Searching for absolute $\mathcal{CR}$-epic spaces

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Abstract

In previous papers, [Barr, Burgess, & Raphael (2003), Barr, Raphael, & Woods (to appear)], two of us have investigated the situation of a topological space Y and a subspace X such that the induced map $C(Y) \longrightarrow C(X)$ is an epimorphism in the category $\mathcal{CR}$ of commutative rings (with units). We call such an embedding a $\mathcal{CR}$-epic embedding and we say that X is absolute $\mathcal{CR}$-epic if every embedding of X is $\mathcal{CR}$-epic. We continue this investigation. Our most notable result shows that a Lindelöf space X is absolute $\mathcal{CR}$-epic if a countable intersection of βX-neighbourhoods of X is a βX-neighbourhood of X. This condition is stable under countable sums, the formation of closed subspaces, cozero-subspaces, and being the domain or codomain of a perfect map. A strengthening of the Lindelöf property leads to a new class with the same closure properties that is also closed under finite products. Moreover, all σ-compact spaces and all Lindelöf P-spaces satisfy this stronger condition. We get some results in the non-Lindelöf case that are sufficient to show that the Dieudonné plank and some closely related spaces are absolute $\mathcal{CR}$-epic.

1 Introduction

In the papers [Barr, Burgess, & Raphael (2003), Barr, Raphael, & Woods (to appear)][1], two of us have participated in the introduction of the concept of an absolute $\mathcal{CR}$-epic space and studied a number of classes of such spaces. In this paper, we continue this study.

For a space X, let $C(X)$ denote the ring of real-valued continuous functions on X and $C^*(X)$ denote the subring of bounded functions. A space X is absolute $\mathcal{CR}$-epic if and only if whenever X is embedded into a space Y, the induced map $C(Y) \longrightarrow C(X)$ is an epimorphism in the category $\mathcal{CR}$ of commutative rings with unit. In order for this to hold, it is necessary and sufficient that it hold whenever Y is a compactification of X.

For a space to be absolute $\mathcal{CR}$-epic, it is necessary that every function in $C^*(X)$ extend to an open set of every compactification, [BBR, 2.6]; for a Lindelöf space, it is also sufficient, [BRW, 2.14]. We show that one way to make this happen is that X satisfy the "CNP" which means that the intersection of

[1] In this paper, we will refer to these papers as [BBR] and [BRW], respectively.

countably many βX-neighbourhoods of X be a βX-neighbourhood of X, see Section 3.

1.1 REMARK. Although we are working in the category $\mathcal{CR}$ of commutative rings with unit (and unit preserving ring homomorphisms), our results are equally valid in the category $\mathcal{CR}_0$ of commutative rings, not necessarily with unit, and homomorphisms that are not required to preserve those that exist. The reason is that whenever X is a subspace of Y, the induced $\phi : C(Y) \longrightarrow C(X)$ is a unit preserving map. If it is an epimorphism in $\mathcal{CR}_0$, it is evidently an epimorphism in $\mathcal{CR}$. If it is an epimorphism in $\mathcal{CR}$, then for any $\mathcal{CR}_0$ homomorphisms $\psi, \rho : C(X) \longrightarrow R$ such that $\psi\phi = \rho\phi$, we have that $e = \psi(1) = \rho(1)$. It is clear that e is idempotent, so that eR is a subring of R on which e is the identity. Moreover, for $f \in C(X)$, $e\psi(f) = \psi(1)\psi(f) = \psi(f)$ and similarly $e\rho(f) = \rho(f)$ so that ψ and ρ can be regarded as unit preserving maps to eR. It follows that $\psi = \rho$.

We show that the class of Lindelöf absolute $\mathcal{CR}$-epic spaces is closed under the formation of closed subspaces and of cozero-subspaces. A Lindelöf space is absolute $\mathcal{CR}$-epic if and only if every point has an absolute $\mathcal{CR}$-epic neighbourhood. It follows that a countable sum of Lindelöf absolute $\mathcal{CR}$-epic spaces is absolute $\mathcal{CR}$-epic. If $f : Y \longrightarrow X$ is a perfect (continuous) surjection (see 2.1), then X is Lindelöf absolute $\mathcal{CR}$-epic if and only if Y is. The one disappointment is that we have not been able to show that a product of two Lindelöf absolute $\mathcal{CR}$-epic spaces is either Lindelöf or absolute $\mathcal{CR}$-epic.

To partially repair that latter defect, we introduce the notion of "amply Lindelöf" (see 4.2). We define a class of Lindelöf CNP spaces which includes the Lindelöf P-spaces and the Lindelöf locally compact spaces and is closed under finite products, countable sums, the formation of closed subsets and cozero-subsets, and other operations. See 4.7 for details. Curiously, we have no example of a pair of Lindelöf CNP spaces whose product is not a Lindelöf absolute $\mathcal{CR}$-epic space, nor even an example of a Lindelöf CNP space that is not amply Lindelöf.[2]

We also consider the non-Lindelöf case. Here the situation is much less clear. We have some results which show that the Dieudonné plank is absolute $\mathcal{CR}$-epic, see 7.15, but very little in the way of genuine understanding. A non-Lindelöf absolute $\mathcal{CR}$-epic space X must be almost Lindelöf. A necessary, but not sufficient, condition that an almost Lindelöf space X be absolute $\mathcal{CR}$-epic is that υX be absolute $\mathcal{CR}$-epic. See 2.1 for the relevant definitions. However, the sufficient conditions we have, 7.11, 7.12, and 7.14 are rather special.

We would like to thank Ronnie Levy for many discussions. In response to an earlier version of this paper, he sent us a proof for the hard half (the "only if")

[2]We have recently discovered that "amply Lindelöf" is not new (although the name of course is), but is the same as the condition called $(*)$ on the first page of [Alster (1988)], in which it is shown that a product of space satisfying $(*)$ and a Lindelöf space is Lindelöf (see 4.5 below) as is a countable product of spaces that satisfy $(*)$, although it need not satisfy $(*)$. In a later paper, [?], we changed "amply Lindelöf" to "Alster".

of Theorem 5.4 and has graciously allowed us to include it. He also contributed Theorem 4.8.

The first person to pay much attention to epimorphisms in general categories was John Isbell, [Isbell (1966)]. As we were preparing the final revision, word of his death reached us and we dedicate this paper to his memory.

2 General results

2.1 Standard defintions and notation All spaces considered in this paper are assumed to be Tychonoff (completely regular Hausdorff) and all functions, unless explicitly stated otherwise, are assumed continuous. As usual, βX denotes the Stone-Čech compactification of the space X. It is the unique compact space in which X is dense and C^*-embedded. See Chapter 6 of [Gillman & Jerison (1960)] for details. We denote by υX the Hewitt realcompactification of a Tychonoff space X; see Chapter 8 of [Gillman & Jerison (1960)] or 5.5(c) and 5.10 of [Porter & Woods (1988)]. A space X is called **realcompact** if $X = \upsilon X$. A space is called **almost Lindelöf** if, of any two disjoint zero-sets, at least one is Lindelöf. It is shown in [Levy & Rice (1981), 5.4] that a space X is almost Lindelöf if and only if it differs from υX by at most one point and υX is Lindelöf.

A subset of a space X is called a **zero-set** if is of the form $f^{-1}(0)$ for some $f \in C(X)$. The complement of a zero-set is called a **cozero-set**. Tychonoff spaces are characterized by having a neighbourhood base of cozero-sets.

A continuous map $\theta : X \longrightarrow Y$ is said to be **perfect** if it is closed and for all $p \in Y$, $\theta^{-1}(p)$ is compact. It can be shown that whenever $K \subseteq Y$ is compact, so is $\theta^{-1}(K)$. The properties of perfect maps are explored in detail in [Porter & Woods (1988)]. However, be warned that they do not invariably assume that their functions are continuous.

2.2 Notation. If $\theta : B \longrightarrow A$ is a function, then we use the same θ for the direct image function $\mathcal{P}(B) \longrightarrow \mathcal{P}(A)$. This has a right adjoint $\theta^{-1} : \mathcal{P}(A) \longrightarrow \mathcal{P}(B)$ and θ^{-1} itself has a right adjoint $\theta_{\#} : \mathcal{P}(B) \longrightarrow \mathcal{P}(A)$ that takes a set $T \subseteq B$ to $\theta_{\#}(T) = A - \theta(B - T)$. It follows that if θ is a closed mapping between topological spaces, then $\theta_{\#}$ takes open sets to open sets, a fact that will turn out to be important. Here are some properties of the adjunctions. Assume that $S \subseteq A$ and $T \subseteq B$. Then

1. $\theta(T) \subseteq S$ if and only if $T \subseteq \theta^{-1}(S)$;
2. $\theta^{-1}(S) \subseteq T$ if and only if $S \subseteq \theta_{\#}(T)$;
3. θ preserves unions, $\theta_{\#}$ preserves intersections and θ^{-1} preserves both;
4. $\theta^{-1}(\theta_{\#}(T)) \subseteq T$, with equality when θ is injective;
5. $\theta_{\#}(\theta^{-1}(S)) \supseteq S$ with equality when θ is surjective;
6. $\theta(\theta^{-1}(S)) \subseteq S$, with equality when θ is surjective;

7. $\theta^{-1}(\theta(T)) \supseteq T$ with equality when θ is injective;

8. $\theta_{\#}(T) \subseteq \theta(T)$ if and only if θ is surjective;

9. $\theta^{-1}(\theta(\theta^{-1}(S))) = \theta^{-1}(\theta_{\#}(\theta^{-1}(S))) = \theta^{-1}(S)$;

10. $\theta(\theta^{-1}(\theta(T))) = \theta(T)$;

11. $\theta_{\#}(\theta^{-1}(\theta_{\#}(T))) = \theta_{\#}(T)$.

