Discrete versus continuous
and the radical approach to infinitesimals.

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Montreal, 2007.

Already the ancient Greeks were intrigued by the opposition between the discrete and the continuous, as well as by that between rest and motion. To resolve the latter contradiction, the idea of infinitely small stretches of space and time seemed to be necessary and, in the seventeenth century, led Cavalieri, Newton and Leibniz to resort to infinitesimals, which were supposed to be infinitely small without being zero. Many attempts have been made to justify this concept. Of special interest is the proposal by Lawvere and Kock to interpret infinitesimals as nilpotent elements in a non-Aristotelian local ring, where they turned out to be not unequal to zero. (Already, Charles Sanders Peirce had observed that the admission of infinitesimals would require the abandonment of the principle of the excluded third.) More generally, Jacques Penon suggested that one admit all elements not unequal to zero as infinitesimals, even in a non-Aristotelian field. While the nilpotent elements of a commutative ring form its *prime radical*, the Penon infinitesimals turn out to constitute the Jacobian radical of a field, or even of a division ring. Moreover, one can prove that, in a *normed* division ring they are precisely the elements which are infinitely small.

1. Everything is water (Thales).

We say "much water" and "many beans". Beans can be counted, but water must be measured. Much of pre-Socratic Greek philosophy was concerned with the question: do things in nature consist of substances to be measured or of indivisible pieces to be counted? On one side of the debate we find Thales, Anaximenes and Empedocles (who was the first to realize that air is a substance). On the other side we find Pythagoras (even though he did not like beans), Democritus and Epicurus. Should we side with the olive oil speculator Thales and say that nature is basically continuous or with the elitist politician and mystic Pythagoras and take nature to be ultimately discrete?

Modern physicists are still in two minds about this question. Fundamental particles, both fermions and bosons, are discrete, but the space-time in which they live is continuous, and so is the probability of their being at any particular location. There may be dissenting voices about the continuity of space-time, but not about that of probability, where real numbers have even been replaced by complex ones.

Parallel to this basic question in what we now call "physics", the ancients also posed one in the foundations of mathematics. Which are more fundamental, the (positive) integers or the (positive) reals, which the Greeks conceived as ratios of geometric quantities? (Zero and negative numbers were only studied systematically in India, around 600 A.D., by Brahmagupta.)

The Pythagoreans discovered to their dismay that the diagonal of a unit square could not be expressed as the ratio of two positive integers, hence what we now call "irrational numbers" had to be admitted. Still, it was soon realized that the positive reals could be constructed from the positive integers.

In fact, there were two ways of doing this, both discussed by Euclid, the Bourbaki of his time.[1] Eudoxus discovered what we now call "Dedekind cuts", while Theaetetus favoured what we now call "continued fractions". The latter approach was forgotten until the seventeenth century and was ultimately absorbed into the modern definition of real numbers as equivalence classes of Cauchy sequences.

The rôle of continued fractions (anthyphareises) at Plato's Academy is now widely recognized (see Fowler [1987]). According to Stelios Negrepontis [2001], traces of these can be found in all the dialectic dialogues of Plato, either literally, as in Meno, or metaphorically elsewhere.

Michael Makkai [2007] has recently observed that positive real numbers could also have been defined by sums of unit fractions, which had been introduced by the ancient Egyptians and were still favoured by Archimedes. In fact, Makkai proved that every real number between 0 and 1 can be written uniquely as a finite or infinite sum of unit fractions $1/p_n$ such that

$$p_{n+1} \geq p_n(p_n - 1) + 1.$$

It seems the ancients missed an opportunity here.

2. You cannot step into the same river twice (Heraclitus).

Another problem of intense debate among the ancients was that of motion versus change. Heraclitus argued that change is ubiquitous and should be explained as resulting from a struggle between opposites. As Lenin would later say: subtraction is the antithesis of addition and arithmetic is their synthesis; similarly calculus results from a struggle between differentiation and integration. On the other hand, according to Parmenides, change and the very flow of time are human illusions. (Modern physicists might interpret this as asserting that time is just another dimension, on par with the three dimensions of space.) His disciple Zeno even argued that motion is impossible, because infinitesimal intervals must be zero.

