Chapter 8
Relations Old and New

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Abstract. We note that (binary) relations on a set form a (partially) ordered monoid with involution, which is also residuated and complete, hence a *quantale*. Relations between sets form an ordered category with involution. If $\rho : A \not\rightarrow B$ and $\sigma : B \not\rightarrow C$, one defines $\sigma\rho : A \not\rightarrow C$ by

$$c(\sigma\rho)a \Leftrightarrow \exists_{b\in B}(c\sigma b \vee b\rho a),$$

for all $a \in A$ and $c \in C$, and $\rho^{\vee} : B \not\rightarrow A$ by

$$a\rho^{\vee}b \Leftrightarrow b\rho a.$$

The partial order between relations $A \not\rightarrow B$ is defined elementwise.

We shall discuss some appearances of relations in anthropology, linguistics, computer science, algebra and category theory.
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1 Kinship relations

If there is such a thing as prehistoric mathematics, it is surely the algebra of kinship relations. *Consanguineous* kinship relations are generated as monoid with involution by a single relation P of parenthood. We adopt the convention that xPy is read "x's parent is y" rather than "x is a parent of y", and write $P^{\vee} = C$, where $C = child$. One avoids PC, which is the disjoint union of the identity relation I and $S = sibling$. S is symmetric, but neither transitive nor reflexive, not having been introduced by a modern mathematician. One also avoids CP, the disjoint union of I and $\Sigma = spouse$, which is crucial for discussing *affine relations*, to be excluded from the present discussion. One also avoids $SP \subseteq P$, $CS \subseteq C$ and SS, the disjoint union of I and S. This leaves P^{m+1}, P^mSC^n and C^{n+1} where m, n are natural numbers. These *kinship descriptions* may be generated by the rewrite rules:

$$R \rightarrow P, S, C, PR, RC.$$

We may decompose I into the disjoint union of two subrelations M and F, denoting equality between males and females respectively. In English, gender is expressed only at the end of a kinship description:

$$R\# \rightarrow RM\#, RF\#.$$

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We have written # for a blank space. Basic English consanguineous kinship terms are then introduced as follows (between blank spaces):

$$
\begin{array}{ll}
PM \to father & PF \to mother \\
CM \to son & CF \to dauther \\
SM \to brother & SF \to sister \\
PSM \to uncle & PSF \to aunt \\
SCM \to nephew & SCF \to niece \\
PSCM \to cousin & PSCF \to cousin
\end{array}
$$

Gender is irrelevant only for cousins; note however that French distinguishes between *cousin* and *cousine* or German between *Vetter* and *Base*. Iterated Ps and Cs are translated with the help of *grand* and *great #*, as in $PPM \to granfather$, $CCCF \to great\#granddauther$. Note, however, that French distinguishes between *grand(e)* and *petit(e)*. For a fuller discussion of English kinship terminology, see [15], where one may also find an explanation why some dialects of English have *grandnephew* but *great # uncle*.

In many languages gender is expressed not only at the end of a kinship description. In Hindi for example [3] one distinguishes between

$$PMSM \to \textit{tāyā} \text{ or } \textit{chāchā}$$

(depending on whether the uncle is older or younger that the father) and

$$PFSM \to \textit{māmā},$$

also between

$$PMSF \to \textit{buā} \text{ or } \textit{phūphī}$$

and

$$PFSF \to \textit{mausī}.$$

In some languages even the ego's sex is relevant, that is, gender may have to be expressed at the beginning of a kinship description. Thus, in the language of the Trobriand islanders [4], one distinguishes between

$$MSF \to luta$$

and

$$FSF \to \textit{tuwa} \text{ or } \textit{bwada}$$

(depending on whether the sibling is older or younger than ego).

