INJECTIVE HULLS OF PARTIALLY ORDERED MONOIDS

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Abstract. We find the injective hulls of partially ordered monoids in the category whose objects are po-monoids and submultiplicative order-preserving functions. These injective hulls are with respect to a special class of monics called "embeddings". We show as well that the injective objects with respect to these embeddings are precisely the quantales.

Introduction

The genesis of this paper is that the first author wrote a draft in 2000. Before getting it typed for publication, he discovered a technical gap that he could not fill. He finally gave it to the other three of us asking if we could fill the gap. We could, by changing a few of the definitions. The result is the paper you see before you. The historical comments and the statements of theorems are all due to Lambek.

The category of po-monoids The subject of this paper is the category of partially ordered monoids (**po-monoids**) and submultiplicative order-preserving maps. If A and B are po-monoids a function $\phi : A \longrightarrow B$ is **submultiplcative** if $\phi(1) = 1$ and $\phi(a)\phi(a') \leq \phi(aa')$ for all $a, a' \in A$.

Another concept that is important to this paper is that of embedding. A morphism $\phi : A \longrightarrow B$ will be called an **embedding** if it is injective (that is, one-one) and if, for all $a_1, a_2, \ldots, a_n, a \in A$, then $\phi(a_1)\phi(a_2)\cdots\phi(a_n) \leq \phi(a)$ implies $a_1 a_2 \cdots a_n \leq a$. We will use the notation $A \hookrightarrow B$ to denote an embedding.

A po-monoid Q will be called **injective** if given any embedding $\phi : A \hookrightarrow B$ and any morphism $\psi : A \longrightarrow Q$ there is a morphism $\theta : B \longrightarrow Q$ such that the triangle

$$\begin{tikzcd} A \arrow[r, hook, "\phi"] \arrow[dr, "\theta"'] & B \arrow[d, "\psi"] \\ & Q \end{tikzcd}$$

commutes.

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We will say that an embedding $A \hookrightarrow B$ is **essential** if every morphism $B \longrightarrow C$ for which the composite $A \longrightarrow B \longrightarrow C$ is an embedding is itself an embedding. We will call an essential embedding $A \hookrightarrow B$ with B injective (with respect to embeddings) an **injective hull** (also known as the **injective envelope**) of A.

A po-monoid is said to be **residuated** if it is equipped with binary operations "/" (**over**) and "\" (**under**) such that

$$ab \leq c \text{ if and only if } a \leq c/b \text{ if and only if } b \leq a\backslash c$$

for all elements a, b, c.[1] By a **quantale**, we mean a complete residuated po-monoid. This is what some others call a unital quantale, for example, [Rosenthal (1990)].

The main results of this paper are:

1. Injectives are the same as quantales (4.1).

2. Every object can be embedded into an injective envelope (5.8).

Our construction is motivated by the 1970 paper of Bruns and Lakser who showed that the injectives in the category of meet-semilattices are exactly the locales and that every object admits an essential embedding into a locale, its injective hull. They constructed this locale without recourse to the axiom of choice. Here their ideas will be extended to the category of partially ordered monoids (and submultiplicative maps and with quantales in place of locales).[2]

1. Historical background.

There is a vast literature on the subject of injective hulls, of which we mention only a few basic items that were noted in the original draft.

The injective hull, though not under that name,[3] was first obtained in [R. Baer (1940)] using transfinite induction.

The best-known early discussion of injective hulls is in [Eckmann and Schopf (1953)], who introduced them as maximal essential extensions. In modern categorical language, an extension (or monomorphism) $A \rightarrowtail B$ is said to be **essential** if any morphism $B \longrightarrow C$ for which the composite $A \longrightarrow B \longrightarrow C$ is monic, is itself monic. They embedded a given module into an injective one and then used Zorn's lemma to pick out a maximal element of the former among the submodules of the latter.

[1] Lambek had used this structure in [Lambek (1959)] in an attempt to give a mathematical description of sentence structure, having observed that the subsets of a free monoid form a complete residuated monoid.

