$$\operatorname{indlim}(C_\omega, D_\psi),\quad \operatorname{indlim}(C_\omega, D),\quad \operatorname{indlim}(C, D_\psi);$$

each group is topologized by the compact open topology.

(4.9) PROPOSITION. *For any* $C \in \mathfrak{C}$, $D \in \mathfrak{D}$, $(C, D) \approx (D^*, C^*)$ *by the natural map.*

PROOF. This is simple to prove for $C \in \mathfrak{C}_0$, $D \in \mathfrak{D}_0$ and the definition gives it for the general case.

(4.10) PROPOSITION. *Let* C_1, $C_2 \in \mathfrak{C}$, $D \in \mathfrak{D}$. *Then the natural interchange gives a 1-1 correspondence between maps*

$$C_1 \to (C_2, D) \quad \textit{and maps} \quad C_2 \to (C_1, D).$$

In fact, $(C_1, (C_2, D)) \approx (C_2, (C_1, D))$.

PROOF. The definition of the hom in terms of the decompositions makes it sufficient to consider each of the 27 possible combinations :

$$C_1, C_2 \text{ compact or } R \text{ or } Z, \quad D \text{ discrete or } R \text{ or } T.$$

As seen below, this is not as formidable as it seems, as there is considerable collapsing of cases.

Let C_1, C_2 compact, D discrete. In this case a map $C_2 \to D$ has an open cofinite kernel and finite range. Since (C_2, D) is discrete, so does a map $C_1 \to (C_2, D)$ which then determines a finite number of maps $C_2 \to D$ which all together have an open cofinite kernel and finite range. Thus the transposed map $C_2 \to (C_1, D)$ has an open kernel, and of course each element of C_2 determines a continuous map $C_1 \to D$.

If C_1 is compact and $C_2 = R$, (C_2, D) is an R-vector space and there are no non-zero maps $C_1 \to (C_2, D)$.

If $C_1 = Z$, there is nothing to prove.

If $C_1 = C_2 = D = R$, the result is trivial.

The statement is symmetric in C_1 and C_2 and, on account of (4.9), dual symmetric in C_2 and D (or C_1 and D). A little reflection will show that every combination can be reduced to one already considered.

5. THE INTERNAL $Hom: \mathfrak{G}^{op} \times \mathfrak{G} \to \mathfrak{G}$.

(5.1) Let G and $H \in \mathfrak{G}$. We topologize $Hom(G, H)$ by the coarsest topology such that $(G, H) \to (C, D)$ is continuous for all

$$C \to G \quad \text{and} \quad H \to D \quad \text{with} \quad C \in \mathfrak{C},\ D \in \mathfrak{D}.$$

Since there will be at most a set of open sets, it will be necessary for a fixed C and D to examine only a set of such pairs, say

$$C_\omega \to G, \quad H \to D_\omega, \quad \omega \in \Omega \,.$$

Thus (G,H) is topologized as a subspace of $\Pi(C_\omega, D_\omega)$. Since $\mathfrak{C}$ and $\mathfrak{D}$ are closed under finite sums (= products), we can suppose for each neighborhood of O, $U \subset (G,H)$, a

$$C \to G, \; H \to D \quad \text{with} \quad C \in \mathfrak{C} \text{ and } D \in \mathfrak{D},$$

and a neighborhood of O, $V \subset (C,D)$, whose inverse image is contained in the neighborhood U.

(5.2) If $G \to G'$ and $H' \to H$, then for any

$$C \to G, \; H \to D \quad \text{with} \quad C \in \mathfrak{C} \text{ and } D \in \mathfrak{D},$$

we also have $C \to G'$, $H' \to D$ so that $(G',H') \to (C,D)$ is continuous. Thus the composite

$$(G',H') \to (G,H) \to \Pi(C_\omega, D_\omega)$$

is continuous and so the first factor is. This shows that this internal Hom is functorial in its arguments.