In many languages, the kinship descriptions are not as freely generated as at first sight in English, but are subject to certain *reduction* rules. For example, Hindi has the rule $PSC \to S$, according to which cousins are regarded as siblings. Even English has the optional rules $PSC^2 \to PSC$ and $P^2SC \to PSC$, according to which ith cousins j times removed may be called cousins. At some stage of its development, Lain had reduction rules

$C^2 \to SC$ and $P^2 \to PS$, according to which *nepos*, originally meaning "grandson", came to mean "nephew" and gave rise to this English word, and *avunculus*, originally meaning "granddaddy", came to mean "uncle" and gave rise to this English word. Anglosaxon had separate words for $PMSM$ and $PFSM$.

Let us pause for a moment to ask: what is an i-th cousin j times removed? One looks at the kinship description $P^{m+1}SC^{n+1}G$, where $G = M$ of F, and calculates

$$i = Min(m, n) + 1, j = |m - n|.$$

It seems that aunt Agatha, the only member of the family who understands this concept, has studied primitive recursive functions!

Reduction rules play an even more prominent role in many languages of interest to anthropologists. For example, the language of the Trobriand islanders had rules, extracted by Lounsbury [22] from data gathered by Malinowski [23]:

$$PMSF \to PMPF, \quad FSMC \to FCMC,$$

$$PGSG \to PG, \quad GSGC \to GC \; (G = M, F),$$

with a resulting collapse of the kinship terminology, and not surprisingly, $PFC \to S$. But, curiously, PMC was not conceived as a kinship description at all, leading some anthropologists to the view that the role of the father in reproduction was not recognized. More realistically, in view of Trobriand women's promiscuous behaviour, it may have been the identity of the father that was not recognized.

For example, Malinowski [23] had observed that what for us is a first cousin might be called by any of the following kinship terms:

tuwa, *bwada*, *luta*, *latu*, *tabu*, *tama*,

whose primary meanings were:

older sibling of same sex, *younger* $\cdots$, *sibling of opposite sex*,

child, *grandparent or grandchild*, *father*

respectively. Not surprisingly, Leach [20] dismissed Malinowski's interpretation of his data by declaring the underlying logic to be utterly incomprehensible. It was to answer this criticism that Lounsbury [22] devised his reduction rules, which enabled him to calculate all the kinship terms belonging to the kinship descriptions

$$G_1PG_2SG_3CG_4,$$

where the G_i are M or F, among others.

For example,

$$G_1PMSFCG_4 \to G_1PMPFCG_4 \to G_1PMSG_4,$$

using the rules $PMSF \to PMPF$ and $PFC \to S$. Now, there are two cases: when $G_4 = M$, one continues thus:

$$\to G_1PM \to tama,$$

using the rule $PMSM \to PM$; but when $G_4 = F$, one immediately translates this into *tabu*.

Similarly, one calculates

$$G_1PFSMCG_4$$

to be *latu* when $G_1 = M$ and *tabu* when $G_1 \neq G_4$, while

$$G_1PMSMCG_4 \to G_1PMCG_4$$

turns out to be undefined.

To quote Chomsky [8]: "[Kinship systems] may be the kind of mathematics you can create if you don't have formal mathematics. The Greeks made up number theory, others made up kinship systems."

2 Syntactic calculus

While kinship grammars may be seen as miniature models of grammars in general, namely production grammars or generative grammars, relations enter the field of linguistics yet through another door.

Inasmuch as the ordered monoid of relations on a set is *residuated*, one can define operations / (*over*) and \ (*under*) such that

$$RS \leq T \Leftrightarrow R \leq T/S \Leftrightarrow S \leq R\backslash T,$$

by stipulating that

$$x(T/S)y \Leftrightarrow \forall_z(ySz \Rightarrow xTz),$$

$$y(R\backslash T)z \Leftrightarrow \forall_x(xRy \Rightarrow xTz).$$

Years ago [13], I had proposed the residuated monoid generated by certain basic types (s = *sentence*, n = *name*, ...) as providing a hierarchy of syntactic types useful for the study of natural language, as illustrated by the sentence

$$John\ sees\ Jane\ today.$$

$$n\ (n\backslash s)/n\ n\ s\backslash s$$

Van Benthem [28] had suggested that relation algebras, viewed as residuated monoids, be studied as models of the syntactic calculus and Andréka and Mikulás [1] actually proved a completeness theorem for the synctatic calculus with respect to these models.

