WHAT IS THE WORLD OF MATHEMATICS?

J. LAMBEK, MCGILL UNIVERSITY

November 2001

Abstract. It may be argued that the language of mathematics is *about the* category of sets, although the definite article requires some justification. As possible worlds of mathematics we may admit all models of type theory, by which we mean all *local toposes*. For an intuitionist, there is a distinguished local topos, namely the so-called *free topos*, which may be constructed as the Tarski-Lindenbaum category of intuitionistic type theory. However, for a classical mathematician, to pick a distinguished model may be as difficult as to define the notion of truth in classical type theory, which Tarski has shown to be impossible.

By 'mathematics' we shall here mean *elementary* mathematics, to include arithmetic and analysis, not metamathematics and category theory, if the former is to be adequate for a proof of Gödel's completeness theorem and the latter for the so-called Yoneda embedding of small categories into functor categories.

We aim to address the question: what is *the world* of mathematics? In particular, we wish to discuss whether the definite article is justified. Mathematicians are by no means in agreement on how this question is to be answered, yet there seems to be a general consensus on the *language* of mathematics, of which the world of mathematics is to be a model. Ever since Cantor and Frege, most mathematicians have agreed that this is some form of *set theory*, although there have been dissenting voices, notably those of Kronecker and Poincaré.

There is less agreement as to the precise form of the language of set theory. Logicists believe that the natural numbers can be *defined*, but at the price of an *axiom of infinity*. This axiom can be avoided if the numeral zero and the successor operation are incorporated into the language to start with, following Peano.

Frege's original attempt was based on his *comprehension scheme*, which asserted that for each formula $\varphi(x)$, x being a free variable, one could construct an entity A such that, for all x, x belongs to A if and only if $\varphi(x)$. Unfortunately, this led to Russell's paradox when one took $\varphi(x)$ to be the formula 'x does not belong to x'. To avoid this, precautions have to be taken, and this is done in a number of proposed languages.

On the whole, mathematicians prefer to work in *Gödel-Bernays* set theory, which distinguishes between small *sets* and large *classes*. In this language one can only say that x belongs to A, in symbols $x \in A$, if x denotes a set and A a class. Logicians generally

I wish to thank Elaine Landry for organizing a symposium at Brock University in 1996, at which an earlier version of this paper was presented, and Phil Scott for helpful discussions. Support from the Social Sciences and Humanities Research Council of Canada is herewith acknowledged.

2010 Mathematics Subject Classification: 03B15, 03G30, 18B05, 18B25..

Key words and phrases: type theory, categorical logic, category of sets, topos theory..

prefer an essentially equivalent first-order language, *Zermelo-Fraenkel* set theory, where the comprehension scheme is restricted to those x which belong to a previously given set. Some philosophers may prefer another formulation of set theory, due to Quine, which restores the legitimacy of the universal set.

However, Russell and Whitehead had previously elaborated a *theory of types*, which only admitted the formula $x \in A$ if A was of an appropriate *higher type* than x. Unfortunately, their original formulation of type theory was too cumbersome to appeal to a wide audience, and type theory did not catch on, even though Church and Henkin gave more elegant formulations later.

Phil Scott and the present author made use of another formulation of type theory, in connection with category theory. It was based on the following *types*:

$$1,\ \Omega,\ N,\ \mathcal{P}(A),\ A \times B\ ,$$

where A and B are given types, Ω is the type of *truth-values* (or *propositions*), N is the type of *natural numbers*, $\mathcal{P}(A)$ is the type of *sets* of entities of type A and $A \times B$ is the type of *pairs* of entities of types A and B respectively. The type 1 is not really necessary, being only introduced for convenience; it is supposed to be the type of a distinguished entity, often denoted by $*$.