(5.3) I do not know whether it is sufficient, in describing (G,H), to have the $C \to G$ and $H \to D$ range over a $\mathfrak{C}$-cover of G and $\mathfrak{D}$-envelopment of H ; or even whether, given G, it is possible to find a family of $C \to G$ which work for all H and dually for H. I also do not know whether for fixed G, the functor $(G,-)$ commutes with projective limits and has an adjoint. This latter lack will be remedied in the next sections as we modify $(-,-)$ and restrict $\mathfrak{G}$.

(5.4) Suppose G and H are reflexive. Then a map $G \to H$ is the same as a map $H^* \to G^*$. If

$$C \to G \text{ and } H \to D \quad \text{with} \quad C \in \mathfrak{C}, \; D \in \mathfrak{D},$$

we have $D^* \to H^*$ and $G^* \to C^*$ so that

$$(H^*, G^*) \to (D^*, C^*) \approx (C,D)$$

is continuous. Since this is true for all $C \in \mathfrak{C}$, $D \in \mathfrak{D}$, it follows that the map $(H^*, G^*) \to (G,H)$ is continuous. By duality so is the opposite direction. Thus

PROPOSITION. *Let G and H be reflexive. Then the natural map induces an isomorphism $(G, H) \approx (H^*, G^*)$.*

(5.5) PROPOSITION. *Let $C \in \mathfrak{C}$ and $H \in \mathfrak{G}$. Then (C, H) is topologized by the finest topology for which every $H \to D$, $D \in \mathfrak{D}$, induces a continuous $(C, H) \to (C, D)$.*

PROOF. Let this topology be denoted $(C, H)_0$. It is clearly coarser than (C, H) so that $(C, H) \to (C, H)_0$ is continuous. To go the other way, suppose $C' \to C$ and $H \to D$ are given with

$$C' \in \mathfrak{C} \text{ and } D \in \mathfrak{D}.$$

Then $(C, H)_0 \to (C, D)$ is continuous by hypothesis and $(C, D) \to (C', D)$ is by the third corollary of (4.6).

6. ζ-COMPLETE AND ζ*-COMPLETE GROUPS.

(6.1) It is well-known that the group G is complete iff whenever $E_0 \to E$ is a dense inclusion, any map $E_0 \to G$ has an extension - necessarily unique - to a map $E \to G$. We modify this definition by restricting the class of E. The resultant notion we term *ζ-completeness*. We require that

i) E be a subgroup of a group in $\mathfrak{C}$,

ii) E^* be complete,

iii) E be prereflexive.

(6.2) PROPOSITION. *Suppose*

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

is a commutative diagram of (separated) uniform spaces and uniformly continuous maps. Suppose the top arrow is a dense inclusion, the bottom is 1-1 and onto and that X has a basis of uniform covers by families of sets closed in Y. Then there is a diagonal fill-in $B \to X$.

PROOF. There is a map $B \to X^{\sim}$ (the uniform completion) gotten by choosing for each $b \in B$ a net in A which converges to b. This is a Cauchy net and so is its image which then converges to a unique point of $X^{\sim}$. It is sufficient to show the image of this lies in X. In view of the continuity of all the maps it is sufficient to show that a Cauchy net in X which converges in the topology of Y does so in X. (Warning: This does not mean that they have the same topology - a net may be non-Cauchy in X and convergent in Y. Consider the example of X uniformly discrete.) So let $\{x_\omega\}$, $\omega \in \Omega$, be a Cauchy net in X and suppose it converges to x in the topology of Y. Let V be an X-neighborhood of x and $\mathfrak{U}$ be an X-uniform cover by Y-closed sets such that

$$V \supset \cup \{ U \mid x \in U \in \mathfrak{U} \}.$$

There is a $U \in \mathfrak{U}$ and an $\omega \in \Omega$ such that

$$\psi > \omega \implies x_\psi \in U.$$

Since U is closed in Y and the net $\{x_\psi\}$, $\psi > \omega$, also converges to x, it follows that $x \in U$ whence

$$U \subset V \quad \text{and} \quad \psi > \omega \implies x_\psi \in V.$$

(6.3) COROLLARY. *If G is ζ-complete, so is G^Δ.*

PROOF. See the proof of (3.9).