However, relation algebras have more structure. Not only is there an involution, the *converse*, but also a *dualizing object* 0 such that

$$(0/R)\backslash 0 = R = 0/(R\backslash 0)$$

making them models of the multiplicative fragment of *classical bilinear* logic. (For this too, a linguistic application has been studied by Claudia Casadio [7]). Moreover, $0/R = R\backslash 0$, hence they are models of *cyclic* bilinear logic, so named by Yetter [29]. Indeed, $x0y \Leftrightarrow x \neq y$, and one easily verifies that $0/R = \neg(R^\vee) = (\neg R)^\vee$, for which one usually writes $R^\perp$, so that $0 = I^\perp$, where I is the identity relation.

3 Recursive functions

Relations can be put to good use in various branches of mathematics. For example, the easiest way to define *partial recursive functions* $N \hookrightarrow N$ is as relations of the form $fg^\vee$, where f and g are primitive recursive functions and $g^\vee g \leq f^\vee f$. Moreover, $fg^\vee$ is then a *total* recursive function if and only if g is surjective, that is, $I \leq gg^\vee$.

Relations on N of the from $fg^\vee$ are precisely those whose graphs are enumerable by primitive recursive functions (or by total recursive functions for that matter). The relations whose graphs are recursively enumerable form a monoid under conjunction. For

$$(fg^\vee)(hk^\vee) = (fp)(kq)^\vee,$$

since the graph of $g^\vee h$ is evidently recursive, hence recursively enumerable, so that $g^\vee h = pq^\vee$, where p and q are primitive recursive functions. The relations whose graphs are recursively enumerable form an ordered monoid with involution.

4 Homomorphic relations

In algebra, equivalence relations compatible with given operations have always been studied as *congruence* relations. Most interesting here is the famous result of Maltsev: pairs of congruence relations in an algebra permute if and only if the algebra possesses a ternary operation m such that $mxyy = x$ and $myyz = z$. As Findlay [9] pointed out, this is also equivalent to saying thet every reflexive homomorphic relation is an congruence. A relation $\rho : A \not\rightarrow B$ between algebras in a variety is said to be *homomorphic* if it is compatible with the operations, equivalently, if its graph is a subalgebra of $B \times A$.

Riguet [27] had called a relation ρ *difunctional* if $\rho\rho^\vee\rho = \rho$, and I pointed out [12] that the existence of a Maltsev operation is also equivalent to the assertion that all homomorphic relations between algebras of the variety are difunctional. This allows one to generalize a theorem, due to Goursat [11] for groups, to arbitrary Maltsev varietes: any homomorphic relation between algebras A and B is induced by an isomorphism between quotient algebras of subalgebras of A and B.

This theorem is easily seen to be equivalent to the Maltsev condition. However, in [14] I established a two-square lemma, which I considered to be a homological version of Goursat's theorem. It was pointed out by Carboni and Pedicchio that this lemma, when generalized to varieties (see Lemma 1 below), holds under more general conditions that those of Maltsev. They named these varietes "Goursat varietes", although Goursat himself was innocent of this concept. What is required is that, for any homomorphic relation ρ, $\rho\rho^\vee$ is tarnsitive, hence idempotent, that is, $\rho\rho^\vee\rho\rho^\vee = \rho\rho^\vee$. This is equivalent to the existence of a quaternary operation h satisfying the identities

$$hxxxy = y, \quad hxxyy = hyxyx.$$

Goursat varietes which are not already Maltsev varietes are not easy to come by. The first example of one was found by Mitschke [26]: weak implicational algebras with a binary operation $\Rightarrow$ satisfying

$$(y \Rightarrow y) \Rightarrow x = x, \quad (y \Rightarrow x) \Rightarrow x = (x \Rightarrow y) \Rightarrow y.$$

As I showed in [12], Goursat's theorem also yields the so-called butterfly lemma used by Zassenhaus to prove the Jordan-Hölder-Schreier theorem for *normal* series. All one has to do is to apply Goursat's theorem to homomorphic relations $\rho = \kappa\lambda$, where κ and λ are subcongruences, that is, homomorphic relations which are transitive and symmetric. Garret Birkhoff had already proved the J-H-S theorem for *principal* series usinfg congruence relations (see his book [5]).

