Six-dimensional Lorentz category

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In memory of Basil Rattray, my friend, colleague and collaborator.

An introduction by R.A.G. Seely

Joachim (Jim) Lambek (1922-2014) is well-known as a mathematician, particularly for his work on ring theory and algebra, and for his work on category theory, especially for deductive systems and for categorical proof theory in general. He is also well-known for his work in mathematical linguistics, particularly for his analysis of categorial grammar, the syntactic calculus, and pregroups. All this is referred to in other papers in this volume. What may be less well known is that he also had a professional interest in theoretical physics. He was particularly interested in the use of the language of quaternions as a tool to explain fundamental aspects of special relativity, for example. In fact the first version of his 1950 Ph.D. thesis included a section on this topic (though he removed that section upon learning that his results had been already proven by A.W. Conway in 1948). He visited this topic several times in his mathematical career, for instance, in a *Mathematical Intelligencer* article "If Hamilton had prevailed: quaternions in physics" (1995), and more recently in a series of articles, referenced in his paper in this volume.

Jim's work in mathematics, logic, and linguistics may be inferred from references to it in other papers in this volume—it seems fitting that this posthumous paper of his should illustrate one of the many other sides of his scholarly interests.

Prologue

Pre-Socratic Greek philosophers were engaged in two intensive debates: Are material objects continuous or discrete, and what is the nature of time? The claim that matter consists of infinitely divisible substances was first made by Thales, who postulated a single basic substance: water. In time, three other substances were added, notably by Empedocles, and even nowadays people accept four states of matter: liquid, solid, gas and energy. The claim that all matter is made up of indivisible units seems to be implicit in the Pythagorean assertion that all things are numbers, but is ultimately replaced by the atomic theory of Democritus and Epicurus. The nature of time was debated by Heraclitus and Parmenides. The former emphasized the importance of time and change in his memorable slogans, while the latter insisted in his famous poem that time was not all that different from space. If I understand him correctly, he claimed that the one-dimensional flow of time is a human illusion not shared by the gods. His pupil Zeno seems to have pointed out that assuming time to be either discrete or continuous leads to contradictions.

Mathematicians too were wondering whether positive reals (which they called geometric quantities) or positive integers are more fundamental. The Pythagoreans at first assumed the latter and only reluctantly admitted the irrationality of the square root of 2. At Plato's Academy two ways of defining positive reals in terms of positive integers were proposed, which are now known as Dedekind reals and Cauchy reals respectively. The former were introduced by Eudoxus and the latter by Theaetatus, who made use of continued fractions. Surprisingly, the ancient Greeks avoided zero and negative numbers, which were only introduced a thousand years later in India.

Modern physicists have definitely decided that the fundamental particles of nature are indivisible objects called fermions and bosons, but the matter of time is still being disputed. In [11] I suggested that all fundamental particles of spin $\frac{1}{2}$ or 1 could be represented by four-vectors with entries $0, 1$ and -1. More recently [12], I observed that six-vectors with the same entries are more suitable if one wishes to distinguish between right-handed and left-handed particles. However, position in space-time is nowadays assumed to be subject to a probability distribution, best expressed as the norm of a quaternion, the Dirac spinor.

For reasons to be discussed below, I have also come to the conclusion that time has more than one dimension. Mathematical elegance would require three dimensions of time, but these may be reduced to two if one insists that Dirac's first-order equation is equivalent to the second-order Klein-Gordon equation. This may be proved as in [10], but better with the help of category theory as below. In [13] I suggested that one should consider a finite additive category with three objects (called a ring with three objects by Barry Mitchell), whose arrows described four-vectors, six-vectors and Dirac spinors of four-dimensional relativistic quantum mechanics. In the present article, the category is generalized to six-dimensional space-time and the six-dimensional classification of fundamental particles is exploited to present a proof of the probability density.

Six-dimensional Lorentz category

Present day theoretical physics relies on the representation of groups and Lie algebras. My personal preference is to make use of the regular representations of the algebra of quaternions instead.

The application of quaternions to Special Relativity has a long history and goes back to Conway [1] and Silberstein [16,17] a century ago. The original idea was to use *biquaternions*, *i.e.* quaternions with complex components. Thus, location in space-time was represented by the *Hermitian* biquaternion (one in which the quaternion and complex conjugates coincide):

$$(i) \qquad x_0 + ii_1x_1 + ii_2x_2 + ii_3x_3$$

or, equivalently, by

$$(ii) \qquad ix_0 + i_1x_1 + i_2x_2 + i_3x_3$$

following Minkowski's suggestion that time be conceived as imaginary space.