(6.4) THEOREM. *Let G be ζ-complete. Then G is quasi-reflexive.*

PROOF. We show that a slight modification of the proof of (3.10) works. Beginning with an $f: G^* \to T$ we find a $C \to G$, $C \in \mathfrak{C}$, such that f factors through a subgroup $H \subset C^*$. I. e. we have the diagramm

$$\begin{tikzcd} G^* \arrow[rr] \arrow[dr, "f"'] & & H \subset C^* \arrow[dl, "e"] \\ & T & \end{tikzcd}$$

Since T is complete we may even suppose that H is closed in C^*, hence complete. Now let $E = H^*$. As observed in the proof of (3.5),

$$E \subset \Pi C_\psi^* \quad \text{where} \quad C_\psi \subset H \quad \text{and} \quad C_\psi \in \mathfrak{C}_0 \cap \mathfrak{D}_0$$

whence $\Pi C^*_\psi \in \mathfrak{C}$. Now H is complete and hence by (3.10) $H^\Delta = H^{**}$. Thus $E^* = H^{**}$ is complete by (3.9). Finally the map $H^{**} \to H$ gives the natural map $E \to E^{**}$ continuous so that E is prereflexive. In view of the fact that G, and hence G^Δ, is ζ-complete, the rest of the proof goes through unchanged.

(6.5) If $G \in \mathfrak{G}$, let ζG denote the intersection of all subgroups of its uniform completion $G^\sim$ which contain G and are ζ-complete. It is clear that $G \subset \zeta G \subset G^\sim$ and that ζG is ζ-complete.

(6.6) PROPOSITION. *A map $G \to H$ has a unique extension to a $\zeta G \to \zeta H$. Thus ζ provides a left adjoint to the inclusion $\zeta \mathfrak{G} \subset \mathfrak{G}$.*

PROOF. Let $f\colon G \to H$. We have a commutative diagram

$$\begin{tikzcd} G \arrow[r] \arrow[d, "f"'] & \zeta G \arrow[r] & G^\sim \arrow[d, "f^\sim"] \\ H \arrow[r] & \zeta H \arrow[r] & H^\sim \end{tikzcd}$$

and it is sufficient to show that $f^{\sim -1}(\zeta H) \supset \zeta G$. Since that inverse image contains G, it is sufficient to show it is ζ-complete. Let $E_0 \subset E$ be as in (6.1) and

$$g\colon E \to G^\sim \quad \text{such that} \quad f^\sim g(E_0) \subset \zeta H.$$

In that case the map $f^\sim g \mid E_0$ has a unique extension to a map $E \to \zeta H$. This map, followed by $\zeta H \to H^\sim$ agrees on E_0 with $f^\sim g$ whence they are the same so that

$$f^\sim g(E) \subset \zeta H \quad \text{and} \quad g(E) \subset f^{\sim -1}(\zeta H).$$

The remaining assertion is trivial.

(6.7) We say that G is ζ^*-complete provided G^* is ζ-complete.

PROPOSITION. *Let G be ζ-complete. Then $(\zeta G^*)^*$ is ζ-ζ^*-complete and reflexive.*

PROOF. Let $E_0 \to E$ be as in (6.1) and $E_0 \to (\zeta G^*)^*$ be given. The dense inclusion $G^* \to \zeta G^*$ gives $(\zeta G^*)^* \to G^{**}$ which is 1-1 and onto. Since

G is ζ-complete we have $G^{**} \to G$, also 1-1 and onto. This gives $E_0 \to G$ which has an extension $E \to G$. The dual $G^* \to E^*$ can be extended, by (6.1.ii), to

$$\zeta G^* \to E^* \text{ and then } E^{**} \to (\zeta G^*)^*.$$

$$\begin{tikzcd} E \arrow[r] \arrow[drr] & E^{**} \arrow[r] & (\zeta G^*)^* \arrow[d] \\ & & G \end{tikzcd}$$

