Compact monoidal categories from Linguistics to Physics.

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This is largely an expository paper, revisiting some ideas about *compact* 2-categories, in which each 1-cell has both a left and a right adjoint. In the special case with only one 0-cell (where the 1-cells are usually called "objects") we obtain a *compact strictly monoidal category*. Assuming furthermore that the 2-cells describe a partial order, we obtain a compact partially ordered monoid, which has been called a *pregroup*. Indeed, a pregroup in which the left and right adjoints coincide is just a partially ordered group (= pogroup).

A brief exposition of recent joint work with Anne Preller [2007] will be given here,investigating *free* compact strictly monoidal categories, which may be said to describe computations in pregroups. Free pregroups lend themselves to the study of grammar in natural languages such as English. While one would not expect to find a connection between linguistics and physics, applications of (free) compact symmetric monoidal categories to physics have been made by Abramsky and Coecke and by Selinger.

Compact symmetric monoidal categories had already been studied by Kelly and Laplaza [1980], who called them "compact closed" and by Barr [1979 etc], who called them "compact star-autonomous". I had intended to show that Feynman diagrams for quantum electro-dynamics (QED) could be described by certain compact Barr-autonomous categories, but was disappointed to find that these reduced to a rather degenerate case, that of partially ordered groups (= pogroups). Still, I will reluctantly present an extension of this idea from QED to the Standard Model. Finally, I will briefly review the transition from 2-categories to the bicategories of Bénabou [1967], using methods of Bourbaki [1948] and Gentzen (see Kleene [1952]), which may also be of interest in physics.

1. Compact 2-categories and pregroups.

A 2-*category* has 0-cells, 1-cells and 2-cells. A typical 2-category (Cat) is that of all small categories with

0-cells = small categories,
1-cells = functors,
2-cells = natural transformations.

We recall that 2-cells have, in addition to the *vertical composition* (represented by a small circle)

$$\frac{t : F \to G \;\; u : G \to H}{u \circ t : F \to H}$$

also a *horizontal composition* (represented by juxtaposition)

$$\frac{s : H \to K \;\; t : F \to G}{st : HF \to KG}$$

defined by the diagonal of the commutative square:

$$\begin{tikzcd} HF \arrow[r, "Ht"] \arrow[d, "sF"'] \arrow[dr, "st"] & HG \arrow[d, "sG"] \\ KF \arrow[r, "Kt"'] & KG \end{tikzcd}$$

The equations in a 2-category are described formally in MacLane's "Categories for the working mathematician", but should be familiar from Cat.

The notion of adjoint functor is known in Cat, but exactly the same definition works in any 2-category. Thus $(F, U, \eta, \varepsilon)$ defines an *adjoint pair* if $\eta : 1 \to FU$ and $\varepsilon : UF \to 1$ are 2-cells such that the following *triangular equations* hold:

$$F \xrightarrow{\eta F} FUF \xrightarrow{F\varepsilon} F = F \xrightarrow{1_F} F \tag{1.1}$$

$$U \xrightarrow{U\eta} UFU \xrightarrow{\varepsilon U} U = U \xrightarrow{1_U} U \tag{1.2}$$

Special cases of 2-categories are
strictly monoidal categories: with only one 0-cell;
partially ordered categories: with only one 2-cell
between any two 1-cells, satisfying the anti-symmetry law;
partially ordered monoids: both of the above.

A 2-category is said to be *compact* if every 1-cell G has both a *left adjoint* G^ℓ and a *right adjoint* G^r. We describe the two adjoint pairs thus:

$$(G, G^r \eta_G, \varepsilon_G),\ (G^\ell, G, \eta^{G^\ell}, \varepsilon_{G^\ell}).$$

Of special interest are compact partially ordered monoids, which I have called *pregroups*. In any pregroup we have

$$GG^r \to 1 \to G^r G,\ G^\ell G \to 1 \to GG^\ell$$

and the triangular equations hold automatically, since the arrow denotes a partial order. If $G^\ell = G^r$ for all 1-cells G, the pregroup is just a *partially ordered group*, more precisely a partially ordered monoid in which each element has an inverse.

Pregroups that are not partially ordered groups are not easy to come by. My favourite example is the monoid of all unbounded order-preserving mappings $\mathbf{Z} \to \mathbf{Z}$, with multiplication and order defined as follows:

$$\begin{aligned} &(fg)(z) = f(g(z)), \\ &f \to g \Leftrightarrow f(z) \le g(z) \end{aligned}$$

for all $z \in \mathbf{Z}$. Adjoints are defined thus:

$$g^\ell(z) = \inf\{y \in \mathbf{Z} | x \leq g(y)\},$$
$$g^r(z) = \sup\{y \in \mathbf{Z} | g(y) \leq z\}.$$