When mathematicians turned their attention to Dirac's equation, they thought it convenient to replace i_1, i_2 and i_3 by their left regular matrix representations and i by the right regular matrix representation of one of them, say i_1. This idea was pursued by Lanczos [14], Conway

[2] and Gürsey [7]. If we also admit the right regular representations of i_2 and i_3, we are led by (i) to think of a 10-dimensional space-time and by (ii) of a 6-dimensional one. It is the latter approach I will pursue here, thus admitting two additional dimensions of time rather than six additional ones of space.

The assumption that time has three dimensions was developed by me for special relativity in [10] and for general relativity in [6] by Gillen. My original motivation for the extra dimensions was based on mathematical elegance. In retrospect, the additional dimensions of time make it easier to understand how Schrödinger's cat can be alive and dead "simultaneously", provided this adverb is interpreted to mean "at the same distance from the origin of temporal three-space".

With any quaternion x we associate two *regular* representations

$$L(x)[\psi] = [x\psi], \ R(x)[\psi] = [\psi x],$$

where $[\psi]$ is the column vector consisting of the coefficients of the quaternion ψ. Evidently

$$L(xy) = L(x)L(y), \ R(xy) = R(y)R(x), \ L(x)R(y) = R(y)L(x).$$

The two representations are related by the diagonal matrix Γ with entries $(1, -1, -1, -1)$:

$$\Gamma L(x)\Gamma = -R(x), \ \Gamma R(y)\Gamma = -L(y),$$

hence

$$\Gamma(L(x) + R(y))\Gamma = -(L(y) + R(x)).$$

Any quaternion may be written as $a_0 + \mathbf{a}$, where a_0 is a scalar and

$$\mathbf{a} = i_1 a_1 + i_2 a_2 + i_3 a_3$$

is called a *three-vector*. It is easily seen that

$$L(\mathbf{a}) + R(\mathbf{b})$$

is a skew symmetric matrix and that every skew-symmetric 4×4 real matrix has this form. See [10].

It is our intention to represent space-time by the skew matrix

$$X = L(\mathbf{x}) + R(\mathbf{t}),$$

where the vector $\mathbf{t}$ now replaces the usual scalar t, and to treat other basic physical entities in the same manner.

Thus we have the *kinetic energy-momentum*

$$P = L(\mathbf{p}) + R(\mathbf{m}),$$

where $\mathbf{p}$ is the usual momentum vector and $\mathbf{m}$ is the three-dimensional analogue of the usual energy = matter = 4π frequency.

The skew matrix replacing the old *four-potential*

$$\Phi = L(\mathbf{A}) + R(\boldsymbol{\phi})$$

is composed of Maxwell's vector potential and the vector analogue of the usual scalar potential. This allows us to describe the *potential energy-momentum* $-e\Phi$ of the electron with charge $-e$, the minus sign being due to a choice made by Benjamin Franklin.

The *charge-current* density is described by

$$J = L(\mathbf{J}) + R(\boldsymbol{\rho}),$$

where $\mathbf{J}$ is the usual current density and $\boldsymbol{\rho}$ replaces the usual charge density.

To the above skew matrices we must add the *partial differentiation operator*

$$D = L(\nabla_x) - R(\nabla_t),$$

where

$$\nabla_x = i_1 \frac{\partial}{\partial x_1} + i_2 \frac{\partial}{\partial x_2} + i_3 \frac{\partial}{\partial x_3}, \;\; \nabla_t = i_1 \frac{\partial}{\partial t_1} + i_2 \frac{\partial}{\partial t_2} + i_3 \frac{\partial}{\partial t_3},$$

the minus sign being due to the contravariance of differentiation.

In addition to the *basic* physical entities discussed so far, others may be represented by conjugation and composition of the above skew matrices, the *conjugate* of $A = L(\mathbf{a}) + R(\mathbf{b})$ being $A^* = L(\mathbf{a}) - R(\mathbf{b})$. Thus every physical entity can be represented by a 4×4 matrix, but this should be accompanied by a *Lorentz transformation*, itself expressed with the help of a 4×4 matrix Q of determinant 1.

To start with, we have the basic entities transforming as follows:

$$\begin{array}{ll} X \mapsto QXQ^T, & \text{space-time,} \\ P \mapsto QPQ^T, & \text{kinetic energy-momentum,} \\ \Phi \mapsto Q\Phi Q^T, & \text{six-potential,} \\ J \mapsto QJQ^T, & \text{charge-current density,} \end{array}$$

where Q^T is the *transposed* matrix of Q. Note that the condition $\det Q = 1$ excludes $Q = \Gamma$, but it does not distinguish between transformations expressed by Q and $-Q$.