By (6.1.iii) we have $E \to E^{**}$. The maps agree on E_0 and hence are equal. Since $(\zeta G^*)^* \to G$ is 1-1 and onto, this map $E \to (\zeta G^*)^*$ extends the given one and shows the latter to be ζ-complete and hence quasi-reflexive. Since ζG^* is ζ-complete, it is quasi-reflexive, so we have

$$(\zeta G^*)^{**} \to \zeta G^* \text{ which gives } (\zeta G^*)^* \to (\zeta G^*)^{***}$$

so $(\zeta G^*)^*$ is also prereflexive and hence reflexive. Finally ζG^* is ζ-complete, hence quasi-reflexive, hence $(\zeta G^*)^{**}$ is ζ-complete by (4.3) and thus $(\zeta G^*)^*$ is ζ^*-complete.

(6.8) PROPOSITION. *Let G be a reflexive ζ^*-group and H a reflexive ζ-group. Then (G, H) is a ζ-group.*

PROOF. Suppose that $E_0 \to E$ is as in (6.1) and $f_0 : E_0 \to (G, H)$ is a map. There is associated a bilinear function which we also call $f_0 : E_0 \times G \to H$. For each $g \in G$ the map $f_0(-, G) : E_0 \to H$ has an extension to a map which we call $f(-, G) : E \to H$. There results a not necessarily continuous bilinear function $f : E \times G \to H$ which is continuous in E for each $g \in G$. Corresponding to f_0 is a map we call

$$\hat{f}_0 : E_0 \to (H^*, G^*) \quad \text{or} \quad \hat{f}_0 : E_0 \times H^* \to G^*.$$

Repeating the argument, we get $\hat{f} : E \times H^* \to G^*$ which is continuous in E for each $\phi \in H^*$. For any $\phi \in H^*$, $g \in G$, the maps

$$e \mapsto \phi f(e, g), \quad e \mapsto \hat{f}(e, \phi)(g)$$

are continuous in E and agree on the dense subgroup E_0, whence they are equal. Thus for all $e \in E$, the map $f(e, -) : G \to H$, whether or not continuous, induces by composition the map

$$\hat{f}(e,-): H^* \to G^*.$$

Since G and H are reflexive, the one is continuous iff the other is. Now suppose temporarily that $G \in \mathfrak{C}$ and $H \in \mathfrak{D}$. Then $(G,H) \in \mathfrak{D}$, and is thus complete. Thus $f_0 : E_0 \to (G,H)$ has a unique extension to a continuous map $f': E \to (G,H)$. The associated map $f': E \times G \to H$ is, at least, separately continous. If $a \in G$, the continuous maps

$$f'(-,a) \text{ and } f(-,a): E \to H$$

agree on E_0 and hence are equal. Thus the maps $f(e,-)$ and $\hat{f}(e,-)$ are continuous whenever $G \in \mathfrak{C}$ and $H \in \mathfrak{D}$. Applying this result to any

$$C \to G,\ H \to D \text{ with } C \in \mathfrak{C},\ D \in \mathfrak{D}$$

we conclude that

$$C \to G \to H \to D$$

is continuous. Since H is topologized by its $\mathfrak{D}$-representations, this implies that $C \to G \to H$ is continuous. Then so is $H^* \to G^* \to C^*$. Since G^* is topologized by its maps to groups of the form C^*, this implies that $H^* \to G^*$ is continuous, whence $G \to H$ is. Thus we have a function $E \to (G,H)$ whose composition with any $(G,H) \to (C,D)$ is continuous and hence is itself continous.

7. THE CATEGORY $\mathfrak{R}$.

(7.1) We let $\mathfrak{R}$ denote the full subcategory of ζ-ζ^*-reflexive groups. The argument of [B_3], 4.1, is readily adaptable to the present situation to give

PROPOSITION. *The functor*

$$G \longmapsto \delta G = (\zeta G^*)^*$$

is right adjoint to the inclusion $\mathfrak{R} \subset \zeta \mathfrak{G}$. *For any* G, *the map* $G^{**} \to G$ *is 1-1 and onto.*

We now define, for G, $H \in \mathfrak{R}$,

$$[G,H] = \delta(G,H).$$

It consists of the same set of maps $G \to H$ but with a possibly finer topology. When $H = T$, however, there is no change, so that the dual is the same. For it follows from (6.3) that when $G \in \mathfrak{R}$, so is G^*.