This ancient problem is still very much with us, as may be seen in the ideas of the influential mathematicians - philosopher Bill Lawvere [1969], who conceived progress in mathematics as resulting from a struggle between "adjoint functors", a categorical generalization of what algebraists call a "Galois connection". In [1980], he dismissed the idealistic world picture, which he ascribed not only to Plato and Berkeley, but also to Mack, Russell, Brouwer and Heyting. Although he himself had shown how to describe the category of sets as an "elementary topos" with a natural numbers object, thus tempting us to introduce a real numbers object by definition, in his 1980 article he favoured a different approach, according to which the real numbers object was incorporated to start with in a category of spaces.

[Without being too formal here, let me remind the reader that an elementary topos is a category with a terminal object 1, with cartesian products and exponentiation, and possessing an object Ω of truth-values, which allows subobjects of any object A to be represented by their characteristic morphisms $A \to \Omega$. A natural numbers object N is accompanied by morphisms $0 : 1 \to N$ and $S : N \to N$, allowing one to define morphisms $N \to A$ by recursion. Each elementary topos possesses an "internal language", a type theory whose terms of type A are morphisms $1 \to A$.]

3. There will be a sea battle tomorrow (Aristotle).

Surprisingly, the internal language of an elementary topos had turned out to be a form of intuitionistic type theory, which Lawvere modestly called "non-Aristotelian", as he did not approve of Brouwer's idealistic philosophy.[2]

In fact, a moderate form of intuitionistic type theory, as conceived by Lambek and Scott [1986], differs from classical type theory only by the absence of a single axiom, Aristotle's axiom of the excluded third:

$$\forall_{x\in\Omega}(x \vee \neg x)$$

or, equivalently,

$$\forall_{x\in\Omega}(\neg\neg x \Rightarrow x),$$

where Ω is the type of truth-values. It must be conceded that Aristotle himself had some doubts about this when talking about the future.

It should be emphasized that their non-Aristotelian language excluded some of the more extreme concepts and views of some intuitionists, such as free choice sequences and the time dependence of truth. As a promising candidate for *the* category of sets, acceptable to moderate intuitionists, they proposed the so-called *free topos*, the initial object in the category of (small) toposes with natural numbers object, which could also be constructed from *pure* higher order intuitionistic arithmetic (not exhibiting any types, terms or postulates that are not strictly necessary) by a familiar Lindenbaum-Tarski construction. For classical mathematicians, any topos with natural numbers object would serve as *a* category of sets, provided the terminal object is a generator (see McLarty [1992]).

In the largely ignored second part of their book, they explored the development of elementary mathematics in an elementary topos with natural numbers object, hence in the language of intuitionistic higher order arithmetic. They did not discuss the real numbers, assuming that they could be introduced either by the Eudoxus-Dedekind method or by that of Theaetetus-Cauchy. It was known that these two methods led to different concepts intuitionistically (see Johnstone [1977]).

To judge only from his 1980 article, Bill Lawvere seems to reject this approach, suggesting instead that a real numbers object ought to be incorporated into the topos to start with. He replaces the topos of sets and mappings by one of geometric manifolds and smooth mappings. In the present article, I prefer to look at the real numbers in the intuitionistic category of sets, without committing myself whether they should be defined or postulated.

4. The moving arrow is at rest (Zeno).

The other ancient problem, how to account for motion and change, was attacked by Newton and Leibniz with the help of a calculus that defined the rate of change as the ratio of two infinitely small quantities. These were conceived as positive real numbers less than $1/n$ for each positive integer n. In the spirit of Zeno's earlier criticism of motion, the idealist philosopher Berkeley dismissed these ratios as "ghosts of departed quantities". It was only in the nineteenth century that this calculus was placed on a solid foundation with the help of epsilons and deltas.[3]