To see that this is not a group, consider $g(x) = gx$, then

$$g^r(x) = [x/2],\ g^\ell(x) = [(x+1)/2].$$

The following equations hold for the elements of any pregroup, or even for the arrows (1-cells) in any compact partially ordered category:

$$1^\ell = 1 = 1^r,\ a^{r\ell} = a = a^{\ell r},\ (ab)^\ell = b^\ell a^\ell,\ (ab)^r = b^r a^r.$$

Moreover, adjoints are unique and

$$a \to b \Rightarrow b^\ell \to a^\ell \Rightarrow a^{\ell\ell} \to b^{\ell\ell} \Rightarrow \cdots,$$
$$a \to b \Rightarrow b^r \to a^r \Rightarrow a^{rr} \to b^{rr} \Rightarrow \cdots\ .$$

2. Pregroups for grammar.

Pregroups freely generated by a partially ordered set have recently found an application to the grammar of natural languages. To illustrate this with a tiny fragment of English grammar, consider the poset of *basic types*:

$\mathbf{q}_1$ = yes-or-no question in present tense,
$\mathbf{q}$ = yes-or-no question in any tense,
$\overline{\mathbf{q}}$ = question (including Wh-question),
$\mathbf{i}$ = infinitive of intransitive verb,
π_3 = third person singular subject,
π_2 = second person singular or any plural subject,
$\mathbf{o}$ = direct object,
$\mathbf{p}$ = plural noun phrase,

with *basic arrows* (inequalities)

$$\mathbf{q}_1 \to \mathbf{q} \to \overline{\mathbf{q}},\ \mathbf{p} \to \pi_2,\ \mathbf{p} \to \mathbf{o}.$$

Here are three sample questions with their associated *types* (elements of the free pregroup):

does he go with her ?

$$(\mathbf{q}_1 \mathbf{i}^\ell \pi_3^\ell)\pi_3\, \mathbf{i}\, (\mathbf{i}^r \mathbf{i} \mathbf{o}^\ell)\mathbf{o} \quad \to \quad \mathbf{q}_1$$

whom does he go with – ?

$$(\overline{\mathbf{q}} \mathbf{o}^{\ell\ell} \mathbf{q}^\ell)(\mathbf{q}_1 \mathbf{i}^\ell \pi_3^\ell)\pi_3\, \mathbf{i}\, (\mathbf{i}^r \mathbf{i} \mathbf{o}^\ell) \quad \to \quad \overline{\mathbf{q}}$$

with whom does he go ?

$$(\overline{\mathbf{q}}\mathbf{o}^{\ell\ell\ell}\overline{\mathbf{q}}^{\ell})(\overline{\mathbf{q}}\mathbf{o}^{\ell\ell}\mathbf{q}^{\ell})(\mathbf{q}_1\mathbf{i}^{\ell}\pi_3^{\ell})\pi_3\mathbf{i} \quad \to \quad \overline{\mathbf{q}}$$

where $\mathbf{q}^\ell\mathbf{q}_1 \to \mathbf{q}^\ell\mathbf{q} \to 1$.

The underlinks in a similar enterprise were first introduced by Zellig Harris [1966]. They may be viewed as degenerate proofnets. The dash at the end of the second question represents what Chomsky calls a *trace*, inserted here to facilitate comparison with mainstream linguistics.

The reader may wonder why the above calculations involve only *contractions* $a^\ell a \to 1$ and $aa^r \to 1$ and no *expansions* $1 \to aa^\ell$ or $1 \to a^r a$. The reason is the following

Switching Lemma. Without loss of generality, one may assume that, in any calculation in a freely generated pregroup, all contractions precede all expansions.

This implies, of course, that, when the right hand side is a *simple type* (obtained from a basic type by adjunctions), no expansions are needed. The proof of the lemma [L1999] depends on the triangular equations.

Note that, already in the first sample question above, the contraction $\mathbf{i}^\ell\mathbf{i} \to 1$ was postponed in order to ensure that the question does not end after *go*. Here the postponement was obligatory, but often it is optional,allowing different interpretations. For example, consider

old men and women

$$(\mathbf{p}\mathbf{p}^\ell)\mathbf{p}\,(\mathbf{p}^r\mathbf{p}\mathbf{p}^\ell)\,\mathbf{p} \qquad \to \mathbf{p}$$

versus

old men and women

$$(\mathbf{p}\mathbf{p}^\ell)\mathbf{p}\,(\mathbf{p}^r\mathbf{p}\mathbf{p}^\ell)\,\mathbf{p} \qquad \to \mathbf{p}$$

In the first noun phrase only the men are described as being old, in the second both men and women are.

This suggests that we should think of the arrow not just as a partial order, but as a *derivation*. In other words, we should replace the pregroup by a compact strictly monoidal category, or even by a compact 2-category.