(7.2) PROPOSITION. *Let $G \to (C, H)$ be given with $C \in \mathfrak{C}$. Then the corresponding function $C \to (G, H)$ is continuous.*

PROOF. $C \to (G, H)$ is continuous iff

$$C \to (G, H) \to (C', D)$$

is for all

$$C' \to G,\ H \to D \quad \text{with} \quad C' \in \mathfrak{C},\ D \in \mathfrak{D}.$$

But then we have

$$C' \to G \to (C, H) \to (C, D),$$

which gives, by (4.10), $C \to (C', D)$.

(7.3) Let $C_1, C_2 \in \mathfrak{C}$. The identity map on (C_1, C_2^*) corresponds, by (4.10), to a map

$$C_1 \to ((C_1, C_2^*), C_2^*) \approx (C_2, (C_1, C_2^*)^*).$$

This determines a bilinear function

$$C_1 \times C_2 \to (C_1, C_2^*)^*.$$

We let $T(C_1, C_2)$ denote the subgroup of $(C_1, C_2^*)^*$ generated by $C_1 \times C_2$, topologized by the induced topology.

(7.4) PROPOSITION. *$T(C_1, C_2)$ is dense in $(C_1, C_2^*)^*$.*

PROOF. If $f: C_1 \to C_2^*$ is considered as an element of $(C_1, C_2^*)^{**}$, the induced character on $T(C_1, C_2)$ sends

$$(c_1, c_2) \text{ to } f(c_1)(c_2).$$

This is 0 for all $(c_1, c_2) \in C_1 \times C_2$ iff $f = 0$. But $(C_1, C_2^*)^*$ belongs to $\mathfrak{C}$ and it follows from [K_2], Theorem 2, that there are enough characters to separate closed subgroups.

(7.5) PROPOSITION. *Let $C_1, C_2 \in \mathfrak{C}$, $G \in \zeta\mathfrak{G}$. Then any map $C_1 \to (C_2, G)$*

extends to a map $C_1 \to (C_2, \delta G)$.

PROOF. We get a function $C_1 \times C_2 \to G$ which induces an additive function $T(C_1, C_2) \to G$. To see that it is continuous, consider $G \to D$ with $D \in \mathfrak{D}$. We have

$$C_1 \to (C_2, G) \to (C_2, D) \to (D^*, C_2^*),$$
$$D^* \to (C_1, C_2^*), \quad (C_1, C_2^*)^* \to D.$$

Thus a commutative square

$$\begin{tikzcd} T(C_1, C_2) \arrow[r] \arrow[d] & (C_1, C_2^*)^* \arrow[d] \\ G \arrow[r] & \Pi D_\omega . \end{tikzcd}$$

That G is topologized as a subspace of ΠD_ω implies the left arrow is continuous. Then since G is ζ-complete, we get

$$(C_1, C_2^*)^* \to G, \quad G^* \to (C_1, C_2^*), \quad \zeta G^* \to (C_1, C_2^*),$$
$$(\delta G)^* = (\zeta G^*)^{**} \to \zeta G^* \to (C_1, C_2^*)$$

and, by (7.2),

$$C_1 \to ((\delta G)^*, C_2^*) \approx (C_2, \delta G).$$

(7.6) PROPOSITION. *Let* $C \in \mathfrak{C}$, $G \in \mathfrak{K}$, $H \in \mathfrak{G}$ *and* $f: C \to (G, H)$ *be given. Then the corresponding function* $G \to (C, H)$ *is continuous.*