In the formal language of type theory we also admit countably many variables of each type, as well as the following *terms* (in addition to the variables):

$$\begin{array}{rll} * & \text{of type} & 1\ , \\ a = a',\ a \in \alpha & \text{of type} & \Omega\ , \\ 0,\ Sn & \text{of type} & N\ , \\ \{x \in A | \varphi(x)\} & \text{of type} & \mathcal{P}(A)\ , \\ (a, b) & \text{of type} & A \times B\ , \end{array}$$

where $a, a', \alpha, n, \varphi(x)$ and b are assumed to be of types $A, A, \mathcal{P}(A), N, \Omega$ and B respectively.

In this language, we can define logical symbols as follows, assuming p, q and $\varphi(x)$ to be of type Ω:

$$\begin{array}{rll} \top & \text{as} & * = * \\ p \wedge q & \text{as} & (p, q) = (\top, \top)\ , \\ p \Rightarrow q & \text{as} & p \wedge q = p\ , \\ \forall_{x \in A}\varphi(x) & \text{as} & \{x \in A | \varphi(x)\} = \{x \in A | \top\}\ , \\ p \vee q & \text{as} & \forall_{t \in \Omega}(((p \Rightarrow t) \wedge (q \Rightarrow t)) \Rightarrow t)\ , \\ \exists_{x \in A}\varphi(x) & \text{as} & \forall_{tt\Omega}(\forall_{x \in A}(\varphi(x) \Rightarrow t) \Rightarrow t)\ , \\ \neg p & \text{as} & \forall_{t \in \Omega}(p \Rightarrow t)\ , \\ \bot & \text{as} & \forall_{t \in \Omega} t\ . \end{array}$$

In such a language, one must also specify the set of *theorems*, as is usually done with the help of a deduction symbol $\vdash$. One writes a *deduction* as follows:

$$p_1, p_2, \cdots, p_n \vdash_X p_{n+1}\ ,$$

where the p_i are terms of type Ω, also called *formulas*, and X is a set of variables containing all the variables occurring freely in the p_i. This is supposed to mean that p_{n+1} is deducible from $p_1, p_2, \cdots, p_n$ and this meaning is laid down by a set of *axioms* and *rules of inference*, which we shall not spell out here. They contain no surprises; the reader interested in the details is referred to Lambek and Scott [1986]. In case $n = 0$ and X is empty, we just write $\vdash p_1$ and call p_1 a *theorem*.

Any formal language $\mathcal{L}$ satisfying these conditions on types, terms and theorems is called *a type theory*. For *pure intuitionistic type theory* $\mathcal{L}_0$ we require that types, terms and theorems are freely generated. This means: (a) there are no types or terms other than those prescribed as above and there are no non-trivial identifications between types or terms, so that, in particular, the types form what Russell calls a *hierarchy*; (b) there are no theorems other than those which follow from the axioms and rules of inference required for intuitionistic arithmetic, so that the set of theorems is recursively enumerable.

Pure classical type theory $\mathcal{L}_1$ is defined likewise, except that classical type theories also require the axiom:

$$\forall_{t\in\Omega}(\neg\neg t \Rightarrow t)$$

or, equivalently,

$$\forall_{t\in\Omega}(t \vee \neg t),$$

the *axiom of the excluded third*.

Type theories form a category if one introduces suitable arrows $\mathcal{L} \to \mathcal{L}'$ between them, called *translations*, which we shall not spell out here. Then $\mathcal{L}_0$ becomes an initial object in this category and $\mathcal{L}_1$ an initial object in the subcategory of classical type theories.

Moderate intuitionists, who agree to a formalized language at all, may accept $\mathcal{L}_0$ as *the* language of mathematics. Classical mathematicians may conceivably accept $\mathcal{L}_1$ in this capacity, but they are more likely to insist on at least one further axiom scheme, the *rule of choice*, really a collection of axioms, one for each type.

Now let us return to our question: what is the world of mathematics? An extremist adherent of the formalist school might claim that there is no such world, whereas other mathematicians may maintain that there are many such worlds. But, according to Plato, there should be a distinguished world, of which others are imperfect copies. The way Leibniz might put it, this distinguished world should be "the best of all possible worlds".