3. Free compact 2-categories.

Free compact 2-categories were studied by Preller and Lambek [2007]. To convey our main ideas, let me sketch briefly here how to construct the compact 2-category with one 0-cell freely generated by a given *basic category*.

basic 1-cells are objects of the basic category;

simple 1-cells have the form $A^{(z)}$, where A is a basic 1-cell and $z \in \mathbf{Z}$;

1-*cells* are strings of simple ones, the empty string to be denoted by 1;

composition of 1-cells is concatenation of strings;

adjoints of 1-cells are formed by reversing the order and decreasing the superscript by one unit for left adjoints, increasing it by 1 for right adjoints, but the empty string is its own left and right adjoint.

A description of 2-cells will be given presently. For this we need to introduce

(3.1) *simple arrows* of the form $f^{(z)} : A^{(z)} \to B^{(z)}$, where

either z is even and $f : A \to B$ is basic

or z is odd and $f : B \to A$ is basic,

(3.2) *contractions* $A^{(z)}A^{(z+1)} \to 1$ and *expansions* $1 \to A^{(z+1)}A^{(z)}$.

For example, if $f^0 = f : A \to B$ is an arrow in the basic category, we obtain $f^r : B^r \to A^r$ as follows:

$$B^r \to A^r A B^r \to A^r B B^r \to A^r.$$

This assumes that we have already introduced contractions $\varepsilon_B : BB^r \to 1$ and expansions $\eta_A : 1 \to A^r A$ when A and B are basic 1-cells. We may then also define

$$\varepsilon_{(A^r)} = (\eta_A)^r, \; \eta_{(B^r)} = (\varepsilon_B)^r.$$

Repeating this process, we obtain $f^{rr} : A^{rr} \to B^{rr}$ as well as $\varepsilon_{(B^{rr})}$ and $\eta_{(A^{rr})}$, etc. This will account for positive z, but negative z may be treated similarly.

The triangular equations for basic 1-cells must be postulated. But then we can infer them also for simple 1-cells, provided we postulate that adjunction acts contravariently on both horizontal and vertical composition. For example,

$$\begin{aligned}
\varepsilon_{(A^r)}A^r \circ A^r\eta_{(A^r)} &= (\eta_A)^r A^r \circ A^r(\varepsilon_A)^r \\
&= (A\eta_A)^r \circ (\varepsilon_A A)^r \\
&= (\varepsilon_A A \circ A\eta_A)^r \\
&= (1_A)^r \\
&= 1_{(A^r)}.
\end{aligned}$$

2-cells from one 1-cell to another are obtained by performing a sequence of "deductions" with the help of simple arrows, contractions and expansions, as follows:

$$\begin{aligned}
&\Gamma A^{(z)}\Delta \to \Gamma B^{(z)}\Delta, \\
&\Gamma A^{(z)}A^{(z+1)}\Delta \to \Gamma\Delta, \\
&\Gamma\Delta \to \Gamma A^{(z+1)}A^{(z)}\Delta,
\end{aligned}$$

where Γ and Δ are strings of 1-cells. However, these 2-cells are subject to the triangular equations discussed earlier. To obtain a canonical representation of 2-cells, it will be convenient to introduce generalized contractions and expansions, which already abort certain simple arrows.

(3.3) *Generalized contractions* have the form ε_f, where $f : A \to B$ is a simple arrow and ε_f is the diagonal of the commutative square

$$\begin{tikzcd}
AB^r \arrow[r, "Af^r"] \arrow[d, "fB^r"'] \arrow[dr, "\varepsilon_f" description] & AA^r \arrow[d, "\varepsilon_A"] \\
BB^r \arrow[r, "\varepsilon_B`\varepsilon_f"'] & 1
\end{tikzcd}$$

and *generalized expansions* have the form η_g, where $g : C \to B$ is a simple arrow and η_g is the diagonal of the commutative square

$$\begin{tikzcd}
1 \arrow[r, "\eta_C"] \arrow[d, "\eta_B"'] \arrow[dr, "\eta_g"] & C^r C \arrow[d, "C^r g"] \\
B^r B \arrow[r, "g^r B"'] & C^r B
\end{tikzcd}$$

The Switching Lemma mentioned earlier for free pregroups can be sharpened to hold also for free compact 2-categories with one 0-cell.

Categorical Switching Lemma.

Without loss of generality one may assume that a 2-cell consists of generalized contractions followed by simple arrows followed by generalized expansions.

Here is an indication of a crucial step in the proof: Suppose a generalized expansion immediately precedes a generalized contraction, as in

$$A \xrightarrow{A\eta_g} AB^r C \xrightarrow{\varepsilon_f C} C$$

where $f : A \to B$ and $g : B \to C$ are simple errors, then the compound arrow

$$(\varepsilon_f C) \circ (A\eta_g) = g \circ f$$

may be replaced by the simple arrow $g \circ f : A \to c$.

To see this look at the following commutative diagram:

and note that the compound arrow on the top is $\varepsilon_f C$ and the compound arrow on the left is A_{η_g}.