Incidentally, $\theta_{\#}$ is called the universal image in topos theory and usually denoted $\forall_\theta$. In contrast, the direct image is called the existential image and denoted $\exists_\theta$.

If $\{X_i \mid i \in I\}$ are disjoint open subspaces of X for which $X = \bigcup_{i\in I} X_i$ we will say that X is the *sum* of the X_i and write $X = \sum_{i\in I} X_i$. We will also write $X = X_1 + \cdots + X_n$ for a finite sum. This is the sum in the category of topological spaces and continuous maps.

2.3 Quotients Since every compactification of a space X is a quotient space of βX and every quotient mapping is the quotient modulo an equivalence relation, we will begin by looking at equivalence relations. Although the results are stated for βX, they are actually valid for any compactification of X.

2.4 Definition. *Let X be a space. An equivalence relation $E \subseteq \beta X \times \beta X$ will be called* **admissible** *if it is a closed subspace of $\beta X \times \beta X$ and if $(X \times \beta X) \cap E = \Delta_X$ (the diagonal of X in $X \times X$).*

Throughout this paper, E will denote an admissible equivalence relation on the Stone-Čech compactification of a space, usually X, and $\theta : \beta X \longrightarrow\!\!\!\!\!\rightarrow \beta X/E = K$ will denote the induced quotient map. The map θ, being continuous between compact sets, is closed. It is an immediate consequence that $\theta_{\#}$ (see 2.2) takes open sets to open sets. Since θ is surjective, $\theta_{\#}(U) \subseteq \theta(U)$, so that when U is a βX-neighbourhood of X, both $\theta_{\#}(U)$ and $\theta(U)$ are K-neighbourhoods of $\theta_{\#}(X)$ and the admissibility of E implies that $\theta_{\#}(X) = X$.

2.5 Proposition. *For any equivalence relation E on βX, the induced map $\theta : X \longrightarrow\!\!\!\!\!\rightarrow K = \beta X/E$ is an embedding into a Tychonoff space if and only if E is admissible.*

Proof. The definition of admissibility implies that in $K = \beta X/E$ no point of X is identified with any other point of βX. Since no two points of X are identified, this implies that $X \longrightarrow K$ is injective. Moreover, we claim that $(X \times \beta X) \cap E = \Delta_X$ if and only if any ultrafilter $\mathbf{u}$ on X which fails to converge to a point $p \in X$ still fails to converge to p when $\mathbf{u}$ is mapped into K. For suppose $\mathbf{u}$ fails to converge to p but its image in K does. Let q be the limit of $\mathbf{u}$ in βX. Then $\theta(q) = p = \theta(p)$ as θ preseves limits of ultrafilters. But this implies that $(p, q) \in E$ contradicting the fact that $(X \times \beta X) \cap E = \Delta_X$. The converse is similar.

It is now clear that X is embedded in K if and only if the same ultrafilters on X converge to the same points of X if and only if $(X \times \beta X) \cap E = \Delta_X$. The

rest of the proof follows from the fact that $\beta X/E$ is Hausdorff if and only if E is closed. ∎

Central to our studies is the following theorem, which is essentially [BRW, Corollary 2.14]. It is used in conjunction with the lemma that follows it, for most of what is proved in this paper concerning Lindelöf spaces.

2.6 THEOREM. *A Lindelöf space X is absolute $\mathcal{CR}$-epic if and only if, for every compactification K of X, every function in $C(X)$ extends to a K-neighbourhood of X.* ∎

2.7 LEMMA. *Suppose E is an admissible equivalence relation on βX. Then for any $f \in C(\beta X)$ and any $n \in \mathbf{N}$, the set*

$$U_n = \{p \in \beta X \,\big|\, (p,q) \in E \Rightarrow |f(p) - f(q)| < 1/n\}$$

is open in βX and contains X.

PROOF. The set $V_n = \{(p,q) \in E \,\big|\, |f(p)-f(q)| < 1/n\}$ is open in E and contains Δ_X. Let $\delta : E \longrightarrow \beta X$ be the first coordinate projection restricted to E. Then $\delta_\#(V_n)$ is open (see 2.2) and contains $\delta_\#(\Delta_X) = \delta_\#((X \times \beta X) \cap E) = X$. But $\delta_\#(V_n) = U_n$ because, by definition, $p \in \delta_\#(V_n)$ if and only if, $V_n \supseteq \delta^{-1}(p) = \{p\} \times \{q \,\big|\, (p,q) \in E\}$, that is, if and only if $p \in U_n$ ∎

We will have need of both halves of the following result from general topology. Although it seems to be well known, we did not find a readily accessible proof and so we sketch one.

2.8 THEOREM. *[Smirnov] If a Tychonoff space X is Lindelöf, then in any compactification K of X any open subset of K that contains X contains a cozero-set containing X. Conversely, if every open subset of βX that contains X contains a cozero-set containing X, then X is Lindelöf.*

PROOF. Any K-open set containing X is a union of cozero-sets. If X is Lindelöf, countably many of those cozero-sets cover X and a countable union of cozero-sets is a cozero-set. Conversely, suppose $\{U_i \,\big|\, i \in I\}$ is an open cover of X. Then $V_i = \mathrm{cl}_K(U_i)$ is a βX-neighbourhood of X whose intersection with X is U_i. Assuming $X \subseteq \mathrm{coz}(u) \subseteq \bigcup V_i$, then $\mathrm{coz}(u)$ is σ-compact and hence Lindelöf, so that countably many of the V_i and hence countably many of the U_i cover X. ∎

3 Spaces satisfying the countable neighbourhood property

In this section, we study a sufficient condition for a Lindelöf space to be absolute $\mathcal{CR}$-epic.

3.1 DEFINITION. *It is standard to call a subset $X \subseteq Y$ a* **P-set in** *Y if the intersection of any countable set of Y-neighbourhoods of X is a Y-neighbourhood of X. We will say that X has the* **countable neighbourhood property** *if X is a P-set in βX. A space that satisfies the countable neighbourhood property will be called a* **CNP space**. *Since a G_δ set is, by definition, a countable intersection of open sets, an equivalent formulation is that every G_δ set of βX that contains X be a neighbourhood of X.*

If $p \in X$ is a point such that $\{p\}$ is a P-set, then p is called a **P-point** *of X. The space X is called a* **P-space** *if every point of X is a P-point of X.*

3.2 PROPOSITION. *A space satisfies the CNP if and only if it is a P-set in any compactification.*

PROOF. Suppose $X \subseteq K$ is a compactification and $\theta : \beta X \longrightarrow K$ is the quotient map. If X satisfies the CNP, suppose that $\{U_n \,\big|\, n \in \mathbf{N}\}$ is a family of K-open sets that contain X. Then $\{\theta^{-1}(U_n)\}$ is a family of βX-open sets that contain X. Thus there is a βX-open set V such that $X \subseteq V \subseteq \theta^{-1}(U_n)$ for all n. But then $X = \theta_\#(X) \subseteq \theta_\#(V) \subseteq \theta_\#(\theta^{-1}(U_n)) = U_n$ for all n. Conversely, suppose that X is a P-set in K and that $\{U_n \,\big|\, n \in \mathbf{N}\}$ is a family of βX-open sets containing X. Then $\{\theta_\#(U_n)\}$ is a family of K-open sets containing X and hence there is a K-open set V such that $X \subseteq V \subseteq \theta_\#(U_n)$ for all n, from which we infer that $X \subseteq \theta^{-1}(V) \subseteq U_n$. ∎

3.3 THEOREM. *Suppose that X is a CNP-space and that E is an admissible equivalence relation on βX. Then for any $f \in C(\beta X)$,*

$$U = \{p \in \beta X \,\big|\, (p, q) \in E \Rightarrow f(p) = f(q)\}$$

is a βX-neighbourhood of X.

PROOF. Let $\{U_n \;\big|\; n \in \mathbf{N}\}$ be the sets constructed in Lemma 2.7. Clearly $U = \bigcap U_n$ and the CNP property implies that U is a βX-neighbourhood of X. ∎

3.4 COROLLARY. *A Lindelöf CNP-space is absolute $\mathcal{CR}$-epic.*

PROOF. Suppose that K is a compactification of the space X. It is sufficient to show that every $f \in C^*(X)$ extends to a K-neighbourhood of X [BBR, 2.1(ii)]. This readily follows from the preceding theorem. ∎

The known permanence properties of CNP spaces are summarized in the following.

3.5 THEOREM.

1. *Any closed subspace of a CNP space is CNP.*
2. *If every point of a space has a CNP neighbourhood then the space is CNP.*
3. *A sum of CNP spaces is CNP.*
4. *Any open subspace of a CNP space is CNP.*

5. *If $\theta : Y \longrightarrow X$ is a perfect surjection, then X is CNP if and only if Y is.*

PROOF. Let X be the given space.

1. Let A be a closed subspace of X and suppose that X is CNP. Let $K = \mathrm{cl}_{\beta X}(A)$. It is sufficient, by 3.2, to show that A is a P-set in K. We claim first that $A = K \cap X$. In fact, A is closed in X, hence closed in $K \cap X$ while A is dense in K, hence dense in $K \cap X$. This implies that $X - A \subseteq \beta X - K$. If U is a G_δ set in K that contains A, then $U \cup (\beta X - K)$ is a G_δ set in βX that contains X. Since X is assumed CNP, it follows that there is a βX-open set V such that $X \subseteq V \subseteq U \cup (\beta X - K)$. But then $K \cap V$ is a K-open set such that $A \subseteq K \cap V \subseteq U$.