[2] In the original draft Lambek claimed that his argument was intuitionistically valid and held in any topos. We have not verified this. Correspondingly, we replace the earlier draft's use of the subobject classifier Ω by the two-element Boolean algebra, denoted 2.

[3] According to [Rotman (2009), page 503], R. Baer jokingly called injectives "fascist", in contrast with projectives, which were usually called "free". He also showed that for the category of groups (not necessarily abelian), no group with more than one element could be injective.

K. Shoda[4] [1952, 1957] treated injective hulls in a wider algebraic context as a generalization of algebraic completions of fields. See also [Cohn (1965)].

A very readable account of injective hulls in various categories is given in [Daigneault (1969)]. Further categorical generalizations are found in [B. Banaschewski (1970), Tholen (1981)], among others. As far as we are aware, all the treatments of the existence of injective hulls referred to above invoke transfinite induction or Zorn's lemma. However, the injective hull of a Boolean algebra or a commutative C^*-algebra can be constructed without relying on any form of the axiom of choice. Lambek would not have been surprised that the construction of the injective hull of a semilattice in [Bruns and Lakser (1970)] was also truly constructive. Reading their paper[5] convinced Lambek that their ideas would carry over from meet semilattices to po-monoids, a meet semilattice being a po-monoid in which the product of two elements is their greatest lower bound.

2. Po-monoids

A WORD ABOUT THE DEFINITION OF MORPHISM. It may seem odd to allow submultiplicativity, but assume that $\phi(1) = 1$. A perhaps more natural definition would to suppose that for any finite set S, we have $\prod_{s\in S} \phi(a_s) \le \phi(\prod_{s\in S} a_s)$. When S is empty, this leads to $1 \le \phi(1)$. This is possible, but there is a complication in the proof of Theorem 4.1. This complication can be dealt with but it appears to outweigh the added generality.

2.1. PROPOSITION. *Injective hulls are unique up to isomorphism.*

PROOF. Suppose that $\phi : A \hookrightarrow B$ and $\psi : A \hookrightarrow C$ are injective hulls. Since C is injective, there is a map $\sigma : B \longrightarrow C$ in the diagram

$$\begin{tikzcd} A \arrow[r, hook, "\phi"] \arrow[dr, "\psi"'] & B \arrow[d, "\sigma"] \\ & C \end{tikzcd}$$

such that $\sigma\phi = \psi$. Since ψ is an essential embedding, the fact that ϕ is an embedding implies that σ is an essential embedding. Since B is injective, there is a map $\tau : C \longrightarrow B$ in the diagram

$$\begin{tikzcd} B \arrow[r, hook, "\sigma"] \arrow[dr, "\mathrm{id}"'] & C \arrow[d, "\tau"] \\ & B \end{tikzcd}$$

[4]Incidentally, the late uncle of the present Empress of Japan.

[5]We are indebted to [Coecke (2000)], who rediscovered their result in the context of quantum logic and for pointing out the crucial reference.

such that $\tau\sigma = \mathrm{id}$. Since σ is essential, τ is also an embedding. This implies that σ and τ are inverse isomorphisms. ∎

3. Embedding po-monoids into quantales

Bruns and Lakser [1970] characterized the injectives in the category of semilattices as being complete Heyting algebras, nowadays called **locales**. Not surprisingly then, the injectives in the category of po-monoids turn out to be complete residuated po-monoids which are called quantales in [Mulvey & Pelletier (2001)].

3.1. It is easily seen (and no surprise to categorists) that, in a residuated monoid, multiplication must distribute over all existing sups. Conversely, completeness and complete distributivity of multiplication over sups, ensure that the operations over and under can be defined (using Freyd's adjoint functor theorem; note that the solution set condition is automatic because a poset is a small category) by

$$c/b = \bigvee\{x \mid xb \le c\}$$

To see this, let $y = \bigvee\{x \mid xb \le c\}$. If $a \le y$ then $ab \le yb \le \bigvee\{yb \mid yb \le c\} \le c$. On the other hand, if $ab \le c$, then a is one of the terms of the sup and so $a \le y$. A similar formula serves to define $a\backslash c$.