To check the commutativity of the above squares, pretend you are in Cat, then apply the naturality of f, ε_B and fB^rB.

We have thus proved the generalized triangle equality

$$(\varepsilon_f C) \circ (A\eta_g) = g \circ f$$

and can show similarly that

$$(A^r \varepsilon_g) \circ (\eta_f C^r) = f^r \circ g^r.$$

We may then represent 2-cells by geometric diagrams called *transition systems* by Preller and Lambek [2007]. For example, given simple arrows

$$f : A \to F,\ g : C \to 0,\ L : B \to E,\ i : G \to H,\ j : I \to J,$$

we obtain a 2-cell

$$ABCD^rE^r \to FG^rHI^rJ$$

as a vertical composition as follows:

$$(FG^rH\eta_j) \circ (F\eta_i) \circ (f\varepsilon_h) \circ (AB\varepsilon_g E^r),$$

which is represented horizontally thus:

When describing a transition system between two 1-cells Γ and Δ, we must ensure that any simple 1-cell of Γ or Δ is at an endpoint of exactly one simple arrow, underlink or overlink, and that these don't cross.

The Switching Lemma ensures that the composition of two transition systems is again a transition system, by a process we called "combing", but which others have called "yanking". For example:

For more details see [loc.cit.], where it is also shown that the free compact 2-category thus constructed has the expected universal property.

Buszkowski [2002] had shown that the original Switching Lemma for free pregroups is essentially a cut-elimination theorem for compact bilinear logic. Our categorical version shows that the composition of 2-cells in free compact 2-categories (with one 0-cell) can be performed without mentioning vertical composition, except that of basic arrows, from which other simple arrows are easily constructed. I believe that this is the true rôle of cut-elimination also in other categorical contexts. (The restriction that there is only one 0-cell was made for expository purposes and may of course be removed.)

4. In search of a compact Feynman category.

From now on let us assume that we are in a compact 2-category with one 0-cell, also known as a compact strictly monoidal category. Let $U = F^r$ be the right adjoint of F, hence $U = F^\ell$ the left adjoint of U. The triangular equation (1.1) may be represented geometrically as an equation between diagrams:

It is tempting to give a physical interpretation to this in quantum electro-dynamics (QED):

$F = e^- =$ electron,
$U = e^+ =$ positron,

$I = \gamma =$ photon.

It does not seem profitable to distinguish between right and left adjoints here, so we will assume from now on that $G^\ell = G^r$ for any 1-cell G.

The equation $G^\ell = G^r$ will hold automatically if the 2-category is *symmetric*, that is, if composition of 1-cells is commutative, as we shall assume from now on. This requires, in particular, that any two composable 1-cells must have the same source and target, as is ensured by our assumption that there is only one 0-cell. Our compact 2-category thus becomes what Kelly and Laplaza [1980] call a *compact closed category* (closure being a consequence of compactness, as defined here) and what Barr calls a *compact star - autonomous* category (the star being the common symbol for the superscripts ℓ and r), although here the tensor product is assumed to be associative on the nose. The second triangular equation (1.2) is now a consequence of the first (1.1).

Diagrams such as (4.1) were introduced by Feynman as an aid to calculating probabilities. For example, the probability of what happens at any vertex of (4.1) is given by the (idealized) charge of the electron.

The equal sign in (4.1) must be taken with a grain of salt. What actually happens is that the electron goes from point x to point y in space-time in many different ways. Each of the ways has associated with it a certain complex number, its *amplitude*, depending on the energy-momentum 4-vector. These amplitudes must be added up and the square of the absolute value of their sum is interpreted as the probability for an electron to go from x to y. Hence the equal sign really holds between equivalence classes of alternative motions.

The easiest way to ensure the equality in (4.1) is to let the arrow stand for a partial order. Then we would also predict

in line with what physicists call "vacuum polarization". Disappointingly, this will imply that our compact 2-category degenerates into a partially ordered group, in which adjoints are just inverses.

5. A progroup for QED.

I had hoped to describe an interesting freely generated compact monoidal category for application to quantum electro-dynamics. But, after all the i-s were dotted and all the t-s were crossed, I realized that all I had was a partially ordered group. I will now describe a provisional version of this progroup, provisional because I have not taken into account the spin of the electron and the polarization of light. These had also been downplayed, if not ignored, by Feynman [1985], whose beautiful exposition I am relying on.

We take as 1-cells all finite multisets (to be explained presently) of *fundamental* 1-cells. There are pairs (x, a), where $x = (x_0, x_1, x_2, x_3) \in \mathbf{R}^4$ is a point in space-time and $a = (a_0, a_1, a_2, a_3) \in \{-1, 0, +1\}^4$ represents a fundamental particle. For expository purposes we will write $(x, a) = x^a$, thus suggesting the $x^0 = 1$ does not depend on x.