2. Suppose that $U \subseteq \beta X$ is a G_δ set that contains X. Every point $p \in X$ has a CNP neighbourhood $V(p)$ whose closure we denote $K(p)$. Since $U \cap K(p)$ is a G_δ in $K(p)$ that contains p, it follows that $U \cap K(p)$ is a $K(p)$-neighbourhood of p. Since $K(p)$ is a βX-neighbourhood of p, it follows that U is a βX-neighbourhood of p. But p is arbitrary and so U is a βX-neighbourhood of X.

3. Immediate from the preceding.

4. Suppose A is open in X. Then each point of A is in the interior of a closed neighbourhood and the first two parts give the result.

5. It is known that θ is perfect if and only if

$$\begin{array}{ccc} Y & \xrightarrow{\theta} & X \\ \downarrow & & \downarrow \\ \beta Y & \xrightarrow[\beta\theta]{} & \beta X \end{array}$$

is a pullback in the category of topological spaces and continuous maps, see [Porter & Woods (1988), Exercise 1J(1)]. This implies not only that $(\beta\theta)^{-1}(X) = Y$ (the definition of pullback) but also that $(\beta\theta)_\#(Y) = X$. Now suppose that X is CNP. If $\{U_n\}$ is a countable set of βY-neighbourhoods of Y, then $\{(\beta\theta)_\#(U_n)\}$ is a countable set of βX-neighbourhoods of $(\beta\theta)_\#(Y) = X$. It follows that $\bigcap(\beta\theta)_\#(U_n)$ is a βX-neighbourhood of X and so $\bigcap(\beta\theta)^{-1}(\beta\theta)_\#(U_n) \subseteq \bigcap U_n$ is a βY-neighbourhood of Y. Conversely, suppose Y is CNP. Given a countable family $\{U_n\}$ of βX-neighbourhoods of X, $\{(\beta\theta)^{-1}(U_n)\}$ is a countable family of βY-neighbourhoods of Y and then $\bigcap(\beta\theta)^{-1}(U_n)$ is a βY-neighbourhood of Y and therefore $\bigcap(\beta\theta)_\#(\beta\theta)^{-1}(U_n) = \bigcap U_n$ is a βX-neighbourhood of X. ∎

Using well-known properties of Lindelöf spaces (for perfect maps, see [Porter & Woods (1988), Corollary 4.2(g)(3)]), we conclude that:

3.6 THEOREM.

1. *Any closed subspace of a Lindelöf CNP space is Lindelöf CNP.*
2. *If every point of a Lindelöf space has a CNP neighbourhood then the space is Lindelöf CNP.*
3. *A countable sum of Lindelöf CNP-spaces is Lindelöf CNP.*
4. *Any cozero-subspace of a Lindelöf CNP space is Lindelöf CNP.*
5. *If $\theta : Y \longrightarrow\!\!\!\!\!\rightarrow X$ is a perfect surjection, then X is Lindelöf CNP if and only if Y is.*

■

4 Amply Lindelöf CNP-spaces

We do not know whether the product of two Lindelöf CNP spaces is Lindelöf CNP. However, if we denote by $\mathcal{C}$ one of the classes: Lindelöf locally compact spaces; or σ-compact CNP spaces; or Lindelöf P-spaces, then the following theorem holds.

4.1 THEOREM. *If X belongs to $\mathcal{C}$ and Y is Lindelöf CNP, then $X \times Y$ is Lindelöf CNP. Moreover,*

1. *If X and Y belong to $\mathcal{C}$, so does $X \times Y$.*
2. *If X belongs to $\mathcal{C}$, so does any closed subspace of X.*
3. *If X is Lindelöf and every point of X has a neighbourhood that belongs to $\mathcal{C}$, then X does as well.*
4. *A sum of countably many objects of $\mathcal{C}$ belongs to $\mathcal{C}$.*
5. *If X belongs to $\mathcal{C}$, so does any cozero-subspace of X.*
6. *If $\theta : Y \longrightarrow\!\!\!\!\!\rightarrow X$ is a perfect surjection, then X belongs to $\mathcal{C}$ if Y does.*

These can be proved using arguments similar to those used to prove Theorem 4.7 below. We mention these facts to motivate a definition that includes the three classes (or two since a Lindelöf locally compact space is already σ-compact).

4.2 DEFINITION. *We will call a cover* $\mathcal{U} = \{U_i \mid i \in I\}$ *of a space* X *an* **ample cover** *if every compact subset of* X *is covered by a finite number of* U_i. *We will say that* X *is* **amply Lindelöf** *if every ample cover by* G_δ *sets has a countable subcover.*

Obviously every open cover is ample so that an amply Lindelöf space is Lindelöf. Another obvious comment is that if $\mathcal{U}$ is an ample G_δ cover, then so is the closure of $\mathcal{U}$ under finite unions. One easily sees that a finite union of G_δ sets is a G_δ and a cover by a countable set of these finite unions is also a cover by a countable set from the original cover. Finally, it is also evident that σ-compact spaces and Lindelöf P-spaces are amply Lindelöf, for trivial reasons: every ample cover of a σ-compact space has a countable subcover, while every G_δ cover of a P-space is an open cover. What is surprising is that the amply Lindelöf condition suffices for the permanence properties we are looking for. See 5.5 for an example of a space that is Lindelöf, but not amply so.

The following proposition and its corollary are the keystones of this section.

4.3 PROPOSITION. *Let* X *be an amply Lindelöf space and* Y *be a Lindelöf CNP space. Then every* $(\beta X \times \beta Y)$*-neighbourhood* W *of* $X \times Y$ *contains a set of the form* $X \times V$ *where* V *is a* βY*-neighbourhood of* Y.

PROOF. From [Kelley (1955), Theorem 5.12], there are, for each compact $A \subseteq X$ and each $y \in Y$, open neighbourhoods $U(A, y)$ and $V(A, y)$ of A and y, respectively, such that $U(A, y) \times V(A, y) \subseteq W$. Let $Y_0(A) \subseteq Y$ be a countable subset such that $Y \subseteq V(A) = \bigcup_{y \in Y_0(A)} V(A, y)$. Then $A \subseteq U(A) = \bigcap_{y \in Y_0(A)} U(A, y)$ and each $U(A)$ is a G_δ. Choose a countable subset $\{A_n \mid n \in \mathbf{N}\}$ of compact sets in X such that $X \subseteq U = \bigcup_{n \in \mathbf{N}} U(A_n)$. Since Y is CNP, the set $V = \bigcap_{n \in \mathbf{N}} V(A_n)$ is a neighbourhood of Y. Clearly $X \times V \subseteq U \times V \subseteq W$. ∎

4.4 COROLLARY. *Let* X *be an amply Lindelöf CNP space and* Y *be a Lindelöf CNP space. Then every* $(\beta X \times \beta Y)$*-neighbourhood* W *of* $X \times Y$ *contains a set of the form* $U \times V$ *where* U *and* V *are* βX*- and* βY*-neighbourhoods of* X *and* Y*, respectively.*

PROOF. The previous proposition provides a βY-neighbourhood V of Y such that $X \times V \subseteq W$. Since Y is Lindelöf, every open set containing Y contains a cozero-set containing Y (Theorem 2.8), so that we may assume that V is a cozero-set. A cozero-set in a compact space is σ-compact, so that V is amply Lindelöf, while X is Lindelöf CNP and a second use of the previous proposition provides the required U. ∎

4.5 THEOREM. *The product of an amply Lindelöf space and a Lindelöf space is Lindelöf; the product of two amply Lindelöf spaces is amply Lindelöf.*

PROOF. Suppose X is amply Lindelöf and Y is Lindelöf. Let $\mathcal{W}$ be an open cover of $X \times Y$. According to the remarks following 4.2, we may assume without loss of generality that $\mathcal{W}$ is closed under finite unions. For each compact set $A \subseteq X$ and each $y \in Y$, choose $W(A, y) \in \mathcal{W}$ such that $A \times \{y\} \subseteq W(A, y)$. From [Kelley (1955), Theorem 5.12], there are open neighbourhoods $U(A, y)$ and $V(A, y)$ of

A and y, respectively, such that $A \times \{y\} \subseteq U(A,y) \times V(A,y) \subseteq W(A,y)$. Let $Y_0(A)$ be a countable subset of Y such that $Y = \bigcup_{y \in Y_0(A)} V(A,y)$ and let $U(A) = \bigcap_{y \in Y_0(A)} U(A,y)$. Then $U(A)$ is a G_δ containing A so that the set of all $U(A)$ is an ample G_δ cover of X such that $A \times \{y\} \subseteq U(A) \times V(A,y) \subseteq W(A,y)$. By hypothesis, there is a countable set $\{A_n \,\big|\, n \in \mathbf{N}\}$ such that $X \subseteq \bigcup_{n \in \mathbf{N}} U(A_n)$. Then

$$\begin{aligned} X \times Y &= \bigcup_{n \in \mathbf{N}} \left(U(A_n) \times \bigcup_{y \in Y_0(A_n)} V(A_n, y) \right) \\ &= \bigcup_{n \in \mathbf{N}} \left(\bigcup_{y \in Y_0(A_n)} U(A_n) \times V(A_n, y) \right) \\ &\subseteq \bigcup_{n \in \mathbf{N}} \left(\bigcup_{y \in Y_0(A_n)} W(A_n, y) \right) \end{aligned}$$

The case that Y is also amply Lindelöf is quite similar. The only changes to be made are to replace $y \in Y$ by a compact subset $B \subseteq Y$ and to show that if $A \subseteq X$ and $B \subseteq Y$ are compact such that $A \times B \subseteq W \subseteq X \times Y$ and W is a G_δ, then there are G_δ sets $U \subseteq X$ and $V \subseteq Y$ such that $A \times B \subseteq U \times V \subseteq W$. But that is immediate from the corresponding result for open sets. ∎