In the second half of the twentieth century, the eminent logician Abraham Robinson pointed out that positive infinitesimals could be assumed to exist, without fear of contradiction. In fact, any potential proof in *first* order arithmetic that

$$0 \leq h < \cdots < \frac{1}{n} < \frac{1}{n-1} < \cdots < \frac{1}{2} < 1$$

implies $h = 0$ can only involve a finite number of steps, hence is contradicted by the fact that

$$0 < \frac{1}{n+1} < \frac{1}{n}.$$

This particular argument breaks down at *higher* order, that is, in type theory, where one can prove that

$$\forall_{h\in\mathbb{R}}((0 \leq h \wedge \forall_{n\in\mathbb{N}}(nh < 1)) \Rightarrow h = 0);$$

but there Robinson's idea has been resurrected by Moerdijk and Reyes [1991].

Lawvere suggested in 1967 (unpublished) that infinitesimals can be defined as nilpotent real numbers, provided one assumes that these form a local ring rather than a field. (Without loss of generality one may assume that infinitesimals have square zero, since $h^{2m} = 0$ implies $(h^m)^2 = 0$, and h^m is infinitesimal if h is.) His program was carried out by Kock [1981] and others.

Jacques Penon [1981] pointed out that even a field, when conceived intuitionistically, may contain nilpotent elements (other than zero) and, more generally, elements not unequal to zero. He suggested that all these be admitted as infinitesimals.

Recalling that the nilpotent elements of a commutative ring from its *prime* radical and that in certain normed rings the infinitely small elements are contained in the *Jacobson* radical (see Fine et al. [1995], Theorem 8.13), I realized that the proposed Penon infinitesimals are precisely the elements of the Jacobson radical.[4]

5. Thou shalt be a witness (Kronecker).

In their book [1986], Lambek and Scott presented a proof that pure intuitionistic type theory is *constructive* in the sense that, whenever the existence of a certain kind of entity is proved, it must be possible to produce an example. As is well-known, classical type theory is non-constructive. This result is usually attributed to the axiom of choice[5]; but, as they pointed out, this is so even without the axiom. For suppose we can prove $\exists_{y\in A}\phi(y)$ using a suitable instance of the axiom of choice. Now, classically, one can prove

$$\exists_{x\in A}(\exists_{y\in A}\phi(y) \Rightarrow \phi(x)).$$

If pure classical type theory were constructive, one could find a closed term a of type A such that

$$\exists_{y\in A}\phi(y) \Rightarrow \phi(a).$$

Therefore, using the given instance of the axiom of choice, one could infer that $\phi(a)$, thus showing that classical type theory with the axiom of choice is constructive.

The question that concerns us here is this: do there exist infinitesimal real numbers other than zero? As constructionists, we cannot say so without being able to exhibit at least one such. This may be difficult, but we can say

(5.1) not every infinitesimal real number is equal to zero.

Unfortunately, we cannot prove this intuitionistically, since it does not hold classically. However, we may expect to show that (5.1) is consistent.

6. What is a field?

In intuitionistic logic, the following statements are equivalent:

$$\neg\neg(p \vee q),\ \neg(\neg p \wedge \neg q),\ \neg p \Rightarrow \neg\neg q,\ \neg q \Rightarrow \neg\neg p.$$

Moreover, we have the following implication (denoted by arrows):

While the reverse implications do not hold, they cannot be disproved intuitionistically, since they hold classically.

Classically, a *field* is a commutative ring in which $1 \neq 0$ and every non-zero element is a *unit*, that is, is invertible. We write $U(x)$ to say that x is a unit.

Intuitionistically, we may consider a number of variants of this notion, in addition to the condition that $1 \neq 0$:

The above names, except for the last, are those given by Johnstone [1977]. He points out that the Dedekind reals form a residue field.

In the present discussion, I find it convenient to use the word "field" for Johnstone's field of fractions.[6] Thus a *field* is a commutative ring A such that

$$\forall_{x \in A}(\neg(x = 0) \Leftrightarrow U(x)).$$

Most of what I plan to say holds also for non-commutative fields, where one assumes that a unit is both left and right invertible. But, in any ring (with 1), one of these assumptions suffices. For suppose that every element x has a left inverse x^ℓ so that $x^\ell x = 1$. Then x^ℓ also has a left inverse $x^{\ell\ell}$, hence

$$x^{\ell\ell} = x^{\ell\ell}(x^\ell x) = (x^{\ell\ell}x^\ell)x = x$$

and therefore $xx^\ell = 1$, showing that x^ℓ is also a right inverse.