Given a po-monoid A, let $\mathcal{P}(A)$ be the set of order ideals (= down-closed subsets) of A. We note that this is the same thing as the set of order-preserving maps $A^{\mathrm{op}} \longrightarrow 2$. In fact, if $\phi : A^{\mathrm{op}} \longrightarrow 2$ is an order-preserving map, then $\phi^{-1}(1)$ is an up-closed subset of A^{op}, which is evidently a down-closed subset of A. The converse is equally obvious. If $a \in A$, we denote by $a{\downarrow}$ the set $\{b \in A \mid b \le a\}$. This is the principal ideal generated by a.

Clearly $\mathcal{P}(A)$ is a complete lattice with sup being union.

3.2. Definition. *If $X \subseteq A$, denote by $X{\downarrow}$ the down-closure of X, that is $\{a \in A \mid a \le x \text{ for some } x \in X\}$. If I and J are ideals, we let $I{\cdot}J = (IJ){\downarrow}$. This works out to*

$$I{\cdot}J = \{c \mid c \le ab \text{ for some } a \in I \text{ and } b \in J\}$$

It is immediate that this multiplication makes $\mathcal{P}(A)$ into a complete po-monoid. Moreover, since the product of two down-ideals is just the down-closure of their pointwise product, it is immediate that products commute with arbitrary union, so that $\mathcal{P}(A)$ is a quantale. If I is an ideal and $a, b \in A$, we note that $aIb \subseteq a{\downarrow}Ib{\downarrow}$, but that $(aIb){\downarrow} = (a{\downarrow}Ib{\downarrow}){\downarrow}$ and so we will usually denote $a{\downarrow}{\cdot}I{\cdot}b{\downarrow}$ by $a{\cdot}I{\cdot}b$.

We define $\mu : A \longrightarrow \mathcal{P}(A)$ by $\mu(a) = a{\downarrow}$.

We will say that a morphism of po-monoids is **multiplicative** if it preserves the multiplication on the nose.

3.3. Proposition. *μ is a multiplicative embedding.*

Proof. Evidently $ab \in a{\downarrow}{\cdot}b{\downarrow}$ so that $(ab){\downarrow} \subseteq a{\downarrow}{\cdot}b{\downarrow}$. It is equally evident that $a{\downarrow}b{\downarrow} \subseteq (ab){\downarrow}$, whence $a{\downarrow}{\cdot}b{\downarrow} \subseteq (ab){\downarrow}$ as well. ∎

4. Quantales and injectivity

4.1. Theorem. *A po-monoid is injective (with respect to embeddings) if and only if it is a quantale.*

Proof. Suppose Q is a quantale, $\mu : A \longrightarrow Q$ a morphism and $\phi : A \hookrightarrow B$ an embedding. Define $\psi : B \longrightarrow Q$ by

$$\psi(b) = \bigvee\{\mu(a_1)\mu(a_2)\cdots\mu(a_n) \mid a_1,\ldots,a_n \in A \text{ such that } \phi(a_1)\phi(a_2)\cdots\phi(a_n) \le b\}$$

It is clear that ψ preserves order. If $\phi(a_1)\cdots\phi(a_n) \le 1 = \phi(1)$, then the definition of embedding implies that $a_1\cdots a_n \le 1$, whence $\mu(a_1)\cdots\mu(a_n) \le \mu(a_1\cdots a_n) \le \mu(1) = 1$ so that every term in the sup that defines $\psi(1)$ is below 1, while one of the terms is $\mu(1) = 1$. Thus $\psi(1) = 1$. Using the fact that Q is a quantale and therefore product distributes over arbitrary sup, we calculate

$$\begin{aligned}\psi(b)\psi(b') &= \bigvee_{\phi(a_1)\cdots\phi(a_n)\le b} \mu(a_1)\cdots\mu(a_n) \bigvee_{\phi(a'_1)\cdots\phi(a'_{n'})\le b'} \mu(a'_1)\cdots\mu(a'_{n'})\\ &= \bigvee_{\substack{\phi(a_1)\cdots\phi(a_n)\le b\\ \phi(a'_1)\cdots\phi(a'_{n'})\le b'}} \mu(a_1)\cdots\mu(a_n)\mu(a'_1)\cdots\mu(a'_{n'}) \le \psi(bb')\end{aligned}$$