4.6 Theorem. *Let X be amply Lindelöf CNP and Y be Lindelöf CNP. Then $X \times Y$ is Lindelöf CNP.*

Proof. By Proposition 3.2, it suffices to show that $X \times Y$ is a P-set in $(\beta X \times \beta Y)$. If W is a G_δ set in $(\beta X \times \beta Y)$ that contains $X \times Y$, let $W = \bigcap W_n$, with each W_n open in $(\beta X \times \beta Y)$. Then 4.4 provides, for each $n \in \mathbf{N}$, open sets $U_n \subseteq \beta X$ and $V_n \subseteq \beta Y$ such that $X \times Y \subseteq U_n \times V_n \subseteq W_n$. But then $U = \bigcap U_n$ and $V = \bigcap V_n$ are βX- and βY-neighbourhoods of X and Y, respectively, such that $U \times V \subseteq W$. ∎

4.7 Theorem.

1. *A product of two amply Lindelöf CNP-spaces is amply Lindelöf CNP.*
2. *A closed subspace of an amply Lindelöf CNP-space is amply Lindelöf CNP.*
3. *A Lindelöf space is amply Lindelöf CNP if every point has an amply Lindelöf CNP neighbourhood.*
4. *A sum of countably many amply Lindelöf CNP-spaces is amply Lindelöf CNP.*
5. *A cozero-subspace of an amply Lindelöf CNP-space is amply Lindelöf CNP.*
6. *If $\theta : Y \longrightarrow\!\!\!\!\!\rightarrow X$ is a perfect surjection, then X is amply Lindelöf CNP if and only if Y is.*

PROOF. In each case, we know either from the preceding theorem or from Theorem 3.6 that the space in question is Lindelöf CNP. So we have to verify the amply Lindelöf condition.

1. This follows from the previous two theorems.

2. Suppose X is a space and $Y \subseteq X$ a closed subspace. Let $\mathcal{V}$ be an ample cover of Y by G_δ sets. It is easy to see that if $V \in \mathcal{V}$, then $V \cup (X - Y)$ is a G_δ set in X and that $\{V \cup (X - Y) \,\big|\, V \in \mathcal{V}\}$ is an ample cover of X. If a subset of these $V \cup (X - Y)$ covers X then the corresponding set of V covers Y.

3. Suppose every point of X has an amply Lindelöf CNP neighbourhood. Since closed neighbourhoods are a basis in any Tychonoff space and since a closed subset of an amply Lindelöf CNP-space is amply Lindelöf CNP, we can assume that each point p has a closed amply Lindelöf CNP neighbourhood $Y(p)$. Choose a countable subset $X_0 \subseteq X$ such that $\{Y(x) \,\big|\, x \in X_0\}$ covers X. If $\mathcal{U}$ is an ample open cover of X by G_δ sets, then for each $x \in X_0$, the set $\{U \cap Y(x) \,\big|\, U \in \mathcal{U}\}$ is an ample cover of $Y(x)$ by G_δ sets and hence has a countable subcover that covers $Y(x)$. Since countably many $Y(x)$ cover X, the conclusion follows.

4. Immediate from the preceding.

5. Let X be amply Lindelöf CNP and suppose $A = \operatorname{coz}(u)$ for $u : X \longrightarrow [0, 1]$ continuous. Then each point of A is in the interior of some closed neighbourhood of the form $u^{-1}[1/n, 1]$ for some n. Then Parts 2 and 3 give the result. ∎

6. In the case that $\theta : Y \longrightarrow X$ is a closed continuous map between topological spaces, both θ^{-1} and $\theta_\#$ take open sets to open sets and preserve meets; hence they take G_δ sets to G_δ sets.

   Let $\theta : Y \longrightarrow X$ be a perfect map. Assume that X is amply Lindelöf CNP. Suppose that $\mathcal{V}$ is an ample cover of Y by G_δ sets. We may assume that $\mathcal{V}$ is closed under finite unions. If $A \subseteq X$ is compact, then $\theta^{-1}(A)$ is compact. If we let $V \in \mathcal{V}$ be such that $\theta^{-1}(A) \subseteq V$, then $A = \theta_\#(\theta^{-1}(A)) \subseteq \theta_\#(V)$ and the latter is a G_δ set. Thus $\{\theta_\#(V) \,\big|\, V \in \mathcal{V}\}$ is an ample open cover of X by G_δ sets and so there is a countable subcover, say $\{\theta_\#(V) \,\big|\, V \in \mathcal{V}_0\}$. But then $\{\theta^{-1}(\theta_\#(V)) \,\big|\, V \in \mathcal{V}_0\}$ covers Y and $V \supseteq \theta^{-1}(\theta_\#(V))$.

   Now suppose that Y is amply Lindelöf CNP. Let $\mathcal{U}$ be an ample cover by G_δ sets in X. Assume $\mathcal{U}$ is closed under finite unions. The set $\mathcal{V} = \{\theta^{-1}(U) \,\big|\, U \in \mathcal{U}\}$ is a cover by G_δ sets. If B is a compact set in Y, $\theta(B)$ is compact in X so there is some $U \in \mathcal{U}$ with $\theta(B) \subseteq U$. But this implies that $B \subseteq \theta^{-1}(U)$ and shows that $\mathcal{V}$ is ample. But then a countable subset of $\mathcal{V}$ covers Y and this is possible only if the corresponding subset of $\mathcal{U}$ covers X. ∎

The following theorem shows that we could define amply Lindelöf by replacing G_δ sets by zero-sets in the definition.

4.8 THEOREM. *A space is amply Lindelöf if and only if every ample cover by zero-sets has a countable subcover.*

PROOF. One way is obvious since every zero-set is a G_δ. Conversely, suppose that every ample cover of X by zero-sets has a countable subcover. Let $\mathcal{U}$ be an ample cover by G_δ sets and suppose that $\mathcal{U}$ is closed under finite unions. Let $A \subseteq X$ be compact and suppose $A \subseteq U \in \mathcal{U}$. Let $U = \bigcap_{n\in\mathbf{N}} U_n$ with each U_n open. Each U_n is a union of cozero-sets, a finite set of which cover A. Since a finite union of cozero-sets is a cozero-set, there is a single $u_n : X \longrightarrow [0,1]$ such that $A \subseteq \mathrm{coz}(u_n) \subseteq U_n$. Since A is compact, u_n has a positive lower bound on A, which means there is an integer m such that $v_n = (1/m - u_n) \vee 0$ vanishes on A. Thus $A \subseteq \mathcal{Z}[v_n] \subseteq \mathrm{coz}(u_n) \subseteq U_n$ and hence $A \subseteq \bigcap \mathcal{Z}[v_n] \subseteq U$. But $\bigcap \mathcal{Z}[v_n] = \mathcal{Z}[\sum 2^{-n} v_n]$ so that there is a zero-set between A and U. Since the zero-sets inside the sets in $\mathcal{U}$ are an ample cover, they have a countable subcover and so does $\mathcal{U}$. ∎

5 Examples and applications: the Lindelöf case

5.1 EXAMPLE. Locally compact spaces are open in their Stone-Čech compactifications and hence are CNP-spaces. This gives another proof of the fact that locally compact Lindelöf spaces are absolute $\mathcal{CR}$-epic, [BRW, 2.15]. A locally compact Lindelöf space is amply Lindelöf. Thus the class of locally compact Lindelöf spaces is included in the class of amply Lindelöf CNP spaces. A second class of amply Lindelöf CNP spaces are the Lindelöf P-spaces, [BRW, Theorem 5.2]. The closure under finite products, countable sums, closed subobjects, cozero-subobjects, and domains and codomains of perfect surjections provides examples of spaces satisfying amply Lindelöf CNP that have only the most remote resemblance to anything in those two classes.

As one example, let X be an uncountable Lindelöf P-space X of cardinality less than a measurable cardinal. We can take an uncountable cardinal smaller than measurable, add one point and declare that proper open sets are those containing the added point whose complement is countable as well all those that do not contain the added point. In [Porter & Woods (1988), Chapter 6] an extremally disconnected space, called EX is constructed along with a perfect surjection $\theta : \mathrm{EX} \longrightarrow X$ such that the only P-points of EX are isolated (Problem 6O). From Theorem 3.6.5, we see that EX is Lindelöf CNP. On the other hand, it is not a P-space since not all points can be isolated (it is uncountable) and it cannot be locally compact, since for locally compact spaces, Lindelöf is equivalent to σ-compact and EX can certainly not be σ-compact since its continuous image X is not.

We claim that EX is also not the product of a Lindelöf P-space and and a Lindelöf locally compact space. We begin by observing that a retract of an

extremally disconnected space is extremally disconnected. This is an easy consequence of the characterization of extremally disconnected spaces by the fact that disjoint open sets are contained in disjoint closed sets (see [Gillman & Jerison (1960), 1H.1]). Suppose now that $\mathrm{EX} = Y \times Z$ where Y is Lindelöf P-space and Z is Lindelöf locally compact. Since there is a retraction $Y \longrightarrow Y \times Z \longrightarrow Y$, Y is extremally disconnected. That makes Y an extremally disconnected P-space of non-measurable cardinality, so it is discrete by a theorem of Isbell's (see [Gillman & Jerison (1960), 12H.6]). Since it is also Lindelöf it is just a copy of $\mathbf{N}$. But that makes $Y \times Z$ σ-compact which is false.

What would really be interesting would be to have an example of an amply Lindelöf CNP space that is not in the closure under all the operations implicit in Theorem 4.7 of the class generated by the σ-compact CNP spaces and the Lindelöf P-spaces.

Here is another source of examples.