5 The connecting homomorphism

Homomorphic relations other than transitive ones (order relations, congruences) and homomorphisms had been eschewed by algebraists until Mac Lane [24] showed that the easiest way to obtain the so-called *connecting homomorphism* in homological algebra was to look at the *zigzag* homomorphic relation

$$\varphi = ee^\vee fg^\vee hi^\vee jk^\vee k$$

and prove that it is a homomorphism, to wit, that $\varphi\varphi^\vee \leq 1$ and $1 \leq \varphi^\vee\varphi$. (From now on, the identity relation will be denoted by 1.). Here e, f, g, h, i, j and k are given homomorphisms between modules embedded in a diagram satisfying certain exactness and commutativity conditions, as in the following diagram:

$$\begin{tikzcd}
 & & & & 0 \arrow[d] & & \\
 & & & A \arrow[r] \arrow[d] & B \arrow[r, dotted, "e"] \arrow[d, "f"] & P \arrow[r] & 0 \\
 & & C \arrow[r] \arrow[d] & D \arrow[r, "g"] \arrow[d, "h"] & E \arrow[r] \arrow[d] & 0 & \\
 & 0 \arrow[r] & F \arrow[r, "i"] \arrow[d, "j"] & G \arrow[r] \arrow[d] & H & & \\
0 \arrow[r] & Q \arrow[r, dotted, "k"] & I \arrow[r] \arrow[d] & J & & & \\
 & & 0 & & & &
\end{tikzcd}$$

It is assumed that all rows and columns are exact and that all squares commute, and it is concluded that the sequence

$$A \to B \xrightarrow{\varphi} I \to J$$

is exact.

This result may be extended to Gousat varietes, provided we suitably extend the notion of exactness and commutativity. Exactness is applied to *forks*:

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, "h"] & C, \end{tikzcd} \quad \begin{tikzcd} D \arrow[r, "u"] & E \arrow[r, shift left, "v"] \arrow[r, shift right, "w"'] & F, \end{tikzcd}$$

called *left forks* and *right aforks* respectively. A left fork is *exact* if

$$\mathrm{Im}(f, g) = \mathrm{Ker} h.$$

A right fork is *exact* if

$$\mathrm{Im} u = \mathrm{Ker}(v, w),$$

where $\mathrm{Ker}(v, w) = \{e \in E | ve = we\}$ is the usual equalizer of v and w. For modules, this may be written

$$\mathrm{Im} u = \mathrm{Ker}(v - w).$$

The appropriate generalization of commutativity will be discussed below.

Instead of repeating the above argument in the more general context, we shall sketch another proof, which generalizes that for groups in [14].

Lemma 1. *In any Goursat variety, if the two forks in the diagram*

$$\begin{tikzcd}
A \arrow[r, shift left] \arrow[r, shift right] & B \arrow[r, "f"] \arrow[d, "g"] & C \\
D \arrow[r, "h"] & E \arrow[r, shift left] \arrow[r, shift right] & F
\end{tikzcd} \tag{1}$$

are exact, then

$$\frac{Img \cap Imh}{gIm(A \rightrightarrows B)g^{\vee}} \cong \frac{g^{\vee} Ker(E \rightrightarrows F)}{Kerf \cup Kerg},$$

where the congruence relations in the denominators are assumed to be restricted to the algebras in the numerators.