Non-commutative fields are called *skew-fields* or *division rings*.

Classically, $\mathbb{R}$ may be defined as a complete ordered field, which turns out to be unique up to isomorphism. We would expect an intuitionistic variant of this definition to serve our present purpose. Classically, there are two constructions which model this definition: the Dedekind reals and the Cauchy reals. Unfortunately, as is well-known, intuitionistic variants of these two constructions do not agree. The Dedekind reals do not form a field in our sense (see Section 7 below), but the constructive Cauchy reals, as presented by Bishop and Bridges [1985], do. Unfortunately, both of them satisfy $\neg\neg a = 0 \Rightarrow a = 0$ (see Bell [2007]p.187).

Let us look at the *smooth reals* introduced by Lawvere and Kock into the category of smooth manifolds, but admitted into an intuitionistic category of sets, subject to a set of axioms, by Moerdijk and Reyes [1991] (see also Bell [2005]p.212). These axioms do describe a field in our sense, which is ordered by a transitive relation $<$ satisfying

$$\neg(a = b) \Leftrightarrow (a < b \vee b < a) \tag{6.1}$$

in place of the usual trichotomy. Bell [loc. cit. p.214], points out in a footnote that (6.1) is satisfied by Bishop's construction reals if one accepts Markov's principle; but this has been proved to hold in pure intuitionistic type theory by Lambek and Scott [1986].

It follows from (6.1) that $<$ is irreflexive and, more surprisingly, that

$$\neg\neg(a < b) \Rightarrow (a < b),$$

as is shown by Bell [2005] (on page 213):

Assume $\neg\neg(a < b)$, then $\neg(a = b]$, since $<$ is irreflexive. Hence $(a < b) \vee (b < a)$ by (6.1). Suppose $b < a$, then $\neg\neg(b = a)$, again because $<$ is irreflexive, contradicting $\neg(a = b)$. Therefore $\neg(b < a)$ and thus $a < b$.

We will also make another assumption, not shared by Moerdijk and Reyes [1991], namely that the real numbers are *bounded*:

(6.3) For every real number r there is a natural number n such that $r^2 < n^2$.

7. The radical of a division ring.

The Jacobson radical of a ring is sometimes attributed to Perlis or Chevalley; we will just call it *the radical*. It may be defined as the set of all elements a such that $1 - ua$ is *left* invertible for every *left* multiple ua of a. The apparent asymmetry of this definition can be removed by observing that either one or both of the occurrences of *left* in the definition can be replaced by *right*. The reader not familiar with this well-known result can find a proof in Section 10 below.

PROPOSITION 7.1. The radical of a division ring consists of all elements not unequal to zero, i.e., of the Penon infinitesimals.

Proof. The element a is in the radical provided, for all elements u, $1 - ua$ is invertible, that is unequal to zero. This is so if and only if a is not a unit, that is, not unequal to zero.

PROPOSITION 7.2. If the element h of a division ring A is a Penon infinitesimal, then so is $f(a+h) - f(a)$, for any $a \in A$ and any mapping $f : A \to A$.

Proof. Obviously, $h = 0$ implies $f(a+h) - f(a) = 0$. Negation is contravariant, hence double negation is covariant; therefore $\neg\neg(h = 0)$ implies $\neg\neg(f(a+h) - f(a)) = 0$.

We may think of Proposition 7.2 as a rudimentary way of asserting the continuity of f. It might have been accepted as such in the seventeenth century. The modern notion of continuity is more sophisticated.

It is customary to abbreviate $f(x+h) - f(x)$ by $df(x)$. If we take f to be the identity function, it follows that $h = (x+h) - x$ may be written as dx. Traditionally, dx and $df(x)$ are called "differentials". Unfortunately, Proposition 7.1 tells us that dx is *not* invertible. Therefore, strictly speaking, the useful Leibniz notation $df(x)/dx$ does not represent an algebraic ratio in a field in our radical approach.