5.2 Theorem. *Suppose X is an absolute $\mathcal{CR}$-epic space and Y is a P-set in $\beta X - X$ for which $X \cup Y$ is Lindelöf. Then $X \cup Y$ is absolute $\mathcal{CR}$-epic.*

Proof. Since $X \subseteq X \cup Y \subseteq \beta X$, it follows that $\beta(X \cup Y) = \beta X$. If $\{U(n)\}$ is a countable collection of βX-neighbourhoods of $X \cup Y$ and $U = \bigcap U(n)$, it follows that $U \cap (\beta X - X)$ is a $(\beta X - X)$-neighbourhood of Y. Since $X \subseteq U$, it is immediate that U is a βX-neighbourhood of Y. Now given an admissible equivalence relation E on βX and an $f \in C(\beta X)$, the fact that X is absolute $\mathcal{CR}$-epic implies that the domain of f is a neighbourhood of X and the above considerations imply that it is also a neighbourhood of Y, whence of $X \cup Y$. ■

With a minor change in the proof of 5.2, one can readily show that if X is CNP and Y is a P-set of $\beta X - X$, then $X \cup Y$ is also CNP. An example is $\mathbf{N} \cup \{p\}$, when p is a P-point of $\beta\mathbf{N} - \mathbf{N}$. Since $\mathbf{N} \cup \{p\}$ is σ-compact, it is amply Lindelöf CNP. By [van Mill (1984)], there are uncountable examples as well

5.3 Corollary. *If X is locally compact Lindelöf, and Y is a cozero-set in $\beta X - X$, then $X \cup Y$ is absolute $\mathcal{CR}$-epic.*

Proof. In that case $\beta X - X$ is compact and a cozero-set is Lindelöf. A cozero-set, being open, is a P-set in $\beta X - X$. ■

One can ask if Y must be Lindelöf whenever X and Y satisfy the conditions of 5.2. Note that such a Y need not be countable. Consider the case that $X = \mathbf{N}$ and Y is any non-empty clopen subset of $\beta\mathbf{N} - \mathbf{N}$. Another possibility is to let Y be the one-point Lindelöfization of the discrete space of size the first uncountable ordinal. See [van Mill (1984), Theorem 4.4.4] for the necessary result.

It is undecidable whether there are P-points in $\beta\mathbf{N} - \mathbf{N}$. Either CH or Martin's axiom imply that there are, but there are other models of set theory in which $\beta\mathbf{N} - \mathbf{N}$ lacks P-points. At any rate, we have the following result, one half of which is an immediate consequence of 5.2. The hard half was supplied by Ronnie Levy in a private communication. It is interesting that this gives a purely ring-theoretic characterization of P-points in $\beta\mathbf{N} - \mathbf{N}$.

5.4 THEOREM. *Let $p \in \beta\mathbf{N} - \mathbf{N}$. Then p is a P-point of $\beta\mathbf{N} - \mathbf{N}$ if and only if $\mathbf{N} \cup \{p\}$ is absolute $\mathcal{CR}$-epic.*

PROOF. A point p of a completely regular space is characterized as being a P-point by the fact that any real-valued function that vanishes at p vanishes on a neighbourhood of p. So if p is a non-P-point, there is a function $f \in C(\beta\mathbf{N} - \mathbf{N})$ that vanishes at p, but does not vanish identically on any neighbourhood of p. If we write $U = \operatorname{coz}(f)$, then $p \notin U$, but $p \in \operatorname{cl}_{\beta\mathbf{N}-\mathbf{N}}(U)$. Since U is a cozero-set in a Lindelöf space, it is Lindelöf. Since $\beta\mathbf{N}$ is 0-dimensional, U is a countable union of clopen sets. We may assume without loss of generality that this union is disjoint, say $U = \sum U_i$. Let $V_1 = U_1 + U_2$, $V_2 = U_3 + U_4$, ..., $V_n = U_{2n-1} + U_{2n}$, and so on. Let $X = \mathbf{N} \cup \{p\}$, $Y = X \cup U$, and $E = \Delta_Y \cup \sum(V_n \times V_n)$. Then E is an equivalence relation on Y. Let $\theta : Y \longrightarrow Z = Y/E$ be the quotient mapping. If V is an open set in Y, it is clear that $\theta^{-1}(\theta(V)) = V \cup \bigcup\{V_n \mid V \cap V_n \neq \emptyset\}$, which is also open. Hence θ is an open mapping. Since θ^{-1} of any point is either a point of X or is one of the clopen sets V_n, it follows that Z is T_1. Since also every point of $\mathbf{N}$ and each $\theta(V_n)$ is open, we see that any subset of Z that does not contain p is open. Therefore of two disjoint closed sets A and B, at least one, say A, is open and then A and $Z - A$ are disjoint open sets containing A and B, respectively. Thus Z is normal and Hausdorff. Since $E \cap (X \times X) = \Delta_X$, X is mapped injectively into Z and we will identify X with its image. Clearly $\theta^{-1}(X) = X$, from which it is easy to see that $\theta | X$ is open and hence θ maps X homeomorphically onto its image. For each $n \in \mathbf{N}$, let $f_n \in C(X)$ be the restriction to X of a function on βX that is 0 on U_{2n-1} and 1 on U_{2n}. Such a function cannot extend to the point of Z that is the image of V_n. Thus the conclusion follows from [BBR, Theorem 2.6]. ∎

The characterization of countable absolute $\mathcal{CR}$-epic spaces is a challenging task. It is shown in [Dow, Gubbi, & Szymanski (1988), Remark 1] that in the presence of P-points in $\beta\mathbf{N} - \mathbf{N}$, there is a countable space without isolated points that is a P-set in its Stone-Čech compactification. Such a space is absolute $\mathcal{CR}$-epic and not locally compact at any point.

5.5 A LINDELÖF, BUT NOT AMPLY LINDELÖF, SPACE In light of 4.5, any space whose product with itself is not Lindelöf will give such an example. Here is an example of a space whose product with itself is Lindelöf but is not amply Lindelöf.

The space $\mathbf{R} - \mathbf{Q}$ of irrationals is separable metric, hence second countable and therefore Lindelöf. On the other hand it is not F_σ since, as is well known, $\mathbf{Q}$ is not G_δ in $\mathbf{R}$. But then it cannot be σ-compact. But every compact set is a G_δ, so the cover by all compact sets is ample, but has no countable subcover.

If X is a space, the space X_δ has the same points as X and the weakest topology for which every G_δ of X is open in X_δ. If a Lindelöf space remains Lindelöf in its δ topology then it is clearly amply Lindelöf. Thus scattered Lindelöf spaces are amply Lindelöf in view of [Levy & Rice (1981), 5.2], which says that X_δ is Lindelöf if X is scattered and Lindelöf. Note that in their Open Question 2, Levy and Rice ask if a functionally countable Lindelöf space is a countable union of closed scattered subspaces. A positive answer to this ques-

tion would clearly show that a functionally countable Lindelöf space is amply Lindelöf (see 8.5, Question 3). Of course many spaces, such as $\mathbf{R}$, are amply Lindelöf CNP and not Lindelöf in their δ topology.

5.6 CNP ALONE DOES NOT IMPLY ABSOLUTE $\mathcal{CR}$-EPIC The existence of locally compact spaces which are not absolute $\mathcal{CR}$-epic shows that the CNP alone does not imply absolute $\mathcal{CR}$-epic. An uncountable discrete set of non-measurable cardinality gives a realcompact example (see [BRW, 2.11.1]), and a version of the Isbell-Mrowka space Ψ ([Gillman & Jerison (1960), 5I]) that is not almost compact (see [Mrowka (1977)] and [Terasawa (1980)]) gives a pseudocompact example.

6 Closed subspaces of absolute $\mathcal{CR}$-epic spaces

Until this point, virtually all the results have been for Lindelöf spaces. The following theorem is easy in the Lindelöf case, so its main interest is when the space is not Lindelöf.

6.1 THEOREM. *A closed C^*-embedded subspace of an absolute $\mathcal{CR}$-epic space is absolute $\mathcal{CR}$-epic.*

PROOF. Suppose $X \subseteq Y$ where Y is absolute $\mathcal{CR}$-epic and X is closed and C^*-embedded. Suppose K is a compactification of X. We form the amalgamated sum (or pushout) $Z = K +_X Y$. This space is the disjoint union of K and Y with the two copies of X identified. It is characterized by the universal mapping property that given a pair of continuous functions $g : Y \longrightarrow W$ and $h : K \longrightarrow W$ whose restrictions to X are equal, then the unique function $f : Z \longrightarrow W$ with $f|Y = g$ and $f|K = h$ is continuous. We let $\widetilde{Z}$ be the complete regularization of Z. That is, $\widetilde{Z}$ has the same underlying set as Z but has the smallest topology for which each $f \in C(Z)$ is continuous from $\widetilde{Z}$ to $\mathbf{R}$. It will follow from the lemmas below that $\widetilde{Z}$ is obtained using the same construction of a completely regular (Hausdorff) space as given in [Gillman & Jerison (1960), Theorem 3.9]. (By Lemma 6.3 below, there will be no need to first identify points of Z.)