This holds because the inequalities $\phi(a_1)\cdots\phi(a_n) \le b$ and $\phi(a'_1)\cdots\phi(a'_{n'}) \le b'$ imply that $\phi(a_1)\cdots\phi(a_n)\phi(a'_1)\cdots\phi(a'_{n'}) \le bb'$. When $b = \phi(a)$, the fact that $\phi(a_1)\cdots\phi(a_n) \le \phi(a)$ which implies that $a_1a_2\cdots a_n \le a$ implies that $\psi\phi(a) \le \mu(a)$, while the opposite inclusion follows from the fact that $\mu(a)$ is one of the terms in the sup that defines $\psi\phi(a)$.

Next suppose that A is injective. Then the embedding $\mu : A \hookrightarrow \mathcal{P}(A)$ is split by $\epsilon : \mathcal{P}(A) \longrightarrow A$. It is immediate that A is complete with $\bigvee a_i = \epsilon(\bigvee \mu(a_i))$. We now compute that if $a = \bigvee a_i$ and $b \in A$,

$$\begin{aligned} ba &= \epsilon(\mu(b))\epsilon\left(\bigvee \mu(a_i)\right) \le \epsilon(\mu(b)\bigvee \mu(a_i))\\ &= \epsilon\left(\bigvee \mu(b)\mu(a_i)\right) = \epsilon\left(\bigvee \mu(ba_i)\right) = \bigvee ba_i\end{aligned}$$

The reverse inclusion is obvious since ba is an upper bound for the set of all ba_i. Similarly we have right distributivity and so A is a quantale. ∎

5. A closer embedding

Although we have embedded the po-monoid A into the quantale $\mathcal{P}(A)$, this embedding is not very close. In this section, we describe an embedding that is, in a sense, closer to A.

5.1. Proposition. *If A is a quantale, then for any $a, b, c \in A$, we have $a\backslash(b/c) = (a\backslash b)/c$.*

Proof. For any $d \in A$, we have $d \le a\backslash(b/c)$ if and only if $ad \le b/c$ if and only if $adc \le b$ and the same is evidently true for the other side of the equation. ∎

We will denote both sides of the equation by $a\backslash b/c$.

5.2. Definition. *For* $I \in \mathcal{P}(A)$*, define* $\mathrm{cl}(I) = \bigcap\{a{\downarrow}\backslash b{\downarrow}/c{\downarrow} \mid a{\cdot}I{\cdot}c \subseteq b{\downarrow}\}$ *and* $\mathcal{Q}(A) = \{I \in \mathcal{P}(A) \mid I = \mathrm{cl}(I)\}$*. The* $I \in \mathcal{Q}(A)$ *will be called the* **closed** *subsets.*

We note that since $b{\downarrow}$ is down-closed, the phrase $a{\cdot}I{\cdot}c \subseteq b{\downarrow}$ is equivalent to $aIc \subseteq b{\downarrow}$.

A word about the empty set. The empty set is an ideal, but might not be closed. An element $a \in A$ is in $\mathrm{cl}(\emptyset)$ if and only if whenever $\emptyset \subseteq b{\downarrow}\backslash c{\downarrow}/d{\downarrow}$, then also $a \in b{\downarrow}\backslash c{\downarrow}/d{\downarrow}$. Obviously, this holds if and only if $a \in b{\downarrow}\backslash c{\downarrow}/d{\downarrow}$ for all $b, c, d \in A$, which will hold if and only if $bad \le c$ for all $b, c, d \in A$. Taking $b = d = 1$, we see that a must be the bottom element of A, denoted $\perp$. In order that this remain true for all $b, d \in A$, we must have that $b{\perp}d = \perp$, that is that $\perp$ be a 2-sided ideal of the monoid. In all other cases, $\emptyset$ will be closed.