A *multiset* is a string, liable to arbitrary permutations of its elements. The composition of 1-cells, usually called "tensor product" in monoidal categories, is obtained by combining two multisets into one. The empty multiset is the unity element 1. (Conceivably, these multisets should be replaced by sets, but this should only be done after the spin of the electron has been taken into account.)

2-cells, that is arrows between multisets, are "made up" from the following:

motions: $x^a \to y^a$,

contractions: $x^a y^b \to x^{a+b}$,

expansions: $x^{a+b} \to x^a x^b$.

The last two are subject to the condition that

$$a_i = 0 \text{ or } b_i = 0 \text{ or } a_i + b_i = 0$$

for all $i \in \{0, 1, 2, 3\}$. We recall that the arrow represents a *partial* order (not a pre-order) so that $\leftrightarrow$ means equality.

We will leave the complete interpretation of the quadruple a until later. For the moment let us only mention that

$e = (1, -1, -1, -1)$ represents the electron e^-,

$-e = (1, 1, 1, 1)$ represents the positron e^+,

$0 = (0, 0, 0, 0)$ represents the photon γ.

Hence the contraction

$$x^e x^0 \to x^e$$

describes an electron at x absorbing a photon. If the arrow is reversed, the expansion describes emission of a photon. The contraction

$$x^\ell x^{-e} \to x^0 = 1$$

describes the annihilation of an electron-positron pair. If the arrow is reversed, the expansion describes pair creation. This should suffice for QED.

What is meant by saying that 2-cells are "made up" from motions, contractions and expansions? Without giving a tedious formal definition, let me illustrate this by a calculation:

$$u^e \to v^e \to v^0 v^\ell \to w^0 v^e \to w^e w^{-e} v^e \to w^e x^{-e} x^e$$
$$\to w^e x^0 \to y^e y^0 \to y^e \to z^e$$

which is furthermore illustrated by the Feynman diagram, the lefthand side of (4.1):

As another illustration, consider two motions $x^a \to y^a$ and $u^a \to v^a$. They generate a 2-cell $x^a u^a \to y^a v^a$. But, since $y^a v^a = v^a y^a$, this 2-cell is also generated by the motions $x^a \to v^a$ and $u^a \to y^a$. According to our interpretation, this should imply that the amplitudes of the two processes are to be added before calculating the probability of the transition $x^a u^a \to y^a v^a$. This is indeed the case, as Feynman pointed out.

Although I had expected to find an interesting compact closed category, all we ended up with was a partially ordered group with $x^0 = y^0 = 1$ and inverse $(x^a)^{-1} = x^{-a}$. It is not the free pogroup generated by the x^a, since we have additional equalities $x^a x^b = x^{a+b}$, when one of a_i, b_i or $a_i + b_i$ is 0 for each i.

We might have obtained a more interesting Feynman 2-category (with one 0-cell), had we not assumed that the 2-cells describe a partial order, but that all "ways" of going from one point in space-time to another count as 2-cells. However, the resulting strictly monoidal category would not be compact and would not be relevant for the present discussion. I have not investigated what happens if we assume that the symmetry is not exact or if it is replaced by braiding, as in Joyal-Street [1986].

Already the ancient philosopher Parmenides believed that the flow of time is an illusion, not shared by the gods. It is therefore of some interest to show formally that $x^a \to y^a$ implies (and is implied by) $y^{-a} \to x^{-a}$, meaning that any particle may be viewed as the corresponding anti-particle moving backwards in time. Assuming $x^a \to y^a$, we calculate

$$\begin{aligned} y^{-a} &\to y^{-a} y^0 \to y^{-a} x^0 \to y^{-a} x^a x^{-a} \\ &\to y^{-a} y^a y^{-a} \to y^0 x^{-a} \to x^0 x^{-a} \to x^{-a}. \end{aligned}$$

To avoid an overabundance of 2-cells, we will not allow $x^a \to y^b$ unless $a = b$ and we will postulate

$$x^a = y^a \text{ if and only if } x = y \text{ or } a = 0. \tag{5.1}$$

The above treatment ignores the Pauli exclusion principle, which asserts that two identical electrons (with the same spin direction) cannot occupy the same position in space-time. We could have overcome this objection had we replaced "multisets" by "sets" in our definition of 1-cells. But this would not do either, since two identical photons or two electrons with opposite spin can be at the same place.

6. From QED to the Standard Model.

Had we only been interested in QED and weak interactions, we could have taken a to be a pair (a_0, a_1) with a_1 representing the electric charge, if the charge of the electron is taken as -1. The minus sign here results from an arbitrary choice by Benjamin Franklin as to what constitutes positive versus negative charge. We have chosen $a = (a_0, a_1, a_2, a_3)$ to account also for strong interactions, with

$$a_1 + a_2 + a_3 = 3 \times \text{ electric charge.}$$

Other "colourless" particles in which $a_1 = a_2 = a_3$ are the following:

neutrino $\nu = (1, 0, 0, 0)$,

anti-neutrino $\overline{\nu} = (-1, 0, 0, 0)$,

and the weak vector bosons

$$\begin{aligned} W^+ &= (0, 1, 1, 1), \\ W^- &= (0, -1, -1, -1), \\ \text{and} \quad Z^0 &= (0, 0, 0, 0) \end{aligned}$$

unfortunately sharing the same quadruple with the photon.