PROOF. The map $G \to (C, H)$ is continuous iff the composite

$$G \to (C, H) \to (C, D)$$

is for all $H \to D$, $D \in \mathfrak{D}$. So we may consider the case $H = D \in \mathfrak{D}$. Let

$$\{ C_\omega \to G \}, \quad C_\omega \in \mathfrak{C},$$

range over a sufficient family so that $G^* \subset \Pi C_\omega^*$. Let G_0 be the inductive limit of the C_ω. Then it is easily seen that G_0 is the same abelian group as G with a topology which is, in general, finer. Since the composite

$$G^* \to G_0^* \to \Pi C_\omega^*$$

is a subspace inclusion, so is $G^* \to G_0{}^*$. We have

$$C \to (G, D) \to (C_\omega, D)$$

and, by (4.10), $C_\omega \to (C, D)$ which gives

$$G_0 \to (C, D) \quad \text{or} \quad (C, D)^* \to G_0^* .$$

The image of $T(C, D^*)$ in G_0^* takes $(c, \phi) \in C \times D^*$ to the character: $g \mapsto \phi(f(c)(g))$ which is the composite

$$G \xrightarrow{f(c)} D \xrightarrow{\phi} T$$

of continuous maps and is hence continuous on G. Thus we have

$$\begin{tikzcd} T(C, D^*) \arrow[r] \arrow[d] & (C, D)^* \arrow[d] \\ G^* \arrow[r] & G_0^* \end{tikzcd}$$

commutative. The left vertical arrow is continuous because the others are and $G^* \subset G_0^*$. Since $T(C, D^*)$ is dense in $(C, D)^*$ and G^* is ζ-complete, it follows that the image of $(C, D)^*$ is in G^*. Since G is reflexive, we get $G \to (C, D)$.

(7.7) COROLLARY. *For $G_1, G_2 \in \mathfrak{R}$ and $H \in \mathfrak{G}$, there is a 1-1 correspondence between maps $G_1 \to (G_2, H)$ and maps $G_2 \to (G_1, H)$.*

PROOF. Given $G_1 \to (G_2, H)$ we know that each element $g_2 \in G_2$ determines a continuous map

$$G_1 \to (G_2, H) \xrightarrow{ev(g_2)} H$$

since the topology on (G_2, H) is at least as fine as that of pointwise convergence. The result is a function $G_2 \to (G_1, H)$ which is continuous iff the composite

$$G_2 \to (G_1, H) \to (C, H)$$

is for all $C \to G_1$, $C \in \mathfrak{C}$. But this is guaranteed by the previous proposition.

(7.8) PROPOSITION. *Let $G, H \in \mathfrak{R}$. Then the evaluation map*

$$G \to ((G, H), H)$$

is continuous.

PROOF. If

$$C \to (G,H),\quad C \in \mathfrak{C},\quad H \to D,\quad D \in \mathfrak{D},$$

then we have that

$$G \to (C,H) \to (C,D)$$

is continuous and hence $G \to ((G,H),\ H)$ is.

(7.9) PROPOSITION. *Let G, H_1, $H_2 \in \mathfrak{R}$. Then composition defines a continuous map*

$$(H_1,H_2) \to ((G,H_1),(G,H_2)).$$

PROOF. A $\mathfrak{D}$-representation of (G,H_2) is determined by the map

$$(G,H_2) \to (C,D) \quad \text{corresponding to} \quad C \to G,\ H_2 \to D$$

with $C \in \mathfrak{C}$, $D \in \mathfrak{D}$. Also let

$$C_1 \to (G,H_1) \quad \text{with} \quad C_1 \in \mathfrak{C}.$$

We must show that $(H_1,H_2) \to (C_1,(C,D))$ is continuous. Using (4.10), (5.4), (6.7) and the preceding, we have

$$C \to G \to ((G,H_1),\ H_1) \to (C_1,H_1) \approx (H_1^*,C_1^*)$$

which gives $H_1^* \to (C,C_1^*)$. This gives

$$(H_1,H_2) \approx (H_2^*,H_1^*) \to (D^*,(C,C_1^*)) \approx$$
$$\approx (D^*,(C_1,C^*)) \approx (C_1,(D^*,C^*)) \approx (C_1,(C,D)).$$