Most mathematicians would agree that *the* world of mathematics is *the category of sets*; but the same doubt attaches to the definite article here. For example, does the category of sets contain the union, or direct limit, of the sets $\mathbf{N}, \mathcal{P}(\mathbf{N}), \mathcal{P}(\mathcal{P}(\mathbf{N})), \cdots$? Does it satisfy the continuum hypothesis, which asserts that there is no set larger than $\mathbf{N}$ yet smaller than $\mathcal{P}(\mathbf{N})$?

Bill Lawvere has given an axiomatic description of those categories which resemble our idea of the category of sets, they are called (elementary) toposes and generalize the toposes introduced by Grothendieck into algebraic geometry. (Some of Lawvere's work was in collaboration with Myles Tierney.)

Without becoming too technical, let us say here that a *topos* is a kind of category in which one can imitate the following properties expected of the category of sets: there should be a *terminal* object 1 corresponding to the one-element set$\{*\}$; one should be able to form the *cartesian product* $A \times B$ of two objects and the *exponential* object C^B of functions from B to C; there should be an object of *truth-values* Ω with a distinguished element $\top$ and a one-to-one correspondence between subobjects B of A and their *characteristic functions* $\chi_B : A \to \Omega$ such that, for any element a of A, a belongs to B if and only if $\chi_B(a) = \top$. Finally, there should be an object of *natural numbers* N, equipped with an element $0 \in N$ and a function $S : N \to N$ allowing definition by *recursion*: for each element a of A and each function $h : A \to A$, one may construct a unique function $f : N \to A$ such that $f(n) = h^n(a)$ for all ordinary natural numbers n.

As Lawvere observed, all these properties can be expressed in the language of category theory, without mentioning *elements*. For example, $\top$ is then seen as an arrow $1 \to \Omega$ and 0 as an arrow $1 \to N$.

Let us now look at the relation between toposes and type theories.

With each topos $\mathcal{T}$ there is associated a type theory $L(\mathcal{T})$, its *internal language*: the types are the objects of $\mathcal{T}$, the closed terms of type A are the arrows $1 \to A$ in $\mathcal{T}$ and the theorems are arrows $p : 1 \to \Omega$ which are *equal* to $\top$ in $\mathcal{T}$. Instead of saying that $\vdash p$ in $L(\mathcal{T})$ we also say that p is *true* in $\mathcal{T}$.

Categorists were somewhat surprised when it turned out that, in general, $L(\mathcal{T})$ is not classical but intuitionistic. In particular, there is no reason why there should be exactly two truth-values, that is, arrows $1 \to \Omega$. Of course, we cannot expect $L(\mathcal{T})$ to be *pure* (see above); it will inherit constraints from $\mathcal{T}$.

Conversely, with each type theory (language) $\mathcal{L}$ there is associated a topos $\top(\mathcal{L})$, the *topos generated* by $\mathcal{L}$, also called the *Tarski-Lindenbaum* category of $\mathcal{L}$. Its objects are names of sets modulo synonymity, that is, terms α of type $\mathcal{P}(A)$ for some type A, such that α is identified with α' of the same type if $\vdash \alpha = \alpha'$. Its arrows are names of binary relations which can be proved to be functions; thus an arrow $\alpha \to \beta$, where β is of type $\mathcal{P}(B)$, say, is a closed term ρ of type $\mathcal{P}(B \times A)$ such that

$$\vdash \forall_{x \in A}(x \in \alpha \Rightarrow \exists!_{y \in B}(y \in \beta \wedge (x, y) \in \rho),$$

where $\exists!$ denotes unique existence. Again, ρ should be identified with $\rho' : \alpha \to \beta$ if $\vdash \rho = \rho'$.