To account for the strong forces, one requires some new fermions, called "quarks", and some new bosons called "gluons", for which a_1, a_2 and a_3 are no longer equal. Thus we have the

$$\text{(red) up-quark } u = (1, 0, 1, 1)$$

and the

$$\text{(red) down-quark } d = (1, -1, 0, 0)$$

with two "colour" variants, depending on the position of the 0 and the -1 respectively, as well as the corresponding anti-particles $-u$ and $-d$. There are six gluons to allow for changes of colour, e.g. $(0, 1, -1, 0)$ permits

$$(1, 0, 1, 1) + (0, 1, -1, 0) \to (1, 1, 0, 1),$$

combining with a red up-quark to yield, say, a blue one. Allegedly, there are also two so-called "diagonal" gluons, which have not bee described here. Altogether, our quadruples represent 25 known fundamental particles and anti-particles: 4 leptons, 12 quarks, 3 weak vector bosons (not distinguishing Z^0 from γ) and 6 gluons.

Let us illustrate this with just one Feynman diagram:

Showing how an up-quark decomposes into a down-quark of the same colour and a positive weak vector boson, which then combines with an electron to form a neutrino. We calculate

$$\begin{aligned} & x^{(1,0,1,1)} t^{(1,-1,-1,-1)} \to y^{(1,0,1,1)} t^{(1,-1,-1,-1)} \\ \to \quad & x'^{(1,-1,0,0)} z^{(0,1,1,1)} t^{(1,-1,-1,-1)} \\ \to \quad & x'^{(1,-1,0,0)} t'^{(1,0,0,0)}. \end{aligned}$$

We have used a_0 to represent the *fermion number*:

$$\begin{aligned} a_0 &= 1 \text{ for fermions,} \\ a_0 &= -1 \text{ for anti-fermions,} \\ a_0 &= 0 \text{ for bosons.} \end{aligned}$$

Actually, only the number of leptons and the number of quarks are preserved separately in known physical interactions. Having adopted the fermion number instead, we allow in principle that quarks can be transformed into leptons with the help of some not yet discovered bosons, e.g.

$$(1, -1, 0, 0) + (0, 1, 0, 0) \to (1, 0, 0, 0).$$

As Feynman points out, this might predict the instability of the proton, which has not yet been verified experimentally.

Our representation of fundamental fermions was inspired by the more concrete representation proposed by Harari [1979] and Shupe [1979], but that of bosons departs from theirs. Here is a rather odd observation, depending on Benjamin Franklin's arbitrary choice: out of a possible 34 = 81 quadruples with components $-1, 0$ and $+1$, the number of $+1$s and the number of -1s in the quadruples occurring above are both odd or zero. This would still be the case if we admitted six additional bosons, variants of the hypothetical $(0, 1, 0, 0)$ mentioned above, making a total of 31. However, we have not accounted for the diagonal gluons and the conjectured graviton and Higgs particle. If our "odd" observation is taken seriously, there would still be six other potential elementary particles, represented by variants of $(1, -1, 1, 1)$, bringing the total up to 37.

I am indebted to Derek Wise for bringing to my attention a recent article by S.O. Bilson Thompson [2006], which also offers an abstract development of the Harari-Shupe model. Rather than invoking a fermion number, he represents a fermion by a braided triple of "helons", namely twists of a ribbon through $\pm 2\pi$ or 0, and he distinguishes fermions from their anti-particles by associating them with braids and anti-braids respectively, thus bringing in the braid group B_3. His ideas are further developed in a joint article with F. Markopoulou and L. Smolin [2006]. If braiding is not used to distinguish electrons from positrons, could it be invoked to distinguish spin-up from spin-down?

7. From 2-categories to bicategories.

Bicategories were introduced by Jean Bénabou [1967]. They are like 2-categories, except that composition of 1-cells, usually called "tensor product", is associated only up to coherent isomorphism. Bicategories with a single 0-cell are better known as monoidal categories. Symmetric monoidal categories, albeit with an additional operation "dagger", play a rôle in the categorical approach to quantum mechanics by Abramsky, Coecke and Selinger. I would like to take a closer look at bicategories, if only to remind people that the usual coherence and other properties need not be postulated, but can be proved if the right definition is adopted. I will follow [L1989 and 2004].

A typical bicategory is that of *bimodules*:

0-cells = rings $R, S, \cdots$,

1-cells = bimodules ${}_RA_S,\ {}_SB_T,\ \cdots$,

2-cells = bimodule homomorphisms (= linear mappings).