The partially doubled squares in the diagram

$$\begin{array}{ccccc} A & \rightrightarrows & B & \xrightarrow{f} & C \\ \downarrow\downarrow & & \downarrow g & & \downarrow\downarrow \\ D & \xrightarrow{h} & E & \rightrightarrows & F \end{array} \tag{2}$$

are said to quasi-commute *if*

$$gIm(A \rightrightarrows B)g^{\vee} = hIm(A \rightrightarrows D)h^{\vee}$$

when restricted to $Img \cap Imh$ *and if*

$$g^{\vee} Ker(E \rightrightarrows F) = f^{\vee} Ker(C \rightrightarrows F)$$

respectively. We note that quasi-commutativity for modules is implied by the usual commutativity if parallel arrows are replaced by their difference.

The so-called *snake lemma* asserting the existence of a connecting homomorphism may be extended to arbitrary Goursat varietes by imposing appropriate exactness and quasi-commutativity conditions on the appropriate diagram. One way of proving it is with the help of Lemma 1. Indeed, a concise way of stating Lemma 1 for the diagram (2) is to say that the *image* of the first square is isomorphic to the *kernel* of the second. Now look at the following diagram:

$$\begin{array}{ccccccccc} & & & & & \rightrightarrows & & \xrightarrow{e} & P \\ & & & & \downarrow\downarrow & 1 & \downarrow f & & \\ & & & \rightrightarrows & \xrightarrow{g} & & & & \\ & & \downarrow\downarrow & 3 & \downarrow & 2 & \downarrow\downarrow & & \\ & & & \xrightarrow{i} & & \rightrightarrows & & & \\ & & \downarrow j & 4 & \downarrow\downarrow & & \downarrow & & \\ Q & \xrightarrow{k} & & \rightrightarrows & & \rightarrow & & & \end{array}$$

It is assumed that all rows and columns are exact, that all squares quasi-commute and that k, i and f are injections and j, g and e are surjections. (An arrow $f : A \to B$ is said to be an *injection* if $f^\vee f = 1_A$ and a *surjection* if $ff^\vee = 1_B$. One then concludes that

$$P \cong \mathrm{Im}(1) \cong \mathrm{Ker}(2) \cong \mathrm{Im}(3) \cong \mathrm{Ker}(4) \cong Q.$$

6 Relations in categories

Let $\mathcal{R}$ be an oredered category with involution. We shall think of the arrows $\rho : A \to B$ in $\mathcal{R}$ as relations. With $\mathcal{R}$ we shall associate two other categories $\mathcal{R}_0$ and $\bar{\mathcal{R}}$.

$\mathcal{R}_0$ is the full subcategory of $\mathcal{R}$ whose arrows: $f : A \to B$ satisfy

$$ff^\vee \leq 1_B, \quad 1_A \leq f^\vee f,$$

that is, for which $f^\vee$ is *right adjoint* to f. The order in $\mathcal{R}_0$ is seen to be discrete.

$\bar{\mathcal{R}}$ is the ordered category with involution whose objects are idempotent and symmetric relations in $\mathcal{R}$. In concrete situations, idempotency here follows from transitivity, hence the objects of $\bar{\mathcal{R}}$ may be described as *partial equivalence relations.* Its arrows ${}_\beta\rho_\alpha : \alpha \to \beta$ are induced by relations $\rho : A \to B$ in $\mathcal{R}$ satisfying $\beta\rho\alpha = \rho$. Composition and converses are defined in the obvious way and identity arrows are ${}_\alpha\alpha_\alpha : \alpha \to \alpha$. The forgetful functor $\bar{\mathcal{R}} \to \mathcal{R}$, which sends the object $\alpha : A \to A$ of $\bar{\mathcal{R}}$ onto A and the relation ${}_\beta\rho_\alpha : \alpha \to \beta$ onto $\rho : A \to B$ is faithful.

Exactness of forks in $\mathcal{R}_0$ is defined as in Sect. 5, provided we define:

$$\mathrm{Ker} h = h^\vee, \quad \mathrm{Im} h = hh^\vee,$$

and call $h^\vee h$ a *congruence*, $hh^\vee$ a *cocongruence*,

$$\mathrm{Im}(f, g) = \text{intersection of all congruences } k^\vee k \text{ containing } fg^\vee,$$

$$\mathrm{Ker}(f, g) = \text{join of all congruences } kk^\vee \text{ contained in } g^\vee f,$$

assuming that these exist. It may be useful here to adopt:

Postulate 1. For each object A of $\mathcal{R}$, $\mathrm{Hom}_{\mathcal{R}}(A, A)$ is a complete lattice.