Toposes themselves are the objects of a category, whose arrows $\mathcal{T} \to \mathcal{T}'$ are functors preserving the logical structure, they are called *logical morphisms*. As was shown by Lambek and Scott [1986], given any type theory $\mathcal{L}$ and any topos $\mathcal{T}$, there is a one-to-one correspondence between translations $\mathcal{L} \to L(\mathcal{T})$ and logical morphisms $T(\mathcal{L}) \to \mathcal{T}$, making L and T what categorists call a pair of *adjoint functors* between the category of type theories and the category of toposes.

By an *interpretation* of $\mathcal{L}$ in $\mathcal{T}$ is meant a translation $\mathcal{L} \to L(\mathcal{T})$ or, equivalently, a logical morphism $T(\mathcal{L}) \to \mathcal{T}$. It turns out that the translation $\mathcal{L} \to LT(\mathcal{L})$, corresponding to the identity functor $T(\mathcal{L}) \to T(\mathcal{L})$, is what logicians call a *conservative extension*, meaning that no closed formula of $\mathcal{L}$ becomes a theorem of $LT(\mathcal{L})$ unless it already was a theorem of $\mathcal{L}$. On the other hand, $TL(\mathcal{T}) \to \mathcal{T}$, corresponding to the identity translation $L(\mathcal{T}) \to L(\mathcal{T})$, is what categorists call an *equivalence* of categories.

We also note that $LT(\mathcal{L})$ has more types and terms, even if not more theorems, than $\mathcal{L}$: any term of type $\mathcal{P}(A)$ in $\mathcal{L}$ becomes a type in $LT(\mathcal{L})$ and, whenever $\vdash \exists!_{x\in A}\varphi(x)$ in $\mathcal{L}$, there will be a term a in $LT(\mathcal{L})$ such that $\vdash \hat{\varphi}(a)$ in $LT(\mathcal{L})$, where $\hat{\varphi}$ is the image of φ under the interpretation $\mathcal{L} \to LT(\mathcal{L})$. Thus we have a categorical version of Russell's *theory of description*: we may think of a as denoting the unique x such that $\hat{\varphi}(x)$.

It is easily seen that, if p and q are closed formulas of $\mathcal{L} = L(\mathcal{T})$, then

(i) $p \wedge q$ is true in $\mathcal{T}$ if and only if p is true in $\mathcal{T}$ and q is true in $\mathcal{T}$.

(ii) Of course, we also know that $\top$ is true in $\mathcal{T}$.

We are therefore justified in saying that the logical symbols $\top$ and $\wedge$ in the language $L(\mathcal{T})$ mean exactly what they are supposed to mean, namely *true* and *and*.

Unfortunately, other logical symbols do not, in general, have the expected meaning; we say that a topos is *local* if the symbols $\bot, \vee$ and $\exists$ do:

(iii) $\bot$ *is not true in* $\mathcal{T}$;

(iv) $p \vee q$ *is true in* $\mathcal{T}$ *if and only if* p *is true in* $\mathcal{T}$ *or* q *is true in* $\mathcal{T}$;

(v) $\exists_{x\in A}\varphi(x)$ *is true in* $\mathcal{T}$ *if and only if* $\varphi(a)$ *is true in* $\mathcal{T}$ *for some term* a *of type* A *in* $L(\mathcal{T})$.

An example of a local topos is the so-called *free topos* $T(\mathcal{L}_0)$, the topos generated by pure intuitionistic type theory. This is not at all obvious and for a proof the reader is referred to Lambek and Scott [1986]. The suggestion was made there, and defended by Couture and Lambek [1991] and Lambek [1994], that the free topos is a suitable candidate for *the* world of mathematics acceptable to members of different philosophical schools, who do not insist on the principle of the excluded middle and who are willing to compromise: (a) to moderate Platonists, because it is an initial object in the category of all toposes; (b) to moderate formalists or even nominalists, because it may be constructed from words, as in the general construction of $\top(\mathcal{L})$ above; (c) to moderate intuitionists, because $\mathcal{L}_0$ is an intuitionistic type theory and properties (iv) and (v) embody constructivist principles.