Composition of 1-cells is the usual tensor product

$$({}_RA_S,\ {}_SB_T)\ \mapsto\ {}_R(A\otimes B)_T.$$

Its many properties can all be deduced from Bourbaki's [1948] definition, which prescribes a bilinear mapping $m_{AB} : AB \to A\otimes B$ with the universal property: given any bilinear mapping $f : AB \to C$, this is a unique linear mapping $f^{\S} : A\otimes B \to C$ such that $f^{\S} m_{AB} = f$.

Given elements $a \in A$ and $b \in B$ and abbreviating

$$m_{AB}ab = (a, b),$$

we may write the above equation as

$$f^{\S}(a, b) = fab.$$

From this the usual properties of the tensor product are easily deduced.

For example, if $f : A \to A'$ and $g : B \to B'$, we may define $f\otimes g : A\otimes B \to A'\otimes B'$ by putting $f\otimes g = h^{\S}$, where $h : AB \to A'\otimes B'$ is given by

$$hab = (fa, gb),$$

hence

$$(f\otimes g)(a, b) = (fa, gb).$$

To show that $\otimes$ is a bifunctor, we require e.g. that

$$(f' \otimes g')(f \otimes g) = f'f \otimes g'g,$$

when $f' : A' \to A''$ and $g' : B' \to B''$. We prove this by calculating

$$\begin{aligned} (f' \otimes g')(f \otimes g)(a,b) &= (f' \otimes g')(fa, gb) \\ &= (f'fa,\ g'gb) \\ &= (f'f \otimes g'g)(a,b). \end{aligned}$$

The associative arrow $\alpha_{ABC} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$ is defined by the equation

$$\alpha_{ABC}((a,b),c) = (a,(b,c)).$$

It is easily checked that α is a natural transformation. Similarly one defines $\alpha^{-1}_{ABC} : A \otimes (B \otimes C) \to (A \otimes B) \otimes C$ and checks that $\alpha\alpha^{-1} = 1$ and $\alpha^{-1}\alpha = 1$ (omitting subscripts). Mac Lane's famous *pentagonal coherence* condition asserts the commutativity of the following diagram:

This is proved by pointing out that there is a unique arrow $f : (((A \otimes B) \otimes C) \otimes D) \to A \otimes (B \otimes (C \otimes D))$ such that

$$f(((a,b),c),d) = (a,(b,(c,d))).$$

The identity 1-cell $I_S : S \to S$ for the tensor product is of course the bimodule ${}_SS_S$ obtained from the ring S.

Passing from the concrete bicategory of bimodules to arbitrary bicategories, we need to treat multilinear maps abstractly. This was done with the help of multicategories (see e.g. [L1989]), called "operads" by some people.

A *multicategory*, as viewed most recently [L2004], es essentially a 2-category, except that 1-cells are freely generated from *basic* 1-cells and 2-cells are restricted to intuitionistic Gentzen sequents (see e.g. Kleene [1952]), which are composed by *cuts* [ibid]:

given basic 1-cells $R \xleftarrow{A} S,\ S \xleftarrow{B} T,\ T \xleftarrow{C} U$,...., we form (compound) 1-cells $R \xleftarrow{AB} T,\ R \xleftarrow{ABC} U$, etc. We must also admit the empty 1-cells $R \xleftarrow{\emptyset_R} R,\ S \xleftarrow{\emptyset_S} S$, etc.

The only 2-cells we retain are of the form $\Gamma \to G$, where Γ is any 1-cell and G is a basic one. A *cut* has the form

$$\frac{f : \Lambda \to A \; g : \Gamma A \Delta \to B}{g\langle f \rangle : \Gamma \Lambda \Delta \to B}$$

where

$$g\langle f \rangle = g \circ \Gamma f \Delta.$$

The equation holding in a multicategory are all inherited from those of the encompassing 2-category, even though we have discarded all 2-cells except those whose targets are basic 1-cells.

A tensor product of 1-cells can be introduced by a 2-cell $\mathbf{m}_{AB} : AB \to A \otimes B$ together with a rule

$$\frac{f : \Gamma A B \Delta \to C}{f^{\S} : \Gamma(A \otimes B)\Delta \to C},$$

which associates to any $f : \Gamma AB\Delta \to C$ a *unique* $f^{\S} : \Gamma(A \otimes B)\Delta \to C$ such that

$$f^{\S}\langle \mathbf{m}_{AB} \rangle = f.$$

The uniqueness may also be expressed equationally by saying that, for any $g : \Gamma(A \otimes B)\Delta \to C$,

$$(g\langle \mathbf{m}_{AB} \rangle)^{\S} = g.$$

With any 0-cell R there is associated an identity 1-cell I_R, introduced by the 2-cell

$$\mathbf{i}_R : \emptyset_R \to I_R$$

and a rule

$$\frac{f : \Gamma\Delta \to C}{f^{\#} : \Gamma I_R \Delta \to C},$$

which associates to any $f : \Gamma\Delta \to C$ a unique $f^{\#} : \Gamma I_R \Delta \to C$ such that

$$f^{\#}\langle \mathbf{i} \rangle = f.$$

The uniqueness amounts to the equation

$$(g\langle \mathbf{i}_R \rangle)^{\#} = g$$

for any $g : \Gamma I_R \Delta \to C$.