If we assume that $\mathcal{R}$ satisfies the Goursat condition that, for any relation ρ, $\rho\rho^\vee$ is idempotent, we may reformulate Lemma 1 of Sect. 5 in a more general context.

Lemma 2. *If $\mathcal{R}$ satisfies the Goursat condition and (1) is a diagram in $\mathcal{R}_0$, then we have the following isomorphism in $\bar{\mathcal{R}}_0$.*

$$gg^\vee h \mathit{Im}(A \rightrightarrows B) h^\vee gg^\vee \cong g^\vee g f^\vee \mathit{Ker}(C \rightrightarrows F) f g^\vee g.$$

Without the Goursat condition, in concrete situations, in which transitive and symmetric relations are necessarily idempotent, we may have recourse to Postulate 1 and replace the two sides of the above isomorphism by their transitive closures.

If we assume the Goursat condition for $\mathcal{R}$, we say that the partially doubled squares in (2) of Sect. 5 for $\mathcal{R}_0$ P*quasi-commute* if

$$hh^\vee g\mathrm{Im}(A \rightrightarrows B)g^\vee hh^\vee = gg^\vee h^\vee \mathrm{Im}(A \rightrightarrows B)h^\vee gg^\vee .$$

and

$$f^\vee fg\mathrm{Ker}(E \rightrightarrows F)gf^\vee f = g^\vee gf^\vee \mathrm{Ker}(C \rightrightarrows F)fg^\vee g$$

respectively. If we assume Postulate 1 instead, in concrete situations, we may again replace the two sides of each equation by their transitive closures.

Example 1. We recall that a category is *regular* in the same sense of Barr et al. [2], if it has finite limits, every kernel pair has a coequalizer and regular epis are stable under pullbacks; it is *exact* if all equivalence relations are kernel pairs.

If $\mathcal{R}$ is the ordered category of relations over a *regular* category $\mathcal{C}$, as in Barr et al. [2], then $\mathcal{C} = \mathcal{R}_0$ and $\bar{\mathcal{R}}_0$ is the *exact completion* of $\mathcal{R}$ (see Calenko et al. [6], Freyd and Scedrov [10] and McLarty [25]).

If $\mathcal{R}_0 = \mathcal{C}$ is a regular category, left forks and right forks are exact if and only if

$$\mathrm{coim}h = \mathrm{coeq}(f, g), \quad \mathrm{im}h = \mathrm{eq}(f, g)$$

respectively.

It is more difficult to express quasi-commutativity for regular categories without invoking zig-zac relations. For exact categories, one can presumably copy the definition for varietes following Lemma 1, which mildly invokes $g^\vee$, $h^\vee$ and $f^\vee$.

7 Partial equivalence relations

Partial equivalence relations are symmetric and transitive, but not necessarily reflexive. In concrete situations and in regular categories, they are also idempotent.

Lemma 3. *Each of the following assertions about relations (arrows in an ordered category $\mathcal{R}$ with involution) implies the next:*

1. *Every relation is representable in the form $gf^\vee$, where f and g are arrows in $\mathcal{R}_0$.*
2. *Every relation ρ is semi-difunctional: $\rho \leq \rho\rho^\vee\rho$.*

3. Every partial equivalence relation is a symmetric idempotent.

Proof. (1) $\Rightarrow$ (2). If $\rho = gf^\vee$ then $\rho = gf^\vee \leq gf^\vee fg^\vee gf^\vee = \rho\rho^\vee\rho$.
(2) $\Rightarrow$ (3) If $\alpha\alpha \leq \alpha$ and $\alpha^\vee \leq \alpha$ then $\alpha \leq \alpha\alpha^\vee\alpha \leq \alpha\alpha\alpha \leq \alpha\alpha$. □