What kind of closure operator is cl? From the previous comment, it is not a Kuratowski closure operator since the empty set is not necessarily closed and the operator does not appear to preserve binary joins. In particular, there is no reason for $a_1{\downarrow}\backslash b_1{\downarrow}/c_1{\downarrow} \cup a_2{\downarrow}\backslash b_2{\downarrow}/c_2{\downarrow}$ to be a closed set. As far as we can tell, it is not the closure operator for a Grothendieck topology either since it does not appear to preserve binary meets. It is obviously expansive, order-preserving, and easily shown to be idempotent and these are the only properties we need. We leave the easy proof of the following to the reader.

5.3. Proposition. *Let* P *be a poset and* $c : P \longrightarrow P$ *be an expansive, order-preserving, and idempotent operator on* P*. Then* c *is a retraction of* P *onto* $Fix(c)$*, the set of elements of* P *fixed by* c*. If* P *is complete, so is* $Fix(c)$*.* ∎

The monoid structure on $\mathcal{Q}(A)$. It is not entirely obvious how to define the monoid structure on $\mathcal{Q}(A)$. We begin with:

5.4. Proposition. *If* $I, J \in \mathcal{P}(A)$ *then* $\mathrm{cl}(I){\cdot}J \subseteq \mathrm{cl}(I{\cdot}J)$*.*

Proof. Suppose $a, b, c \in A$ are such that $I{\cdot}J \subseteq a{\downarrow}\backslash b{\downarrow}/c{\downarrow}$. For all $x \in I$ and $y \in J$, we have that $xy \in a{\downarrow}\backslash b{\downarrow}/c{\downarrow}$ which implies that $axyc \in b{\downarrow}$. This implies that $x \in a{\downarrow}\backslash b{\downarrow}/(yc){\downarrow}$. Since this is true for all $x \in I$, we must have that $I \subseteq a{\downarrow}\backslash b{\downarrow}/(yc){\downarrow}$ so that $\mathrm{cl}(I) \subseteq a{\downarrow}\backslash b{\downarrow}/(yc){\downarrow}$. This implies that $\mathrm{cl}(I)yc \subseteq a{\downarrow}\backslash b{\downarrow}$ or that $\mathrm{cl}(I)y \subseteq a{\downarrow}\backslash b{\downarrow}/c{\downarrow}$. Since this is true for all $y \in J$ and $a{\downarrow}\backslash b{\downarrow}/c{\downarrow}$ is down-closed, we conclude that $\mathrm{cl}(I){\cdot}J \subseteq a{\downarrow}\backslash b{\downarrow}/c{\downarrow}$. This is true whenever $I{\cdot}J \subseteq a{\downarrow}\backslash b{\downarrow}/c{\downarrow}$ and hence $\mathrm{cl}(I){\cdot}J \subseteq \mathrm{cl}(I{\cdot}J)$. ∎

5.5. Corollary.

1. *For any* $I, J \in \mathcal{P}(A)$*,* $I{\cdot}\,\mathrm{cl}(J) \subseteq \mathrm{cl}(I{\cdot}J)$*.*
2. *For any* $I, J \in \mathcal{P}(A)$*,* $\mathrm{cl}(I){\cdot}\,\mathrm{cl}(J) \subseteq \mathrm{cl}(I{\cdot}J)$*.*
3. *For any* $I, J \in \mathcal{P}(A)$*,* $\mathrm{cl}(\mathrm{cl}(I){\cdot}\,\mathrm{cl}(J)) = \mathrm{cl}(I{\cdot}J)$*.*

PROOF.