The arguments we employed for bimodules carry over to any multicategory, provided we replace elements $a \in A$ by *indeterminate* arrows $a : \Lambda \to A$, or better by *variables* of type A. This can be done formally by invoking the *internal language* of a multicategory, see [L1989] for details of this approach.

8. Other operators in bilinear logic.

It may be of interest to point out that other operations occurring in bilinear (= non-commutative linear) logic can be introduced in the same way (see e.g. [L1993]). For example, the operation "over" whose dual operation "under" is represented by a lollipop by Girard (see e.g. Troelstra [1992]), is introduced as follows:

$$\mathbf{e}_{DA} : (D/A)A \to D,$$

the rule

$$\frac{f : \Gamma A \to D}{f^* : \Gamma \to D/A} ,$$

which associates to every 1-cell $f : \Gamma A \to D$ a unique 1-cell $f^* : \Gamma \to D/A$ such that

$$\mathbf{e}_{DA}\langle f^* \rangle = f.$$

The uniqueness can be expressed by the equation

$$(\mathbf{e}_{DA}\langle g \rangle)^* = g$$

for any $g : \Gamma \to D/A$.

The logical conjunction (= categorical direct product) can be introduced by two 1-cells

$$\mathbf{p}_{AB} : A \wedge B \to A, \ \mathbf{q}_{AB} : A \wedge B \to B$$

and the rule

$$\frac{f : \Lambda \to A \ \ g : \Lambda \to B}{\langle f, g \rangle : \Lambda \to A \wedge B} ,$$

which associates to any pair of 1-cells $f : \lambda \to A$ and $g : \Lambda \to B$ a unique 1-cell $\langle f, g \rangle : \Lambda \to A \wedge B$ such that

$$\mathbf{p}_{AB}\langle f, g \rangle = f, \ \mathbf{q}_{AB}\langle f, g \rangle = g,$$

The uniqueness can be expressed by the equation

$$\langle \mathbf{p}_{AB}, \ \mathbf{q}_{AB} \rangle \ = \ 1_{A \wedge B}.$$

For a discussion of these and other operations see e.g. [L1993]. It was then assumed that there is only one 0-cell, but the arguments carry over to the general case.

Adjoints of 1-cells can be defined in any bicategory. Thus $(F, U, \eta, \varepsilon)$ is an *adjoint pair* if

$$F : R \to S, \ U : S \to R, \ \eta : I_S \to F \otimes U, \ \varepsilon : U \otimes F \to I_R$$

such that

$$\begin{aligned} U \xrightarrow{\sim} U \otimes I_S \xrightarrow{1_U \otimes \eta} U \otimes (F \otimes U) \xrightarrow{\alpha} (U \otimes F) \otimes U \\ \xrightarrow{\varepsilon} I_R \otimes U \xrightarrow{\sim} U = U \xrightarrow{1_U} U \end{aligned}$$

and a similar equation holds for F.

A bicategory is said to be *compact* if every 1-cell has both a left and a right adjoint. To exhibit a concrete example of a compact bicategory, I find myself turning to the exercises in [L1966]. It is shown there that a right module A_S has a left adjoint S^{A^ℓ} if and only if it is finitely generated and projective. The left module ${}_SA^\ell$ is again finitely generated and may be identified with S_S/A_S. If S is a division ring, all one-sided S-modules are automatically projective. Similar considerations apply to left modules ${}_RA$ and their right adjoints $A^r_R = {}_RA\backslash_R R$. We infer that the following concrete bicategory is compact:

0-cells = division rings,

1-cells = bimodules finitely generated on both sides,

2-cells = bimodule homomorphisms.

A compact monoidal category of possible interest in Physics is the category of all $\mathbf{H}-\mathbf{H}$-bimodules finitely generated on both sides, when $\mathbf{H}$ is the division ring of quaternions. A special object of this monoidal category is the ring of all 4×4 real matrices, which is known to be isomorphic to $\mathbf{H} \otimes \mathbf{H}^{\mathrm{op}}$.

9. Postscript.

After writing this paper, I became aware of the article by Joyal and Street [1991]. They carried out something resembling what I have been trying to do in Section 3, but for monoidal categories which are not strictly monoidal. They also constructed free monoidal, symmetric monoidal and braided monoidal categories. They had in mind an (as yet unpublished) application to Feynman diagrams. Their work involves many technical details and definitions, which I admit not having had the patience to absorb.

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