8. Normed division rings.

To support Penon's decision to call the elements not unequal to zero "infinitesimals", we turn to *normed division rings*, that is, division rings equipped with a norm. By the *norm* of a ring A we understand a function from (the underlying set of) A into the set of squares of real numbers such that

$$N(n^1) = n^2, \quad N(ab) = N(a)N(b),$$

for all $a, b \in A$ and all natural numbers n.

Examples of normal division rings are: the field of real numbers with $N(a) = a^2$, the field of complex numbers and the division ring of quaternions with $N(a) = a\overline{a}$, $\overline{a}$ being the *conjugate* of a.

At this point, it will be convenient to introduce the assumption $(6,3)$ about the real numbers, to ensure that they are bounded: for every real number r there exists a natural number n such that $r^2 < n^2$.

We might expect that infinitesimals are *infinitely small*. In a normed division ring, this property can be expressed by the formula $\forall_{n\in\mathbb{N}}(N(nh) < 1)$.

PROPOSITION 8.1. The element h of a normed division ring is not unequal to zero if and only if it is infinitely small.

Proof. Clearly, $h = 0$ implies $nh = 0$, hence $N(nh) = N(01) = 0^2 = 0 < 1$.

To prove the converse, first assume that $\neg(h = 0)$, so that h has a left inverse x and $xh = 1$. By $(6,3)$ we can find a natural number n such that $N(x) < n^2$, and so

$$\begin{aligned} 1 = \;& N(1) = N(xh) = N(x)N(h) \\ < \;& n^2 N(h) = N(n+)N(h) = N(nh), \end{aligned}$$

hence $\neg(N(nh) < 1)$. Thus

$$\neg(h = 0) \Rightarrow \exists_{n\in\mathbb{N}}\neg(N(nh) < 1)$$

and therefore

$$\neg\exists_{n\in\mathbb{N}}\neg(N(nh) < 1) \Rightarrow \neg\neg(h = 0),$$

i.e.,

$$\forall_{n\in\mathbb{N}}\neg\neg(N(nh) < 1) \Rightarrow \neg\neg(h = 0).$$

By (6.2), the double negation here may be omitted.

9. Differentiation revisited.

While the Leibniz notation $df(x)/dx$ is very useful, it cannot justifiably be viewed as denoting the ratio of two Penon infinitesimals since these are *not* invertible by Proposition 7.1. Nonetheless, differentiation can be treated constructively, as is well-known.

Consider a continuous real valued function f defined on a suitable domain, say an open interval V. We say that f is *differentiable* at $a \in V$ if there is a continuous real-valued function f^a defined on a neighbourhood of 0 such that, for all h in this neighbourhood,

$$f(a+h) - f(a) = f_1^a(h)h. \tag{9.1}$$

We define the *derivative* f' of f by

$$f'(a) = f_1^a(0). \tag{9.2}$$

The usual formulas for the derivatives of products and of polynomials are easily obtained.

How unique is the auxiliary function f_1^a? When h is invertible, we can calculate $f_1^a(h)$ from (9.1). Since f_1^a is continuous, we can also calculate f_1^a is continuous, we can also calculate $f_1^a(0)$ as the limit of the $f_j^a(h)$ as h tends to 0. Unfortunately, $f_1^a(h)$ is not determined for infinitesimal h, except for saying that it is not unequal to $f_1^a(0)$. Fortunately, $f^1(a) = f_1^a(0)$ is uniquely determined.