1. This follows by symmetry.
2. We have $\mathrm{cl}(I)\cdot \mathrm{cl}(J) \subseteq \mathrm{cl}(I\cdot \mathrm{cl}(J)) \subseteq \mathrm{cl}(\mathrm{cl}(I\cdot J)) = \mathrm{cl}(I\cdot J)$.
3. $I\cdot J \subseteq \mathrm{cl}(I)\cdot \mathrm{cl}(J)$, implies one inclusion, while the previous claim, together with the idempotence of cl, implies the other. ■

We extend the multiplication on A to $\mathcal{Q}(A)$ by letting $I\circ J = \mathrm{cl}(I\cdot J)$. It is clear that $1{\downarrow}$ is a unit for the multiplication. Associativity is not immediate, however. Assuming $K = \mathrm{cl}(K)$, we get $(I\circ J)\circ K = \mathrm{cl}(\mathrm{cl}(I\cdot J)\cdot \mathrm{cl}(K)) = \mathrm{cl}(I\cdot J\cdot K)$ and, assuming $I = \mathrm{cl}(I)$, we also have $I\circ(J\circ K) = \mathrm{cl}(I\cdot J\cdot K)$. In particular, when $I, J, K \in \mathcal{Q}(A)$, we conclude that $(I\circ J)\circ K = \mathrm{cl}(I\cdot J\cdot K) = I\circ(J\circ K)$.

5.6. PROPOSITION. *The inclusion* $\mathcal{Q}(A)\hookrightarrow\mathcal{P}(A)$ *is an embedding, while* $\mathrm{cl} : \mathcal{P}(A)\longrightarrow\mathcal{Q}(A)$ *is a retraction of the inclusion.*

PROOF. For the first claim observe that if $I_1, \ldots, I_n$ and J are in $\mathcal{Q}(A)$ such that $I_1 \cdots I_n \subseteq J$, then $I_1\circ \cdots \circ I_n = \mathrm{cl}(I_1\cdot \cdots \cdot I_n) \subseteq J$ since J is closed. The second claim is simply that $\mathrm{cl}(I\cdot J) = I\circ J$ (and so the retraction is actually multiplicative). ■

5.7. COROLLARY. $\mathcal{Q}(A)$ *is injective.*

PROOF. A retract of an injective is injective. ■

It is clear that every $a{\downarrow}$ is a closed ideal. We let $\eta : A\longrightarrow\mathcal{Q}(A)$ be given by $\eta(a) = a{\downarrow}$.

5.8. THEOREM. *The map* $\eta : A \longrightarrow \mathcal{Q}(A)$ *is an essential embedding and thus is an embedding into an injective hull.*

PROOF. Suppose that $\psi : \mathcal{Q}(A) \longrightarrow B$ is a morphism such that $\psi\eta$ is an embedding. We must show that ψ is an embedding. Suppose $\psi(I_1)\psi(I_2)\cdots\psi(I_n) \le \psi(J)$. Then for all elements $a_1, a_2, \cdots, a_n, b, c, d$ of A such that $a_1 \in I_1, a_2 \in I_2, \cdots, a_n \in I_n$ and $J \subseteq b{\downarrow}\backslash c{\downarrow}/d{\downarrow}$, we have

$$\begin{aligned}\psi\eta(b)\psi\eta(a_1)\cdots\psi\eta(a_n)\psi\eta(d) &= \psi(b{\downarrow})\psi(a_1{\downarrow})\cdots\psi(a_n{\downarrow})\psi(d{\downarrow})\\ &\le \psi(b{\downarrow})\psi(I_1)\cdots\psi(I_n)\psi(d{\downarrow}) \le \psi(b{\downarrow})\psi(J)\psi(d{\downarrow})\\ &\le \psi(b{\downarrow}\cdot J\cdot d{\downarrow}) \le \psi(c{\downarrow}) = \psi\eta(c)\end{aligned}$$

Since $\psi\eta$ is an embedding, it follows that $ba_1\cdots a_n d \le c$ which implies that $a_1a_2\cdots a_n \in b{\downarrow}\backslash c{\downarrow}/d{\downarrow}$. Thus $I_1I_2\cdots I_n \subseteq b{\downarrow}\backslash c{\downarrow}/d{\downarrow}$. Since $b{\downarrow}\backslash c{\downarrow}/d{\downarrow}$ is both down-closed and closed, it follows that $I_1\circ I_2\circ\cdots\circ I_n$ is included in $b{\downarrow}\backslash c{\downarrow}/d{\downarrow}$ whenever J is, and hence that $I_1\circ I_2\circ\cdots I_n \subseteq J$, as required. ■