Needless to say, such a compromise will be rejected by extreme Platonists, who may believe that mathematical entities are thoughts in the mind of a demiurge, by extreme nominalists, who may believe that only words are real, but that equivalence classes of synonymous words are not, and by extreme intuitionists, who may believe that infinite sets do not exist or that truth varies with historical time.[1]

Another nice property of the free topos is that all numerals in its internal language are *standard*, that is, all closed terms of type N have the form $S^n 0$ for some ordinary natural number n. Thus numerals too have the expected meaning. Unfortunately, the logical symbols $\neg, \Rightarrow$ and $\forall$ do not, as follows from Gödel's incompleteness theorem (see below).

A topos $\mathcal{T}$ is called *Boolean* if its internal language is classical, that is, if the principle of the excluded third holds in $\mathcal{T}$, equivalently, if $\forall_{t\in\Omega}(\neg\neg t \Rightarrow t)$ is true in $\mathcal{T}$. In a Boolean

[1]See also the Concluding Remarks and Footnote 4.

local topos *all* logical symbols have the expected meaning. Thus, in addition to (i) to (v), a Boolean local topos also satisfies:

(vi) $\neg p$ *is true in* $\mathcal{T}$ *if and only if* p *is not true in* $\mathcal{T}$;

(vii) $p \Rightarrow q$ *is true in* $\mathcal{T}$ *if and only if, if* p *is true in* $\mathcal{T}$, *then* q *is true in* $\mathcal{T}$;

(viii) $\forall_{x\in A}\varphi(x)$ *is true in* $\mathcal{T}$ *if and only if* $\varphi(a)$ *is true in* $\mathcal{T}$ *for all terms* a *of type* A *in* $L(\mathcal{T})$.

In view of (v) and (viii), the *substitutional interpretation* of quantifiers, discussed by Russell and Kripke [1976], holds for the internal language of a Boolean local topos.

Note that any non-trivial well-pointed topos is a Boolean local topos, see McLarty [1992]. Here *non-trivial* means that the terminal object is not initial and *well-pointed* means that it is a *generator*, in the sense that two arrows $f, g : A \to B$ will be equal if $fa = ga$ for all $a : 1 \to A$. Actually, *non-trivial* is equivalent to (iii) and *well-pointed* to (viii).

It had been suggested earlier that the free topos $T(\mathcal{L}_0)$ may be a suitable candidate for *the* world of mathematics, acceptable to moderate intuitionists, formalists and Platonists alike, provided only that the last two are willing to forget the principle of the excluded third. Unfortunately, most mathematicians refuse to do so and insist on working with classical logic.

To a classical mathematician, *a* world of mathematics should presumably be a Boolean local topos and *the* world of mathematics a distinguished one. In view of Gödel's completeness theorem (see below), applied to pure classical type theory $\mathcal{L}_1$, there exist plenty of Boolean local toposes. Unfortunately, the *free Boolean topos*, namely the topos $T(\mathcal{L}_1)$ generated by $\mathcal{L}_1$, is not local, in view of Gödel's incompleteness theorem: if g is Gödel's undecidable statement, then $g \vee \neg g$ is true in $T(\mathcal{L}_1)$ but neither g nor $\neg g$ are. Thus, since the usual proof of the completeness theorem is non-constructive, it becomes desirable to exhibit a Boolean local topos or to show that none can be constructed.