I have elsewhere [17] constructed a category $\mathcal{R}_0^{\mathrm{per}}$ as follows:

- its objects are partial equivalence relations;
- its arrows ${}_\beta\rho_\alpha : \alpha \to \beta$ are relations $\rho : A \to B$ such that $\rho\alpha\rho^\vee \leq \beta$ and $\alpha \leq \rho^\vee\beta\rho$;
- equality between arrows ${}_\beta\rho_\alpha$ and ${}_\beta\rho_\alpha$ is defined to mean $\rho\alpha\sigma^\vee \leq \beta$ or, equivalently, $\alpha \leq \rho^\vee\beta\sigma$.

Under the assumption (3) of Lemma 2, it is not difficult to see that $\mathcal{R}_0^{\mathrm{per}}$ is equivalent ro $\bar{\mathcal{R}}_0$. All one has to observe is that every arrow ${}_\beta\rho_\alpha$ in $\mathcal{R}_0^{\mathrm{per}}$ is equal to exactly one arrow ${}_\beta\bar{\rho}_\alpha$ in $\bar{\mathcal{R}}_0$, namely when $\bar{\rho} = \beta\rho\alpha$. Indeed, one easily verifies that $\bar{\rho}\bar{\rho}^\vee \leq \beta$ and $\alpha \leq \bar{\rho}^\vee\bar{\rho}$, as well as

$$\rho\alpha\bar{\rho}^\vee = \rho\alpha\rho^\vee\beta \leq \beta\beta \leq \beta.$$

If $\mathcal{R}_0$ is regular, all partial equivalence relations are induced by equivalence relations on subobjects [17], Prop. 3. Hence $\mathcal{R}_0^{\mathrm{per}}$ is equivalent to $\mathcal{R}_0^{\mathrm{eq}}$, whose objects are equivalence relations. According to Freyd and Scedrov [10] or McLarty [25], $\mathcal{R}_0^{\mathrm{eq}}$ is exact completion of $\mathcal{R}_0$.

Example 2. Let $\mathcal{R}$ be the ordered monoid of all relations on N whose graphs are recursively enumerable (see Sect. 3 above), so that $\mathcal{R}_0$ is the monoid of totally recursive functions $N \to N$. It is not difficult to show that any arrow ${}_\beta\rho_\alpha : \alpha \to \beta$ of $\mathcal{R}_0^{\mathrm{per}}$ is equal to the arrow ${}_\beta\varphi_\alpha : \alpha \to \beta$, where φ is the partial recursive function defined as follows:

$$\varphi x = \text{ smallest } y \text{ such that } y\rho x.$$

Indeed, one immediately verifies that

$$\varphi\alpha\varphi^\vee \leq \rho\alpha\rho^\vee \leq \beta.$$

To prove that $\alpha \leq \varphi^\vee\beta\varphi$ takes a bit longer [17].

Constructed thus, $\mathcal{R}_0^{\mathrm{per}}$ resembles the category PER, which plays a role in theoretical computer science [21]. The only difference is that PER has more objects than $\mathcal{R}_0^{\mathrm{per}}$: all partial equivalence relations on N are object of the former, while only those with recursively enumerable graphs are object of the latter, it being crucial to the intended applications that the set of objects of PER is closed under arbitrary intersections.

8 Summary

Binary relations play a prominent role in anthropological linguistics in connection with the algebra of kinship relations, while in categorial linguistics they enter only marginally as models of the syntactic calculus. Whereas, in mathematics, relations have been largely replaced by functions, we have seen that often the very definition of a function is greatly simplified by viewing it as a relation. this is so for recursive functions and for the connecting homomorphism in homological algebra. the best known relations in algebra are congruence relations, but other "homomorphic" relations also turn out to be useful in proving basic results such as the Zassenhaus Lemma. The construction of the connecting homomorphism has been generalized from module categories to other varietes and categories. Finally, relations are crucial for obtaining the exact completion of a regular category and for describing the category PER in theoretical computer science.

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