According to tradition, a Lorentz transformation is supposed to preserve the expression

$$X \odot X = \mathbf{x} \circ \mathbf{x} - \mathbf{t} \circ \mathbf{t} = -XX^*,$$

where $\mathbf{x} \circ \mathbf{x}$ is the usual Heaviside scalar product and $X \odot X$ is its extension to six dimensions.

The condition that $\det Q = 1$ ensures that $X \mapsto QXQ^T$ preserves the determinant of X, hence the square of $X \odot X$. But, if we also wish to preserve the sign of $X \odot X$, we can achieve this by postulating

$$(iii) \qquad X^* \mapsto Q^{\#} X^* Q^{-1}$$

where $Q^{\#}$ is the matrix of *cofactors* of Q, so that

$$Q^T = (Q^{-1})^{\#} = Q^{-\#}.$$

In fact, (iii) is a necessary and sufficient condition for $X \odot X$ to be Lorentz invariant. See [10].

We are now in a position to introduce the *Lorentz category* as an additive category (also called a *ring* by Barry Mitchell) with three objects $1, \#$ and 0, where

$$\#\# = 1, \;\; u\# = 0 = \#u, \;\; 1u = u = u1$$

for all objects u. The arrows $A : u \to v$ are matrices A such that

$$A \mapsto Q^u A Q^{-v},$$

where $Q^{-v} = (Q^v)^{-1}$. If $B : v \to w$ is another such arrow, so is the matrix product $AB : u \to w$, where we have reversed the conventional order of composition of arrows.

In particular, the basic entities X, P, Φ, J and D all describe arrows $1 \to \#$, whereas X^*, P^* *etc* are arrows $\# \to 1$. This had been done under the assumption that there was only one dimension of time in [13], where the usual four-vectors, six-vectors and Dirac spinors were represented by arrows $1 \to \#$, $\# \to \#$ and $1 \to 0$ respectively. The same category had been employed in [13], where the basic entities turned out to be Hermitian biquaternions, but here they are skew-symmetric 4×4 matrices.

Most (if not all) physical entities in pre-quantum physics live already in a ring with two objects, 1 and #, called a *Morita context*, but an understanding of the Dirac equation requires a third object 0 to admit the so-called *Dirac spinors* $1 \to 0$, see below.

To get an idea of how useful calculations are carried out in the Lorentz category, consider skew matrices $A, B, C : 1 \to \#$. Then $AB^* : 1 \to 1$ and $AB^*C : 1 \to \#$; but these arrows can be decomposed. Thus

$$AB^* = \tfrac{1}{2}(AB^* + BA^*) + \tfrac{1}{2}(AB^* - BA^*),$$

where the first summand is the *trace* or scalar part of AB^*:

$$\tfrac{1}{2}(AB^* + BA^*) = -A \odot B : 1 \to 1.$$

Moreover

$$AB^*C = \tfrac{1}{2}(AB^*C + CB^*A) + \tfrac{1}{2}(AB^*C - CB^*A),$$

where the first summand is the *skew* part of AB^*C, which may be calculated as follows:

$$\mathrm{skew}(AB^*C) = -A(B \odot C) + B(C \odot A) - C(A \odot B).$$

This happens to be useful in discussing the Maxwell-Lorentz treatment of the electron, which may be summarized by the equation

$$d(P - e\Phi) = 0,$$

expressing the conservation of the total energy-momentum, where Φ may be subject to a gauge transformation. See [10].

Maxwell had defined the electro-magnetic field F acting on the charged particle in four dimensions as the vector part of $D^*\Phi$. In six dimensions this becomes the symmetric part:

$$F = \tfrac{1}{2}(\overrightarrow{D^*}\Phi - \Phi\overleftarrow{D}^*).$$

It is supposed to be caused by J according to Maxwell's equation $DF = J$, where J satisfies the *equation of continuity* $D \odot J = 0$. On the other hand, the force of the field on an electron as described by Lorentz becomes

$$\frac{dP}{ds} = \mathrm{skew}\left(-e\frac{dX}{ds}F\right) = \frac{d}{ds}(e\Phi),$$

where $(ds)^2 = dXdX^*$.

The relativistic treatment of quantum mechanics begins with the so-called *Klein-Gordon* equation, already known to Schrödinger:

$$DD^*[\psi] = -\mu^2[\psi],$$

where $\mu = PP^* : 1 \to 1$ is the rest-mass of a particle and $[\psi] : 1 \to 0$ is a *Dirac spinor*. If $\mu \neq 0$, this is equivalent to two first order equations:

$$D^*[\psi_1] = \mu[\overline{\psi}_2],\ D[\overline{\psi}_2] = -\mu[\psi_1].$$

Penrose [15] might call $[\psi_1]$ the *zig* and $[\psi_2]$ the *zag*. However, Dirac would combine them into a single first order equation.