6.2 LEMMA. *The map $Y \longrightarrow \widetilde{Z}$ is an embedding.*

PROOF. It suffices to show that if $A \subseteq Y$ is closed and $p \in Y - A$ then there is a function in $C^*(Z) = C^*(\widetilde{Z})$ which, when composed with $Y \longrightarrow \widetilde{Z}$, is 0 on A and 1 at p. For this readily implies both that the map $Y \longrightarrow \widetilde{Z}$ is one-to-one and that the closed sets of Y coincide with the closed sets in the relative topology when we regard Y as a subset of $\widetilde{Z}$. If $p \notin X$, there is a function in $C^*(Y)$ that is 0 on $A \cup X$ and 1 at p. Since this function and the function that is identically 0 on K agree on X, the pushout property implies that there is a continuous function on Z whose restriction to Y is 0 on A and whose value at p is 1. Now suppose that $p \in X$. Since $X \subseteq Y$ is closed and C^*-embedded, the induced map $\beta X \longrightarrow \beta Y$ is also an embedding and we will regard βX as

a subspace of βY. Let $W = \beta X \cap \mathrm{cl}_{\beta Y} A$ and let V be the image of W under the map $\theta : \beta X \longrightarrow K$ induced by the inclusion $X \subseteq K$. Notice that V is compact, hence closed in K. Also $p \notin W$ since there is a function on Y that is 0 on A and 1 at p and this function extends to βY. We claim that $p \notin V$. In fact, writing $K = \beta X/E$, then $p \in \theta(W)$ if and only if there exists $w \in W$ with $\theta(w) = p$ if and only if $(w, p) \in E$, which would contradict the fact that E is admissible.

Now choose an $h \in C^*(K)$ that is 0 on V and 1 at p. Then there is a continuous function $u \in C^*(\beta X \cup \mathrm{cl}_{\beta Y}(A))$ such that $u|\beta X = h \circ \theta$ and $u|\mathrm{cl}_{\beta Y}(A) = 0$. In fact, the individual restrictions to closed sets are continuous and both vanish on W, see [Gillman & Jerison (1960), 1A.1]. Since βY is normal, βX is C^*-embedded, and hence u extends to a function $g \in C^*(\beta Y)$. Then there exists $f \in C^*(Z)$ such that $f|K = h$ and $f|Y = g|Y$ because $h|X = g|X = u|X$. It is then readily verified that f is 0 on A and 1 at p which, as indicated above, completes the proof of the lemma. ∎

6.3 Lemma. *$\widetilde{Z}$ is Hausdorff.*

Proof. Let p and q be distinct points of $\widetilde{Z}$. We will find a function in $C(\widetilde{Z}) = C(Z)$ which is 0 at p and 1 at q. If p and q are both in Y, then the proof of the above lemma applies. If p and q are both in K, we can find a function $f \in C(K)$ with $f(p) = 0$ and $f(q) = 1$ and extend $f|X$ to all of Y. Finally suppose that $p \in K$ and $q \in Y$. We may as well assume that neither p nor q is in X; otherwise one of the above cases applies. Find $h \in C(K)$ with $h(p) = 0$ and $h = 1$ on X. Let $g = 1$ on Y and let $f : Z \longrightarrow \mathbf{R}$ be the map obtained by the universal mapping property of Z. ∎

We return to the proof of the theorem. Since Y is absolute $\mathcal{CR}$-epic and $Y \longrightarrow \widetilde{Z}$ is an embedding in the category $\mathcal{CR}$, it follows that $C(\widetilde{Z}) \longrightarrow C(Y)$ is epic. It follows from [BBR, 2.1] that $C(Y) \longrightarrow C(X)$ is epic in the category $\mathcal{CR}$ and the result can be now read from the commutative diagram

$$\begin{array}{ccc} C(X) & \longleftarrow & C(Y) \\ \uparrow & & \uparrow \\ C(K) & \longleftarrow & C(\widetilde{Z}) \end{array}$$

since the composite and left factor of epimorphisms are epimorphisms. ∎

6.4 Corollary. *If $X \times Y$ is absolute $\mathcal{CR}$-epic, so is X.* ∎

6.5 Corollary. *A closed subspace of a normal absolute $\mathcal{CR}$-epic space is absolute $\mathcal{CR}$-epic; therefore a closed discrete subspace of a normal absolute $\mathcal{CR}$-epic space is at most countable ([BRW, 2.14]).* ∎

In the cases where the space Ψ of [Gillman & Jerison (1960), 5I] is almost compact, it is absolute $\mathcal{CR}$-epic, but has an uncountable closed discrete subset, and is not normal ([Gillman & Jerison (1960), 5I.5]). Thus the preceding corollary requires normality.

An anonymous reader of an earlier version of this paper suggested the proof of the following corollary. A family of non-empty subsets of a space is called **discrete** if every point of the space has a neighbourhood that intersects at most one member of the family.

6.6 Corollary. *If X is absolute $\mathcal{CR}$-epic, every discrete family of open subsets is at most countable; equivalently every discrete family of cozero-sets is at most countable.*

Proof. If $\mathcal{U}$ were an uncountable discrete family of open subsets, then any set D that contains exactly one element from each $U \in \mathcal{U}$ is C-embedded by [Gillman & Jerison (1960), 3L]. This is a closed uncountable discrete space, which cannot be absolute $\mathcal{CR}$-epic ([BRW, 2.14]). ∎

The converse to the corollary is false. Since a non-Lindelöf absolute $\mathcal{CR}$-epic space must be almost Lindelöf and therefore not realcompact (see 2.1 along with 7.2 below), any realcompact separable non-Lindelöf space gives a counterexample. So does a functionally countable P-space that is not Lindelöf. See [Levy & Rice (1981), Proposition 3.2 and Example 6] for the facts that discrete families of cozero-sets are countable and that such spaces exist, and [BRW 5.2] for the fact that they are not absolute $\mathcal{CR}$-epic.

Recall that a subspace $X \subseteq Y$ is said to be **z-embedded** if every zero-set of X is the intersection with X of a zero-set of Y.

6.7 Corollary. *A z-embedded zero-set of an absolute $\mathcal{CR}$-epic space is absolute $\mathcal{CR}$-epic. In particular, a Lindelöf zero-set in an absolute $\mathcal{CR}$-epic space is absolute $\mathcal{CR}$-epic.*

Proof. By [Blair & Hager (1974), 4.4] a z-embedded zero-set is C^*-embedded. Lindelöf spaces are always z-embedded by a result of Jerison that first appeared in [Henriksen & Johnson (1961)]. ∎

We note that an almost Lindelöf space contains many Lindelöf zero-sets. In fact, one definition of almost Lindelöf is that of any two disjoint zero-sets, at least one is Lindelöf.

7 Punctured spaces: non-Lindelöf examples of absolute $\mathcal{CR}$-epic spaces

The next two theorems are known (see [Hager & Martinez (2002), Theorem 9,10] and also see [BRW, 2.10] for an explanation of why that result applies) but the proofs there are rather technical and here we give more straightforward proofs.

7.1 Theorem. *If $|\upsilon X - X| \geq 2$, then X is not absolute $\mathcal{CR}$-epic.*

Proof. Suppose $p \neq q \in \upsilon X - X$. Then $E = \Delta_{\beta X} \cup \{(p,q),(q,p)\}$ is an admissible equivalence relation on βX. Let f be the restriction to X of any function on βX for which $f(p) = 0$ and $f(q) = 1$. Suppose that $f = GAH$ with A, GA, and AH all defined mod E. Since X is C-embedded in υX, G and H are

both defined at p and q. But we must have $A(p) = A(q)$, $G(p)A(p) = G(q)A(q)$, and $A(p)H(p) = A(q)H(q)$. Let B be a quasi-inverse for $A(p) = A(q)$. Then

$$f(p) = G(p)A(p)H(p) = G(p)A(p)BA(p)H(p)$$
$$= G(q)A(q)BA(q)H(q) = G(q)A(q)H(q) = f(q)$$

a contradiction. ∎

7.2 THEOREM. *An absolute $\mathcal{CR}$-epic, realcompact space is Lindelöf. Hence an absolute $\mathcal{CR}$-epic space is almost Lindelöf.*

PROOF. Assume that X is absolute $\mathcal{CR}$-epic, realcompact and not Lindelöf. Then there exists a collection of closed subsets with the countable intersection property but no point in common. The filter base of countable intersections of these sets generates a σ-filter $\mathcal{F}$ of non-empty closed sets with empty intersection. Then $S = \bigcap_{Z\in\mathcal{F}} \mathrm{cl}_{\beta X}(Z) \neq \emptyset$ by the compactness of βX. Since X is realcompact, there exists, for any $p \in S$, an $f \in C(X)$ that does not extend to p. Since every function extends to $[-\infty, +\infty]$ (which is isomorphic to the unit interval), it follows that such an f is unbounded in every neighbourhood of p. Therefore, f must be unbounded on every $Z \in \mathcal{F}$.

Let K be the compactification of X obtained from βX by identifying S to a single point. It is clear that $C(K)$ can be identified with

$$\{h \in C(\beta X) \,\big|\, h \text{ is constant on } S\}$$

We now claim that if $h(S) = c$, then for each $n \in \mathbf{N}$ there is some $Z_n \in \mathcal{F}$ such that $|h(x) - c| < 1/n$ for all $x \in Z_n$. If not then, for some $n \in \mathbf{N}$, the set

$$\{\{z \in Z \,\big|\, |h(z) - c| \geq 1/n\} \,\big|\, Z \in \mathcal{F}\}$$

is a σ-filter of closed non-empty sets. But then the intersection of their closures in βX is a non-empty subset of S and $|h(q) - c| \geq 1/n$ on any element of the intersection, contradicting the fact that $h = c$ on S. If we let $Z = \bigcap Z_n$, we have that $Z \in \mathcal{F}$ and that h is constant on Z. And given any finite (or even countable) set of functions in $C(K)$, there is still some $Z \in \mathcal{F}$ such that each one of that finite set of functions is constant on Z. Choose an $f \in C(X)$ that is unbounded on each element of $\mathcal{F}$. Since X is absolute $\mathcal{CR}$-epic, we can write $f = GAH$ with the entries of GA, A, and AH in $C(K)$. Choose $Z \in \mathcal{F}$ so that the entries of GA, A, and AH are constant on Z. Let B be a quasi-inverse of $A|Z$. Then on Z, we have $f = GAH = (GA)B(AH)$ and all terms are constant there, which contradicts the fact that f is unbounded on Z. ∎

7.3 NOTATION. The only way an absolute $\mathcal{CR}$-epic space D can fail to be Lindelöf is if there is a Lindelöf space X and a point $p \in X$ such that $D = X - \{p\}$ and $X = \upsilon D$. If D is absolute $\mathcal{CR}$-epic, we know that X must be, [BRW, 2.20], but the converse fails in general. Our results will take the form of conditions on the Lindelöf space X and the point $p \in X$ such that $X = \upsilon D$ and that $D = X - \{p\}$ is absolute $\mathcal{CR}$-epic. The fact that $X = \upsilon D$ implies that

$\beta X = \beta D$. We will call D a **punctured space**. Examples of punctured spaces include the Dieudonné plank, see 7.15, and some closely related spaces.