5.9. THEOREM. *Any object with more than one element is not injective with respect to all monics.*

PROOF. Let Q have more than one element. We first consider the case that 1 is not the bottom element of Q so that there is a $q \in Q$ with $1 \not\leq q$. Let $\mathbf{N}$ denote the non-negative integers with the usual order and addition as monoid operation. Let $|\mathbf{N}|$ denote the same monoid but with the discrete order. Define $\psi : |\mathbf{N}| \longrightarrow Q$ by $\psi(n) = q^n$, with, as usual, $q^0 = 1$. Then there is no possible way to complete the diagram

$$\begin{tikzcd} |\mathbf{N}| \arrow[r, "\mathrm{id}"] \arrow[dr, "\psi"'] & \mathbf{N} \\ & Q \end{tikzcd}$$

Next, we consider the case that 1 is the bottom element of Q. Now there is a $q \in Q$ with $q \not\leq 1$. Simply replace $\mathbf{N}$ by $\mathbf{N}^{\mathrm{op}}$ in the same diagram. ∎

6. Examples

In the original article, the candidate for the injective hull was the object $\mathcal{R}(A)$ which consisted of all down-closed sets that were closed under distributive sups. In order to gain some insight into the meaning of this, we tried to find examples. It is clear that every set of the form $a{\downarrow}$ is closed not only under distributive sups, but also closed under all sups. We soon realized that all sets of the form $a{\downarrow}\backslash b{\downarrow}/c{\downarrow}$ were also closed under distributive sups and the fact that those sups were distributive was necessary. This led us to try the definition of $\mathcal{Q}(A)$ that we have given and, *mirabile dictu*, it worked!

Clearly $\mathcal{Q}(A) \subseteq \mathcal{R}(A) \subseteq \mathcal{P}(A)$. In the examples below we see that either or both of these inclusions can be equality. In all the examples below, the multiplication is commutative and hence we need consider only the operation $a\backslash b$.

6.1. THERE ARE NO NON-TRIVIAL INJECTIVES IN THE CATEGORY OF PO-MONOIDS AND MULTIPLICATIVE ORDER-PRESERVING FUNCTIONS. Suppose Q was injective in that category. Then if B is any discrete monoid and $A \hookrightarrow B$ a submonoid, the inclusion ϕ is an embedding in our sense since $\phi(a_1)\phi(a_2)\cdots\phi(a_n) = \phi(a)$ if and only if $a_1a_2\cdots a_n = a$. Thus any map $A \longrightarrow Q$ would have to extend to a map on B and we conclude that the discrete monoid $|Q|$ underlying Q would be injective with respect to all inclusions in the category of monoids. We will now show that Q cannot have more than one element.

First observe from the inclusion $\mathbf{N} \hookrightarrow \mathbf{Z}$ that Q must in fact be a group and hence injective in the category of groups. It is known (see [Eilenberg & Moore (1965)]) that there are no non-trivial injective groups, but we will sketch an easier argument that there are no non-trivial injectives for monoids. For let A be $\mathbf{N} \cup \{\infty\}$ with the monoid structure in which $n\infty = \infty n = \infty$ for all $n \in A$. Clearly, the injection $\mathbf{N} \hookrightarrow A$ cannot be extended across arbitrary maps $\mathbf{N} \longrightarrow Q$.

6.2. The case of a total order. If A is totally ordered then so is $\mathcal{P}(A)$. For if I and J are down-closed subsets and there are elements $a \in I - J$ and $b \in J - I$, neither $a \leq b$ nor $b \leq a$ is possible. Since $\mathcal{Q}(A)$ is also ordered by inclusion, the same is true for that object. Thus $\mathcal{Q}(A)$ is an injective hull for A in the category of totally ordered monoids.