Assuming that one time coordinate, say t_3, is redundant in a certain frame of reference, we may take $K = R(i_3)$ in this coordinate system and verify that

$$X^* = -K^{-1}XK^{-1}$$

for $K : 1 \to \#$, and similarly for $P^*,\ D^*$ *etc.* Now let

$$[\psi] = [\psi_1] - K[\overline{\psi}_2],$$

then we may calculate

$$(iv) \qquad D^*[\psi] = -\mu K^*[\psi]$$

and take this to be the six-dimensional Dirac equation.

An explicit solution of (iv) is given by

$$[\psi] = \exp(\eta(X \odot P))\ [\psi_0],$$

where

$$\eta = \mu^{-1}KP^* = -\mu^{-1}PK^*$$

satisfies

$$\eta^2 = -1.$$

On the other hand, (iv) may be written in purely quaternionic form:

$$\overrightarrow{\nabla}_x\psi + \psi\overleftarrow{\nabla}_t + \mu(\mathbf{k}\psi - \psi\mathbf{k}') = 0,$$

provided

$$K = L(\mathbf{k}) + R(\mathbf{k}').$$

Multiplying (iv) by the row vector $[\psi]^T$ on the left, we obtain

$$[\psi]^T\overrightarrow{D^*}[\psi] = -\mu[\psi]^T K^*[\psi].$$

Here the right side is skew symmetric, hence so must be the left side, so that

$$[\psi]^T\overleftrightarrow{D^*}[\psi] = [\psi]^T\overrightarrow{D}[\psi] + [\psi]^T\overleftarrow{D}[\psi] = 0.$$

Multiplying this by a constant skew-symmetric matrix S on the left and assuming that $S : 0 \to 0$ is Lorentz invariant, we infer that

$$(v) \qquad \mathrm{trace}(\overrightarrow{D}^*[\psi]S[\psi]^T) = \mathrm{trace}([\psi]^T \overleftrightarrow{D}^*[\psi]S) = 0.$$

Write $J_S = [\psi]S[\psi]^T$, check that $J_S : 1 \to 0 \to 0 \to \#$ and note that (v) then asserts that $D \odot J_S = 0$, which resembles Maxwell's equation of continuity and suggests a comparison of J_S with the electric charge-current density. A proof of this when time has only one dimensions is found in [18].

It remains to identify S. I will speculate that S is the quaternionic version of the six-vector characterization of fundamental particles of spin 1 and 1/2 that I have discussed in [12]:

$$\begin{aligned} S = \; & L(s_1 i_1 + s_2 i_2 + s_1 i_3) + R(s'_1 i_1 + s'_2 i_2 + s'_3 i_3) \\ \sim \; & (S_1, S_2, S_3;\; s'_1, s'_2, s'_3) \end{aligned}$$

where the S_α and S'_β are all equal to $0, 1$ or -1. Addition of these six-vectors or their skew-symmetric analogues helps to justify Feynman diagrams for fundamental particles. For example,

$$U = L(i_1 + i_3) + R(i_1) \;\sim\; (1, 0, 1; -1, 0, 0)$$

characterizes a first generation left-handed blue up-quark and

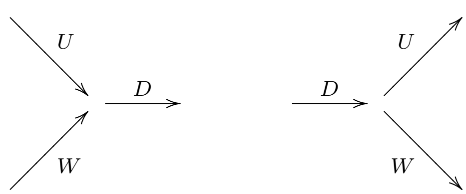
$$W = L(-i_1 - i_2 - i_3) + R(-i_1 - i_2 - i_3) \sim (-1, -1, -1; -1, -1, -1)$$

characterizes a weak vector boson W^-. Adding these two expressions, we obtain

$$D = U + W = L(-i_2) + R(-i_2 - i_3) \sim (0, -1, 0; -0, -1, -1),$$

which characterizes a first generation right-handed blue down-quark.

The equation $S_D = S_U + S_W$ serves to justify the Feynman diagram



and allows us to infer the equation $J_D = J_U + J_W$. This seems reasonable, but fails to bring the coupling constant into the picture.

The two extra dimensions of time had been introduced for the sake of mathematical elegance and I have not settled on their physical meaning. For a while I had hoped that they might help to incorporate the direction of the spin axis, but did not succeed to make this idea work.

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