We call f *infinitely differentiable* or *smooth* at a if there exist auxiliary continuous functions f_k^a (for all natural numbers k) on a neighbourhood W of zero such that, for all $h \in W$ and $k \in \mathbb{N}$,

$$f_0^a(h) = f(a+h),\; f_k^a(h) = f_k^a(0) + f_{k+1}^a(h)h. \tag{9.3}$$

We then have the Taylor expansion

$$f(a+h) = \sum_{k=0}^{n} f_k^a(0)h^k + f_{n+1}^a(h)h^{n+1}. \tag{9.4}$$

By repeated differentiation of (9.4) with respect to h, we can obtain in the usual way that

$$f^{(n)}(a+h) = f_n^a(0)n! + hg(a,h) \tag{9.5}$$

for some $g(a,h)$. Putting $h=0$, we get

$$f^{(n)}(a) = f_n^a(0)n! \tag{9.6}$$

and so (9.4) can be written as

$$f(a+h) = \sum_{k=0}^{n} (f^{(k)}(a)/k!)h^k + f_{n+1}^a(h)h^{n+1}. \tag{9.7}$$

For invertible h, we can then calculate

$$f_{k+1}^a(h) = (f_k^a(h) - f_k^a(0)h^{-k}, \tag{9.8}$$

assuming that $f_k^a(k)$ has already been calculated. By continuity, we can also calculate $f_{k+1}^a(0)$. But what about infinitesimal h? One can easily obtain

$$f_m^a(h) = \sum_{k=0}^{m+n} f_{m+k}^a(0)h^n + f_{m+n+1}^a(h)h^{n+1}. \tag{9.9}$$

The partial sums of (9.9) form a Cauchy sequence, so one ought to be able to deduce

$$f_m^a(h) = \sum_{k=0}^{\infty} f_{m+k}^a(0)h^k.$$

We have taken f and f_k^a to be continuous. This may be an unnecessary assumption if Brouwer is correct in asserting that all real valued functions defined on a closed interval of the real line are continuous. This has been proved for second order arithmetic by Hayashi [1977], who asserts that the proof would become even easier at higher order. A proof for the free topos has also been presented by Joyal in a lecture, though never published.

10. The algebraic background.

Although the elementary properties of the Jacobian radical are well-known and have been discussed both classically (e.g. by Lambek [1980]) and intuitionistically (by Mines et al. [1988]), the reader unfamiliar with the background may appreciate a formal résumé of some relevant results.

PROPOSITION 10.1. The following four assertions about the element a of a ring A are equivalent:

(1) $1 - ua$ is left invertible for all $u \in A$,

(2) $1 - au$ is left invertible for all $u \in A$,

(3) $1 - au$ is right invertible for all $u \in A$,

(4) $1 - ua$ is right invertible for all $u \in A$,

Proof. We begin by showing that (1) $\Leftrightarrow$ (2). Assuming that $x(1-ua) = 1$, one easily calculates that $(1 - axu)(1 - ax) = 1$. Conversely, assuming that $x(1 - au) = 1$, one calculates that $(1 + uxu)(1 - ua) = 1$. Thus (1) $\Leftrightarrow$ (2). By mirror symmetry, also (2) $\Leftrightarrow$ (3). It remains to show, for example, that (1) $\Leftrightarrow$ (4). (1) $\Rightarrow$ (4) follows by checking that the left inverse of $1 - ua$ is also a right inverse. Indeed, by (1), $x = 1 + xua$ has a left inverse y, so that

$$1 = yx = y + yxua = y + ua,$$

hence

$$x(1 - ua) = xy = 1.$$

Thus (1) $\Rightarrow$ (4) and, by mirror symmetry also (3) $\Rightarrow$ (2).

COROLLARY 10.2. The radical of a ring is an ideal.

Proof. It remains to show that it is closed under addition. Suppose a and b are in the radical. Then there exist elements x and y such that, for all u,

$$x(1 - ua) = 1,\ y(1 - xub) = 1,$$

hence

$$yx(1 - u(a + b)) = 1.$$

Here is another well-known characterization of the radical, where "right" may be replaced by "left".

PROPOSITION 10.3. The radical possesses a unary operation $(\quad)^\dagger$ such that $h+h^\dagger = hh^\dagger$ for all its elements h. Moreover, the radical is the smallest right ideal with this property.

Proof. If h is an element of the radical, then $1 - h$ has a right inverse $1 - h^\dagger$, so that $h + h^\dagger = hh^\dagger$. To see that $h^\dagger$ is also in the radical, we need only show that $1 - h^\dagger u$ is right invertible for any u. This follows from

$$1 - h^\dagger u = 1 - h(h^\dagger u - u),$$

which is right invertible by Proposition 10.1.