As usual, we will let K be a compactification of D and $\theta : \beta X = \beta D \longrightarrow\!\!\!\!\!\rightarrow K = \beta D/E$ be the quotient mapping. Since no element of D has the same image under θ as any other element of βD, it follows that $\theta|X$ is injective. However, it is not generally an embedding. Everything we do is trivial in that case.

7.4 Definition. *If R is any commutative ring, and T is any subset of R, we will, following [Isbell (1966), Introduction], say that an element $r \in R$ is in the* **dominion** *of T if for any commutative ring S and any pair of homomorphisms $\phi, \psi : R \longrightarrow S$, $\phi|T = \psi|T$ implies that $\phi(r) = \psi(r)$. The set of elements in the dominion of T will be denoted* $\mathrm{dmn}(T)$. *It is clearly a subring of R.*

There is a useful refinement of the standard theorem that characterizes epics in the category $\mathcal{CR}$.

7.5 Proposition. *Let R be a commutative ring and T a subset. Suppose $f \in R$ can be written $f = GAH$, where G is a $1 \times n$ matrix of elements of R, A is an $n \times n$ matrix of elements of R, and H is an $n \times 1$ matrix of elements of R. If all the elements of GA, A, and AH are in* $\mathrm{dmn}(T)$, *then $f \in \mathrm{dmn}(T)$.*

Proof. Suppose ϕ and ψ are a pair of commutative ring homomorphisms out of R that agree on T, then they agree on GA, A, and AH from which an obvious argument implies that they agree on f. ∎

7.6 Definition. *If R is a commutative ring and $T \subseteq R$ a subring we will say that a subset $F \subseteq R$ is* **T-adequate** *(or simply* **adequate** *if T is understood) if* $\mathrm{dmn}(T \cup F) = R$.

To show that a subring $T \subseteq R$ is embedded epimorphically, it is sufficient to find an adequate subset of R that is a subset of the dominion of T. We are interested in the case where $R = C(X)$ and $T = C(K)$ for K some compactification of X. Much of this section is therefore devoted to shrinking a $C(K)$-adequate subset of $C(X)$ and enlarging the dominion of $C(K)$ until the dominion contains the adequate set. In practice, the $C(K)$ is understood and will not be made explicit.

In the situtation under discussion, 7.3, all four of the rings $C^*(X)$, $C^*(D)$, $C(\beta X)$, and $C(\beta D)$ are isomorphic, with the isomorphisms induced by the inclusions, and we will identify them. We will also identify $C(K)$ as a subring of all of them; namely as the subring consisting of those functions that are defined modulo the equivalence relation E. The statement of the following proposition makes sense since a function in $C(X)$ extends continuously to a unique largest subset of K, [BBR, 2.2]. Among other things, this proposition justifies the restriction to bounded functions.

7.7 Proposition. *Suppose $\mathcal{V}$ is a family of subsets of βX and $F \subseteq C(X)$ consists of those functions that extend to and vanish on some $V \in \mathcal{V}$. If F is adequate, then the set of non-negative bounded functions in F is also adequate.*

PROOF. Suppose that ϕ, ψ is a pair of commutative ring homomorphisms out of $C(X)$ that agree on all the non-negative bounded functions in F. Since every function is the difference of two non-negative ones that vanish exactly where they do, the reduction to the non-negative ones is clear. That is, when ϕ and ψ agree on all non-negative functions in F, then they agree on all functions in F. So, assuming f is a non-negative member of F, we need to prove that $\phi(f) = \psi(f)$. Then $f/(1+f)$ is non-negative, bounded, and vanishes whenever f does and hence belongs to F. We have $\phi(f/(1+f)) = \psi(f/(1+f))$. Expanding and cross-multiplying, we get $\phi(f) + \phi(f)\psi(f) = \psi(f) + \psi(f)\phi(f)$ so that $\phi(f) = \psi(f)$. ∎

7.8 PROPOSITION. *Let X be Lindelöf and let W be an open, locally compact subset of X. Let $u : \beta X \longrightarrow [0,1]$ be such that $X \subseteq W \cup \operatorname{coz}(u)$. Then there exists $w : \beta X \longrightarrow [0,1]$ such that $\operatorname{coz}(w) \subseteq W \subseteq X \subseteq \operatorname{coz}(w) \cup \operatorname{coz}(u)$.*

PROOF. Let $Y = W - \operatorname{coz}(u)$. Then Y is closed in X and therefore Lindelöf. For each $y \in Y$, let K_y be a compact X-neighbourhood of y with $K_y \subseteq W$. Since K_y is closed in βX, it follows that K_y is also a βX-neighbourhood of y. So there exists $w_y : \beta X \longrightarrow [0,1]$ such that $w_y(y) = 1$ and $w_y = 0$ on $\beta X - K_y$. Since Y is Lindelöf, there exist $y(1), \ldots, y(n), \ldots$ such that $Y \subseteq \bigcup \operatorname{coz}(w_{y(n)})$. Let $w = \sum 2^{-n} w_{y(n)}$. ∎

7.9 PROPOSITION. *Let A be a compact subspace of X. Then the family of all non-negative functions in $C^*(X)$ that vanish on A is adequate.*

PROOF. Given any function $f \in C(X)$, its restriction to A can be extended to $g \in C(K)$. The difference $f - g$ vanishes on A. Thus the functions that vanish on A form an adequate set and 7.7 implies that the non-negative bounded ones do too. ∎

7.10 PROPOSITION. *In the notation of 7.3, a CNP space X has the property that any function in $C^*(X)$ that vanishes on a $(\beta X - D)$-neighbourhood of $S = \theta^{-1}(\theta(p))$ is in the dominion of $C(K)$.*

PROOF. Let f be such a function and suppose V is a $(\beta X - D)$-neighbourhood of S on which f vanishes. It follows that $V \cup D$ is a βX-neighbourhood of S. From 2.7 applied to D, the set

$$U_n = \{p \in \beta X \,\big|\, (p,q) \in E \Rightarrow |f(p) - f(q)| < 1/n\}$$

is an open βX-neighbourhood of D. Then $V \cup U_n$ is a βX-neighbourhood of $S \cup D = S \cup X$. Let $U = \bigcap_{n \in \mathbf{N}} U_n$. Then $V \cup U = \bigcap (V \cup U_n)$ is a βX-neighbourhood of X and also of S, so that $V \cup U$ is a βX-neighbourhood of $S \cup X$. It follows that $\theta_\#(U)$ is a K-neighbourhood of $\theta_\#(S \cup X) = \theta(X)$. Since $\theta(X)$ is Lindelöf, any function that extends to a neighbourhood of $\theta(X)$ has a zig-zag in $C(\theta(X))$ over $C(K)$. But $\theta | (X \cup S)$ is a closed mapping, so that $C(\theta(X))$ can be identified as the subring of $C(X)$ consisting of the functions that are constant on S, so f has a zig-zag in $C(X)$ over $C(K)$ and hence belongs to $\operatorname{dmn}(C(K))$.

7.11 THEOREM. *In the notation of 7.3, suppose that the CNP space $X = W \cup A$ where W is locally compact and A is compact. Then for any $p \in X$ for which $\upsilon(X - \{p\}) = X$, the space $D = X - \{p\}$ is absolute $\mathcal{CR}$-epic.*

We note that a space that is the union of a compact set and a locally compact set is not necessarily locally compact.

PROOF. By replacing A by $A \cup \{p\}$ and then replacing W by $W - A = X - A$, we can suppose that W is open and disjoint from A and that $p \in A$. Since, by 7.7 and 7.9, the functions that are non-negative, bounded, and vanish on A are adequate, it suffices to show that all such functions are in $\mathrm{dmn}(C(K))$. So let $f \in C(X)$ be such a function. The fact that a compact neighbourhood of a point in X is also a neighbourhood in βX implies that W is open in βX. We claim that $W \cup \mathcal{Z}(f)$ is a βX-neighbourhood of X. For each $n \in \mathbf{N}$, the set $f^{-1}[0, 1/n)$ is an open neighbourhood of A and hence $W \cup f^{-1}[0, 1/n)$ is an open neighbourhood of X. The CNP implies that $\bigcap_{n\in\mathbf{N}}(W \cup f^{-1}[0, 1/n)) = W \cup \mathcal{Z}(f)$ is a neighbourhood of X.

Let V be a βX-open set such that $X \subseteq V \subseteq W \cup \mathcal{Z}(f)$. Since $W \subseteq D$, $V - D \subseteq V - W$. Let $B = S - V$. Since A and B are disjoint compact sets, there is a $u : \beta X \longrightarrow [0, 1]$ that is 1 on A and vanishes on a *neighbourhood* of B. Since f vanishes on $V - D$, so does fu. But u vanishes on a βX-neighbourhood of B so fu vanishes on $(V - D) \cup U$ which is clearly a $(\beta X - D)$-neighbourhood of S. It follows from the preceding proposition that $fu \in \mathrm{dmn}(C(K))$.