6.3. It can happen that $\mathcal{Q}(A) = \mathcal{R}(A) = \mathcal{P}(A)$. Let A be the set of positive integers with the usual multiplication and usual order. As just observed, $\mathcal{P}(A)$ is a total order. It consists of the empty set we will denote $0{\downarrow}$, all the $a{\downarrow}$, and all of A, which we will denote $\infty{\downarrow}$. Then $\mathcal{P}(A) = \{0{\downarrow} < 1{\downarrow} < 2{\downarrow} < \cdots < \infty{\downarrow}\}$. One easily sees that $a{\downarrow}\backslash b{\downarrow} = \lfloor b/a \rfloor{\downarrow}$, the downset of the floor of b/a. From this it is easy to see that every element of $\mathcal{P}(A)$ is closed. In particular, $2{\downarrow}\backslash 1{\downarrow} = 0{\downarrow}$ so that the $0{\downarrow}$ is closed while $\mathrm{cl}(\infty{\downarrow})$ is the empty inf, that is $\infty{\downarrow}$.

6.4. We can have $\mathcal{Q}(A)$ be much smaller than $\mathcal{R}(A)$. Let A be the set of positive integers with the discrete order (no two distinct elements comparable) and the usual multiplication. Every ideal is down-closed and so $\mathcal{P}(A)$ is just the power set of A. For $a \in A$, the set $a{\downarrow} = \{a\}$. If $a, b \in A$, the set $a{\downarrow}\backslash b{\downarrow}$ is empty unless $a|b$, in which case it is $\{a/b\}$. It follows that the only closed sets are the empty set, all singletons, and A itself. If we denote the empty set by 0 and A by ∞, then we can think of $\mathcal{Q}(A)$ as the set $\{0 < 1, 2, 3, \ldots < \infty\}$. Here we are denoting $\{a\}$ by a. We see in what sense $\mathcal{Q}(A)$ is "close" to A, while $\mathcal{P}(A)$ is not. Since there are no non-trivial sups, every ideal is closed under distributive sups so $\mathcal{R}(A) = \mathcal{P}(A)$. In this example we see that $\mathcal{R}(A)$ is not the injective hull of A.

6.5. $\mathcal{Q}(A) = \mathcal{R}(A)$ can be much larger than A and nearly as large as $\mathcal{P}(A)$. Take A to be the positive rational numbers with the usual order and multiplication. Then a down-closed set is nearly the same thing as a cut. The empty set and A itself are down-closed and we also have to distinguish between $a{\downarrow}$ and what we will call $a^< = \{b \mid b < a\}$. The following facts are easy to verify:

1. $a{\downarrow}\backslash b{\downarrow} = (b/a){\downarrow}$.
2. $\mathrm{cl}(a^<) = a{\downarrow}$.
3. Irrational cuts are closed and also sup-closed.

Thus all cuts (including $\emptyset$ and A) are closed except those of the form $a^<$, for $a \in A$. It readily follows that $\mathcal{Q}(A)$ can be identified as the one-point compactification of the non-negative reals, while in $\mathcal{P}(A)$, the cuts $a{\downarrow} \neq a^<$. In both $\mathcal{Q}(A)$ and $\mathcal{P}(A)$, the monoid structure is such that 0 times anything is 0, while ∞ times any non-zero element is ∞. Since all sups in A are distributive, $\mathcal{R}(A)$ contains only the sup-closed sets, that is the sets in $\mathcal{Q}(A)$.

6.6. $\mathcal{R}(A)$ CAN LIE STRICTLY BETWEEN $\mathcal{Q}(A)$ AND $\mathcal{P}(A)$. Let $A = \mathbf{Q}^+ \times \mathbf{Z}_2$ where $\mathbf{Q}^+$ is the positive rationals as above and $\mathbf{Z}_2$ is the 2-element group with the discrete order. Then one easily sees that the elements of $\mathcal{Q}(A)$ and $\mathcal{R}(A)$ do not include any element of the form $a^< \times \{z\}$, for $z \in \mathbf{Z}_2$, so that $\mathcal{R}(A) \neq \mathcal{P}(A)$, while $\mathcal{Q}(A)$ does not contain any element of the form $a{\downarrow} \times \mathbf{Z}_2$, all of which belong to $\mathcal{R}(A)$ so that $\mathcal{Q}(A) \neq \mathcal{R}(A)$.

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