To see that the radical is the smallest such right ideal, just observe that, if h is any element of such a right ideal, then so is hv, for any v, hence $1 - hv$ has right inverse $1 - (hv)^\dagger$, and therefore h is in the radical.

If the proposed radical approach to calculus is taken seriously, we are faced with the question: given an infinitesimal h, what is the meaning of $h^\dagger$? Of course, when $h^2 = 0$, $h^\dagger = -h$ is easily interpreted. This may be an argument in favour of the nilpotent radical of Lawvere and Kock.

11. Conclusion.

I have presented a rather idiosyncratic account of the opposition between the discrete and the continuous, which has its origin in pre-Socratic Greek philosophy and mathematics, and which has not yet been resolved by modern physics.

The main result of this paper may be summarized as follows:

THEOREM 11.1. In a non-Aristotelian division ring the following subsets coincide:

(1) the set of elements not unequal to zero,

(2) the set of non-units,

(3) the Jacobian radical.

In a normed division ring, this is the same as (

4) the set of infinitely small elements.

Proof. (1) $\Leftrightarrow$ (2), by definition of division ring. (1) $\Leftrightarrow$ (3), by Proposition 7.1, making use of the assumption (6.3) that the real numbers are bounded.

If the division ring itself is taken to be the field of real numbers, this theorem offers several characterizations of the Penon infinitesimals.

If axiom (6,3) is dropped, infinitely large real numbers may make their appearance, and their inverses will serve as invertible infinitesimals outside the radical. They have been exploited by Moerdijk and Reyes [1991] and would help to justify the Leibniz notation dy/dx. However, to discuss them any further here would take us away from our radical approach.

In our radical approach, we cannot prove that infinitesimals other than zero exist, or even that the radical is not zero, because such a proof would remain valid classically. However, it ought to be possible to prove the consistency of such an assumption by exhibiting a suitable model of the real numbers in an appropriate category.

Fine et al. [1965] introduced the ring $Q(X)$ of all (equivalence classes) of continuous functions from dense open subsets of the topological space X into the (classical) field of real numbers. They showed that it was the so-called "complete" ring of quotients of $C(X)$, the ring of all continuous real-valued functions on X. Similarly, they showed that $Q^*(X)$ is the complete ring of quotients of $C^*(X)$, where the star indicates that only bounded functions are admitted.

Johnstone [1977, Example 6.66 ii) points out that Q is an object in the category of sheaves over $\mathbb{R}$ and that q is a unit if and only if the zero-set $\{x \in R | q(x) = 0\}$ is nowhere dense. Therefore, Q is a field (of fractions), it satisfies (6.1) and its radical is not zero. Unfortunately for our approach, Q does not satisfy (6.3), suggesting that we look at Q^* instead.

Acknowledgements.

I wish to thank John Bell for sending me a copy of his beautiful treatise "The continuous and the infinitesimal in mathematics", just as I was putting the last touches on the present article. He goes much deeper into the historical background than I did in my brief sketch here and discusses different candidates for the real numbers object at length. Thanks are also due to Phil Scott for fruitful conversations.

Endnotes.

1. Some ancient historians even believed that Euclid was a committee.

2. I recall how pleased L.E.J. Brouwer was when he learned about a science fiction novel "The world of null-A", which explored a non-Aristotelian universe.

3. Still, infinitesimals survived in physics courses until quite recently. I remember losing some marks in an examination after facetiously declaring that an alleged infinitesimal physical quantity had to be zero.

4. For all I know, this observation may be implicit in the interesting monograph by Mines et al. [1985], which differs from the present account by admitting an *apartness* relation other than the negation of equality.

5. In type theory, instances of the axiom of choice may arise at different types, hence in their 1986 monograph, Lambek and Scott speak of a *rule of choice*.

6. This is also the definition of a field given by Mines et al. [1988], provided their apartness is taken to be the negation of equality.

7. Lawvere [1980] takes an infinitesimal neighbourhood.

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