From 7.8, there is a function $w : \beta X \longrightarrow [0, 1]$ such that $\mathrm{coz}(w) \subseteq W$ and $X \subseteq \mathrm{coz}(u) \cup \mathrm{coz}(w) = \mathrm{coz}(u + w)$. Since w vanishes outside W so does fw, which is then in $\mathrm{dmn}(C(K))$, and then so is $f(u + w)$. The same arguments show that fu^2, fuv, and fv^2 are all in the dominion of $C(K)$. Then we have the zig-zag

$$f = \frac{1}{u+w} \cdot f(u+w)^2 \cdot \frac{1}{u+w}$$

which, by 7.5, implies that $f \in \mathrm{dmn}(C(K))$. ∎

7.12 COROLLARY. *If D is an almost Lindelöf space for which υD is locally compact, then D is absolute $\mathcal{CR}$-epic.*

PROOF. Assuming that D is not already Lindelöf, let $\upsilon D = D \cup \{p\}$. Then $D = \upsilon D - \{p\}$ is still locally compact and $\{p\}$ is compact. ∎

7.13 REMARK. It is possible for D to be almost Lindelöf and not absolute $\mathcal{CR}$-epic, and have $\upsilon(D)$ absolute $\mathcal{CR}$-epic with CNP. By [BRW, Theorem 5.2] any almost Lindelöf, non-Lindelöf P-space serves as an example. For such a space see [Levy & Rice (1981), Example 4].

7.14 THEOREM. *Suppose that the Lindelöf space Y contains a locally compact subspace whose complement is a non-isolated P-point y_0 and that Z is a separable compact space. For any non-isolated point $z_0 \in Z$, the space $(Y \times Z) - \{(y_0, z_0)\}$ is absolute $\mathcal{CR}$-epic.*

PROOF. Since y_0 is a P-point, any G_δ set of βY that contains Y contains a neighbourhood of p. Since all other points have compact neighbourhoods, any

such set contains a neighbourhood of each other point and so Y satisfies the CNP. Let $X = Y \times Z$, $p_0 = (y_0, z_0)$, and $D = X - \{p_0\}$. The first thing to show is that $vD = X$. Clearly p_0 is not isolated, so D is dense in X. Thus it is sufficient to show that D is C-embedded in X. Suppose that $\{z_1, z_2, \ldots\}$ is a countable dense subset of Z. We may suppose without loss of generality that z_0 is not in this dense set. Let $f \in C(D)$. For each $n, m \in \mathbf{N}$ there is a neighbourhood $U_{n,m}$ of y_0 such that $y \in U_{n,m}$ implies that $|f(y, z_n) - f(y_0, z_n)| < 1/m$. Then for $U_n = \bigcap_{m \in \mathbf{N}} U_{n,m}$ it is clear that $y \in U_n$ implies that $f(y, z_n) = f(y_0, z_n)$. Since y_0 is a P-point, U_n is a Y-neighbourhood of y_0 as is $U = \bigcap U_n$. For all $y \in U$ and all $n \in \mathbf{N}$, we have $f(y, z_n) = f(y_0, z_n)$. The function $f(y, \text{-}) - f(y_0, \text{-}) : Z - \{z_0\} \longrightarrow \mathbf{R}$, vanishes on the dense set $\{z_n\}$ and hence on all of $Z - \{z_0\}$. So $f(y, z) = f(y_0, z)$ for all $y \in U$ and all $z \in Z - \{z_0\}$. Similarly, for $y, y' \in U - \{y_0\}$, we see that $f(y, z_0) = f(y', z_0)$. We extend f to $Y \times Z$ by letting $f(y_0, z_0) = f(y, z_0)$ where $y \in U - \{y_0\}$ (which is non-empty as y_0 is non-isolated). Note that the extended f when restricted to $U \times Z$ factors through the projection $U \times Z \longrightarrow Z$, so is continuous.

Now let $W = (Y - \{y_0\}) \times Z$. It is the product of a locally compact space and a compact space, hence is locally compact. The complement, $\{y_0\} \times Z$, is obviously compact. Hence the result follows from the preceding theorem. ∎

7.15 Example. The Dieudonné plank is described as follows. Let ω_1 denote the first uncountable ordinal and let $\omega_1^+ = \omega_1 \cup \{\omega_1\}$ with the elements of ω_1 open, while a neighbourhood of ω_1 is a set containing ω_1 whose complement is countable. Let ω be the first countable ordinal and ω^* be its one-point compactification. Then $X = \omega_1^+ \times \omega^*$, $p = (\omega_1, \omega)$ and $D = X - \{p\}$. The punctured space D is the Dieudonné plank. The preceding theorem implies it is absolute $\mathcal{CR}$-epic. The topology on ω_1 is just the P-space topology generated by the order topology. However, it is clear that neither the order nor the size of ω_1 actually matters and it could be replaced by any uncountable discrete space.

8 Miscellaneous results

8.1 Derived sets For a space Y, define, for each ordinal α, a space $Y^{(\alpha)}$ by letting $Y^{(0)} = Y$, $Y^{(\alpha+1)}$ is the derived set of $Y^{(\alpha)}$ and for a limit ordinal α, $Y^{(\alpha)} = \bigcap_{\beta<\alpha} Y^{(\beta)}$.

8.2 Theorem. *Suppose every point of the space X is a G_δ. Let K be a compactification of X and assume that there exist an ordinal α and an open subset $U \subseteq K - X$ such that $(K - X)^{(\alpha)} \subseteq U$ while, for each $\beta < \alpha$, $(K - X)^{(\beta)} - U$ is infinite. Suppose further that α is either a successor ordinal or a limit ordinal of countable cofinality. Then X is not absolute $\mathcal{CR}$-epic.*

Proof. Let $Y = K - X$. First consider the case $\alpha = \beta + 1$. We claim that all limit points of $Y^{(\beta)} - U$ lie in X. Suppose y is such a limit point which lies in $K - X = Y$. Then, by a straightforward induction, $y \in Y^{(\gamma)}$ for all $\gamma \leq \alpha$. But $y \in Y^{(\alpha)}$ implies $y \in U$ and U is then a neighbourhood of y which misses $Y^{(\beta)} - U$ contradicting the assumption that y was a limit point of $Y^{(\beta)} - U$. It

follows that any countably infinite subset of $Y^{(\beta)} - U$ has all limit points in X, so [BRW, Corollary 2.29] applies.

If α is a limit ordinal of countable cofinality, let $\beta_1 < \beta_2 < \cdots < \alpha$ be a countable sequence of ordinals such that $\sup \beta_i = \alpha$. Let $z_1 \in Y^{(\beta_1)} - U$ and, having chosen a set of distinct points $z_i \in Y^{(\beta_i)} - U$ for all $i < n$, the fact that $Y^{(\beta_n)} - U$ is infinite allows us to choose $z_n \in Y^{(\beta_n)} - U$ different from all of $z_1, \ldots, z_{n-1}$. The set $Z = \{z_1, z_2, \ldots\}$ is countable, closed, and discrete in $Y - U$. Since only finitely many terms of Z lie in $Y^{(\beta_n)}$ no limit point of Z can be in $Y^{(\beta_n)}$ and hence any limit point must lie in $Y^{(\alpha)}$. But no limit point of Z in Y can lie in U and hence all limit points in K are in X and again [BRW, Corollary 2.29] applies. ∎

8.3 Corollary. *Suppose every point of the space X is a G_δ. Let K be a compactification of X. Suppose there is a countable ordinal α such that $(K - X)^{(\alpha)}$ is compact. Then either X is locally compact or X is not absolute $\mathcal{CR}$-epic.*

Proof. If X is not locally compact, it is not open in K and so $Y = K - X$ is not compact. Let $\mathcal{U}$ be an open cover of Y that has no finite subcover. Assume that $\mathcal{U}$ is closed under finite unions and let α be the smallest ordinal such that $Y^{(\alpha)}$ is in some member of $\mathcal{U}$. Since α is countable, it is either a successor or of countable cofinality and then the prededing theorem applies. ∎

8.4 Some answers In an earlier version of [BRW] there were a number of questions that we can now answer. The questions themselves have disappeared (because they were answered) from the final version of the paper, so we repeat them here along with the answers.

1. We asked if a closed C-embedded subspace of an absolute $\mathcal{CR}$-epic space is necessarily absolute $\mathcal{CR}$-epic. Theorem 6.1 provides the answer and requires only C^*-embedding.

2. We asked if X must be absolute $\mathcal{CR}$-epic if $\beta X - X$ is a disjoint union of countably many zero-sets of βX. The answer is no. The space X of irrational numbers in the unit interval is not absolute $\mathcal{CR}$-epic (see [BRW, 2.27] and observe that the set X of irrational numbers is first countable and not locally compact). However, there is a surjection $\beta X \longrightarrow [0, 1]$ and the image of $\beta X - X$ is just the rational numbers. Since there are countably many of them and each rational is a zero-set, it follows that $\beta X - X$ is the union of countably many zero-sets.

3. We asked about the Dieudonné plank $(\omega_1^+ \times \omega^*) - \{(\omega_1, \omega)\}$. The example 7.15 provides the positive answer.

8.5 some questions

1. The most outstanding question is whether the product of two Lindelöf CNP-spaces is Lindelöf CNP. It might fail to be Lindelöf or fail to be CNP or both. Closely related is the question of whether every Lindelöf CNP-space is amply Lindelöf.

2. Must a realcompact P-space all of whose real-valued functions have countable range (such a space is called **functionally countable**) be Lindelöf? One way to prove this would be to show that such a space must be absolute $\mathcal{CR}$-epic.

3. Must a functionally countable Lindelöf space be amply Lindelöf?

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