Before stating Gödel's theorems in a categorical context, we must define the notion of a *model* for type theory or higher order logic. At first sight it might seem that we should regard as a model of $\mathcal{L}$ an interpretation of $\mathcal{L}$ in *any* topos $\mathcal{T}$ whatsoever. Since $\mathcal{L} \to LT(\mathcal{L})$ is a conservative extension, this single interpretation would suffice to decide provability in $\mathcal{L}$ by looking at truth in $T(\mathcal{L})$, making the completeness theorem quite trivial. However, models had previously been defined for classical type theories by Leon Henkin more narrowly (see Hintikka [1969]), and his definition was extended in Lambek and Scott [1986] to intuitionistic type theories: a *model* of $\mathcal{L}$ is an interpretation of $\mathcal{L}$ in a local topos. Henkin models then are essentially Boolean local toposes. Gödel's completeness theorem now asserts less trivially:

A closed formula in a type theory $\mathcal{L}$ *is a theorem if and only if it becomes true in every model of* $\mathcal{L}$.

To state Gödel's incompleteness theorem, we need another definition. A topos $\mathcal{T}$ (and a model $\mathcal{L} \to L(\mathcal{T})$) is said to be ω-*complete* if the truth of $\varphi(S^n 0)$ for all ordinary natural numbers n implies the truth of $\forall_{x\in N}\varphi(x)$. For a Boolean local topos, ω-completeness is equivalent to the statement that all numerals are standard. The incompleteness theorem now asserts:

Assuming that $\mathcal{L}$ is consistent, meaning that not all closed formulas are theorems, *and that the set of theorems is recursively enumerable, then it does not suffice to look at ω-complete models only.*

Gödel exhibited a formula $\varphi(x)$, where x is a free variable of type N, such that $\varphi(S^n 0)$ is a theorem for all ordinary natural numbers n, hence $\forall_{x \in N} \varphi(x)$ is true in any ω-complete model, yet is not a theorem of $\mathcal{L}$. Gödel was a Platonist, believing in a *real* world of mathematics which he assumes to be ω-complete, and he claimed to have shown the existence of *true* statements (i.e. true in the real world) which are not provable. But what is the real world if not a distinguished local topos? When it comes to pure intuitionistic type theory $\mathcal{L}_0$, such a model can be constructed, namely the free topos.[2)] But what about a distinguished model of pure classical type theory $\mathcal{L}_1$?

Let us now return to the question of whether a Boolean local topos can be constructed. If this were the case, we would have a classical model $\mathcal{L}_1 \to L(\mathcal{T})$ and this would enable us to define *truth* in the language $\mathcal{L}_1$ to mean truth in the topos $\mathcal{T}$. Conversely, if we could define truth in $\mathcal{L}_1$, by adjoining the set θ of all true formulas as axioms, we would obtain a new language $\mathcal{L}_1/\theta$ and we would be able to construct the Boolean (though not yet local) topos $T(\mathcal{L}_1/\theta)$. Thus, the problem of constructing a Boolean local topos is related to the problem of defining truth in $\mathcal{L}_1$.

What is truth? This question was raised by one Pontius Pilatus two thousand years ago and answered only recently by Alfred Tarski [1956]. He showed that truth in languages such as $\mathcal{L}_1$ cannot be defined, provided the word 'defined' is formally interpreted in a certain way. If Tarski's notion of 'defined' correctly captures the informal notion of this word corresponding to our informal notion of 'constructed', then indeed we may conclude that a distinguished local topos cannot be constructed and that it makes no sense for a classical mathematician to speak of *the* category of sets or *the* world of mathematics.

Without a distinguished model, we are thus led to consider the totality of all possible worlds, or all models of $\mathcal{L}_1$, simultaneously. But first let us make an algebraic analogy.

It had been noticed by Peter Freyd that for a topos to be local could be expressed algebraically thus: its terminal object is not initial and it is an *indecomposable projective.* Gödel's completeness theorem may also be expressed algebraically: every topos is contained (up to isomorphism) in a product of local toposes. A similar theorem holds in commutative algebra: every commutative ring is contained in a product of local rings. It is not a complete accident that the word 'local' appears in both contexts. The choice of this word in connections with toposes seems to have originated with Alexander Grothendieck, who also found that the assertion about rings can be sharpened as follows:

Every commutative ring is the ring of continuous sections of a sheaf of local rings.