(7.10) PROPOSITION. *Let G, H_1, $H_2 \in \mathfrak{R}$. Then composition defines a continuous map*

$$[H_1,H_2] \to [[G,H_1],[G,H_2]].$$

PROOF. We begin by observing that with $\mathfrak{C} \subset \mathfrak{R}$, a map $C \to (G,H)$ with $C \in \mathfrak{C}$ is the same as a map $C \to [G,H]$. Now let

$$C \to [H_1,H_2],\quad C_1 \to [G,H_1] \quad \text{with} \quad C,\ C_1 \in \mathfrak{C}.$$

Then using the preceding, we have

$$C \to [H_1,H_2] \to (H_1,H_2) \to ((G,H_1),(G,H_2)) \to (C_1,(G,H_2))$$

which gives, by (7.5), $C \to (C_1,[G,H_2])$. Now every map $H_1 \to H_2$ gives a continuous $(G, H_1) \to (G, H_2)$ and extends, by functoriality, to a continuous $[G, H_1] \to [G, H_2]$. In particular this is true of maps in C and so we have a function

$$C \to ([G, H_1], [G, H_2])$$

for which, for any $C_1 \to [G, H_1]$, the composite

$$C \to ([G, H_1], [G, H_2]) \to (C_1, [G, H_2])$$

is continuous, which means the given function is. It corresponds by (7.7) to a continuous

$$C \to [H_1, H_2] \to (H_1, H_2) \to ((G, H_1), (G, H_2)) \to ((G, H_2)^*, C_1^*)$$

which gives $(G, H_2)^* \to (C, C_1^*)$. Now $(C, C_1^*) \in \mathfrak{D}$ and is ζ-complete so we get

$$\zeta(G, H_2)^* = [G, H_2]^* \to (C, C_1^*),$$

and reversing the above gives $C \to (C_1, [G, H_2])$. Now every map $H_1 \to H_2$ gives by composition a continuous $(G, H_1) \to (G, H_2)$ which extends by functoriality to $[G, H_1] \to [G, H_2]$. Thus there is a not necessarily continuous function of the desired kind which restricts to $C \to (C_1, [G, H_2])$. This is continuous whenever

$$C_1 \to [G, H_1] \quad \text{with} \quad C_1 \in \mathfrak{C},$$

and thus $C \to ([G, H_1], [G, H_2])$ is continuous. Also by (7.3) we have $[G, H_1] \to (C, [G, H_2])$ which is continuous and is the restriction of a function

$$[G, H_1] \to ([H_1, H_2], [G, H_2])$$

which again is well defined but not *a priori* continuous. But it is when composed with any $C \to [H_1, H_2]$ and hence is continuous. We then apply the corollary of (7.3) once more to get

$$[H_1, H_2] \to ([G, H_1], [G, H_2]).$$

Finally, the right adjointness properties of δ allow us to get

$$[H_1, H_2] \to [[G, H_1], [G, H_2]].$$

(7.11) Up until now we have not defined any tensor product. At this point we define

$$\text{for } G, H \in \mathfrak{R}, \quad G \otimes H = [G, H^*]^*.$$

PROPOSITION. *For any* $G_1, G_2, H \in \mathfrak{R}$, *there is a 1-1 correspondence between maps*

$$G_1 \otimes G_2 \to H \quad \textit{and} \quad G_1 \to [G_2, H].$$

The tensor product is commutative and associative and Z *is the unit.*

PROOF.

$$[G_1, G_2^*]^* \to H, \quad H^* \to [G_1, G_2^*], \quad G_1 \to [H^*, G_2^*] \approx [G_2, H]$$

are all equivalent. The commutativity and associativity follow from the corollary to (7.3) and the preceding composition law (see [EK], Chapter II, Section 3, for details).

(7.12) THEOREM. *The category* $\mathfrak{R}$ *equipped with* $[-,-]$ *and* $\otimes$ *is a closed monoidal category in which every group is reflexive.*

(7.13) One word of warning should be given. The tensor product should not be confused with that of [G] and [H], which was constructed to represent bilinear maps continuous on the cartesian product. The tensor product here does represent bilinear maps but they are not generally jointly continuous. Even the evaluation map $G^* \times G \to T$ is not generally jointly continuous, as remarked at the end of $[K_1]$.

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