As Bill Lawvere [1975] might put it less formally:

Every commutative ring is a continuously variable local ring.

One would like to prove a similar theorem for toposes. In fact, a completely analogous result holds for toposes which obey the rule of choice (see Lambek and Scott [1986]) and even for arbitrary toposes (Awodey [2000]): [2]

Every topos is equivalent to the topos of continuous sections of a sheaf of local toposes.

It is tempting to abandon the search for a distinguished Boolean local topos and be satisfied with the sheaf of Boolean local toposes instead. But, if we ask in which world this sheaf lives, we are back at square 1.

Concluding remarks.

I have tried to show that one cannot construct a distinguished world of mathematics to satisfy a classical mathematician, who thus would reject a Platonic world and might accept a sheaf of possible worlds instead.

On the other hand, it has been suggested by Lambek [1994] that there is a constructible candidate for such a Platonic world, namely the free topos, that ought to please moderate Platonists, formalists and intuitionists, even if not logicists. However, I doubt whether most Platonists, formalists and intuitionists would accept this proposal, which, for this reason, was called '*constructive nominalism*' in Couture and Lambek [1991] to distinguish it from the views advocated by the traditional schools.

Followers of the first two schools would certainly reject it, if only because they are committed to the principle of the excluded third. Platonists will have other objections, perhaps of a quasi-mystical nature, and formalists might not be happy with the necessity of introducing equivalence classes of formal expressions.

Intuitionists appear to be divided.[3] Many follow Brouwer in rejecting Cantorian sets altogether, replacing them by *spreads* (see Van Dalen [1999]), hence would not even accept pure intuitionistic type theory $\mathcal{L}_0$. Others are inclined to accept $\mathcal{L}_0$ and are pleased with the free topos $T(\mathcal{L}_0)$ for exhibiting a number of intuitionistic principles, e.g. that any true existential statement can be witnessed by a concrete example. On the other hand, some intuitionists see a problem with the notion of truth, here identified with provability.

Gödel, in proving his incompleteness theorems produced a non-provable formula of the form $\forall_{x\in N}\varphi(x)$, even though all instances $\varphi(S^n0)$ are provable. According to the so-called *Brouwer-Heyting-Kolmogorov interpretation* of intuitionistic logic, there is no reason for $\forall_{x\in N}\varphi(x)$ to be true unless the truths of all instances $\varphi(S^n0)$ have been established in a *uniform* manner; for example, the proofs of $\varphi(S^n0)$ might increase in length as n increases. However, some intuitionists would insist that truth is language-independent and that the truth of $\forall_{x\in N}\varphi(x)$ has been established by Gödel metamathematically by showing that each $\varphi(S^n0)$ is true. These intuitionists are committed to ω-completeness and would

[2] I had conceived such a result and presented it at the Brouwer conference in Amsterdam, giving rise to a joint article with Iecke Moerdijk [1982], who had pointed out that I had used an awkward topology. We there used the word "local" in presence of the disjunction property only, but the existence property held automatically for toposes with the rule of choice (see also Lambek and Scott [1986]). I later [1989] used "local" in the present sense, but had to adjoin a sufficiently large set of so-called *Henkin constants*. Awodey [2000] managed to avoid these constants, but used "completely local" for what is here called "local".

[3] I have been told that present day intuitionists even in Holland are divided. Those in Amsterdam and Utrecht might accept the proposal compromise, while those in Nijmegon, though not extremists, would be inclined to reject it.

reject the free topos for not being ω-complete.

Let me point out, however, that the free topos is ω^*-*complete* in the following sense:

$$\textit{if } \exists_{x \in N}\varphi(x) \textit{ is true, then so is } \varphi(S^n 0) \textit{ for some } n \in \mathbf{N}.$$

This principle is equivalent to ω-completeness classically, but not intuitionistically. One should note that for $\mathcal{L}_0$ substitutional interpretation holds for existential, but not for universal quantifiers!

Now what about logicists who are not committed to the principle of the excluded third? Their first objection might be that the language $\mathcal{L}_0$, as formulated above, does not start with logical symbols as given, but treats them as defined. This objection can easily be overcome by formulating $\mathcal{L}_0$ differently, beginning with the logical symbols representing truth, conjunction, implication and universal quantification, as well as the symbol of membership. There remains the objection that the natural numbers are not definable in either presentation, but must be postulated à la Peano or, equivalently, derived from a so-called *axiom of infinity*. To overcome this objection we would have to replace $\mathcal{L}_0$ by a more powerful language, such as Quine's *New Foundations* or *polymorphic type theory*, which allows quantification over variable types. But then we have no ready-made models like Lawvere's elementary toposes. One would have to adjoin formal products of variable objects to a topos.

Finally, let me address the question: what rôle should category theory play in the foundations of mathematics? I recall that, at the 1963 meeting devoted to Logic, Methodology and Philosophy of Science in Jerusalem, Bill Lawvere proposed basing mathematics on categories rather than sets. Alfred Tarski, who was in the audience, objected: what is a category if not a set of objects and a set of arrows? Lawvere replied: set theory deals with the binary relation of membership, category theory with the ternary relation of composition. Apparently, Tarski was satisfied with the answer.

In my above account, I have presented the language of mathematics in traditional non-categorical terms, but its models as certain categories, namely local toposes. Some categoraphobes have asked whether they are to be excluded from understanding the foundations of mathematics. I apologize to them for describing the models as categories; after all, Leon Henkin managed to describe the models of classical type theory without mentioning categories and the same could be done for intuitionistic type theory. However, the categorical description is so much more concise and allows one to say in one page what otherwise might take ten.

References

S. Awodey, *Sheaf representation for topoi*, J. Pure and applied algebra, 145 (2000), 107-121.

J. Couture and J. Lambek, "Philosophical reflections on the foundations of mathematics", *Erkenntnis* 34 (1991), 187-209.

J. Hintikka, *The philosophy of mathematics*, Oxford University Press, Oxford 1969.

S. Kleene, *Introduction to metamathematics*, Van Nostrand, New York 1952.

S. Kripke, "Is there a problem about substitutional quantification?" in: Evans and MacDowell (eds.), *Truth and meaning*, Clarendon Press, Oxford 1976, 325-419.

J. Lambek, "On the sheaf of possible worlds", in: Adamek and Mac Lane (eds.), *Categorical topology*, World Scientific, Singapore 1989.

J. Lambek, "Are the traditional philosophies of mathematics really incompatible?" *The Mathematical Intelligencer* 16 (1994), 56-62.

J. Lambek and I. Moerdijk, "Two sheaf representations of elementary toposes, in: A.S. Troelstra and D. van Dalen (eds), *The L.E.J. Brouwer Centenary Symposium*, Studies in Logic and the Foundations of Mathematics 110, North-Holland Publishing Company, Amsterdam 1982.

J. Lambek and P.J. Scott, *Introduction to higher order categorical logic*, Cambridge University Press, Cambridge 1986.

F.W. Lawvere, "Continuously variable sets: algebraic geometry = algebraic logic", in: Rose and Sheperdson (eds.), *Proc. A.S.L. Logic Colloquium, Bristol 1974*, North Holland, Amsterdam 1975, 135-156.

C. McLarty, *Elementary categories, elementary toposes*, Clarendon Press, Oxford 1992.

A. Tarski, *Logic, Semantics, Metamathematics*, Clarendon Press, Oxford 1956.

D. van Dalen, Mystic, Geometer and Intuitionist: the life of L.E.J. Brouwer, Vol. 1, Clarendon Press, Oxford 1999.

J. van Heijenoort (ed.), *From Frege to Gödel*, Harvard University Press, Cambridge MA 1967.