# BECK DISTRIBUTIVITY

*Dedicated to the memory of Jon Beck*

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## 1. Introduction

Models of linear logic are generally assumed to have two binary connectives we will denote by $\otimes$ and $\oplus$, along with a natural transformation, natural in all three arguments, $A \otimes (B \oplus C) \longrightarrow (A \otimes B) \oplus C$. This used to be called a "mixed associativity" but seemed to be more like a distributive law than an associative law and is now called "linear distributivity". The more usual distributive law is not linear because one argument appears once on one side and twice on the other. I began to wonder if this was actually an instance of Beck distributivity. The purpose of this note is to pursue Beck distributivity to its lair and show that the answer to the question raised above is affirmative.

When Beck first defined distributive laws (see [Beck, 1969]) it was of one triple over another and was used to show lift one of the two triples to the category of Eilenberg-Moore algebras for the other and thereby equip the composite of the two functors with a triple structure that was compatible with those of the two constituents. However, this turns out to generalize to any two endofunctors on the same category.

## 2. Distributive laws

Let $T : \mathcal{C} \longrightarrow \mathcal{C}$ be an endofunctor on the category $\mathcal{C}$. We define the category $\mathcal{C}^T$ of $T$-algebras in $\mathcal{C}$. An object is a pair $(C, \alpha : TC \longrightarrow C)$. A morphism $f : (C, \alpha) \longrightarrow (C', \alpha')$ is a morphism $f : C \longrightarrow C'$ such that

$$\begin{tikzcd} TC \arrow[r, "Tf"] \arrow[d, "\alpha"'] & TC' \arrow[d, "\alpha'"] \\ C \arrow[r, "f"'] & C' \end{tikzcd}$$

commutes. The forgetful functor $U^T : \mathcal{C}^T \longrightarrow \mathcal{C}$ may or may not have an adjoint but it certainly preserves all limits and otherwise satisfies the condition of Beck's tripleableness

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theorem. If there is an adjoint, the triple associated to the adjoint pair is called the **free triple generated by** $T$ and its category of algebras is just $\mathcal{C}^T$. See [Barr, 1970] for more details on this.

Dually, the category $\mathcal{C}_T$ of $T$-coalgebras has as objects pairs $(C, \beta : C \longrightarrow TC)$. A map $f : (C, \beta) \longrightarrow (C', \beta')$ is a map $f : C \longrightarrow C'$ such that

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\beta"'] & C' \arrow[d, "\beta'"] \\ TC \arrow[r, "Tf"'] & TC' \end{tikzcd}$$

commutes.

If $S$ and $T$ are endofunctors on the same category, a **distributive law** (in the most general sense) of $S$ over $T$ is a natural transformation $\lambda : ST \longrightarrow TS$. For special kinds of functors, one might require more of a distributive law. For example, if $\mathbf{S} = (S, \eta_S, \mu_S)$ and $\mathbf{T} = (T, \eta_T, \mu_T)$ are triples, then a distributive law of $\mathbf{S}$ over $\mathbf{T}$ is a distributive law of $S$ over $T$ that satisfies additional conditions (see [Beck, 1969] or [TTT, 1984]).

2.1. THEOREM. *Suppose that $S$ and $T$ are endofunctors on $\mathcal{C}$. Then any distributive law $\lambda : ST \longrightarrow TS$ determines a lifting of $T$ to an endofunctor $T^\lambda$ on the category $\mathcal{C}^S$ by the formula $T^\lambda(C, \beta : SC \longrightarrow C) = (TC, \beta^\lambda = T\beta.\lambda)$. The category $(\mathcal{C}^S)^{T^\lambda}$ has as objects $(C, \alpha, \beta)$ such that $(C, \alpha)$ is a $T$-algebra, $(C, \beta)$ is an $S$-algebra subject to the additional compatibility condition that*

$$\begin{tikzcd} STC \arrow[r, "\lambda"] \arrow[d, "S\alpha"'] & TSC \arrow[r, "T\beta"] & TC \arrow[d, "\alpha"] \\ SC \arrow[rr, "\beta"'] & & C \end{tikzcd}$$

*commutes.*

This condition can reasonably be interpreted as saying that $\alpha$ is a morphism of $S$-algebras or that $\beta$ is a morphism of $T$-algebras.

PROOF. If $(C, \beta)$ is an $S$-algebra, then $(TC, T\beta.\lambda C)$ puts an $S$-structure on $TC$. So if we define $T^\lambda(C, \beta) = (TC, T\beta.\lambda C)$, then $T^\lambda$ is a lifting of $T$ to $\mathcal{C}^S$. A $T^\lambda$-structure on $(C, \beta)$ is given by a morphism $\alpha : (TC, T\beta.\lambda C) \longrightarrow (C, \beta)$ in $\mathcal{C}^S$. This condition is exactly the commutativity diagram of the statement. ∎

We can dualize everything as follows:

2.2. THEOREM. *Suppose that $S$ and $T$ are endofunctors on $\mathcal{C}$. Then any distributive law $\lambda : ST \longrightarrow TS$ determines a lifting of $S$ to an endofunctor $S_\lambda$ on the category $\mathcal{C}_T$ by the formula $S_\lambda(C, \alpha : C \longrightarrow TC) = (SC, \alpha_\lambda = S\alpha.\lambda)$. The category $(\mathcal{C}_T)_{S_\lambda}$ has as objects $(C, \alpha, \beta)$ such that $(C, \alpha)$ is a $T$-coalgebra, $(C, \beta)$ is an $S$-coalgebra subject to the additional compatibility condition that*

$$\begin{tikzcd} C \arrow[rr, "\alpha"] \arrow[d, "\beta"'] & & TC \arrow[d, "T\beta"] \\ SC \arrow[r, "S\alpha"] & STC \arrow[r, "\lambda"] & TSC \end{tikzcd}$$

*commutes.* ∎

This condition can reasonably be interpreted as saying that $\alpha$ is a morphism of $S$-coalgebras or that $\beta$ is a morphism of $T$-coalgebras.

Finally, we can look at mixed algebras. We have seen that a distributive law $\lambda : ST \longrightarrow TS$ leads to a lifting of $T$ to $\mathcal{C}^S$ and of $S$ to $\mathcal{C}_T$. Therefore it it is reasonable to ask about $(\mathcal{C}^S)_{T^\lambda}$ and $(\mathcal{C}_T)^{S_\lambda}$.

2.3. THEOREM. *A distributive law of $S$ over $T$ allows the lifting of $T$ to an endofunctor $T^\lambda$ on $\mathcal{C}^S$ and of $S$ to an endofunctor $S_\lambda$ on $\mathcal{C}_T$. Moreover, $(\mathcal{C}_T)^{S_\lambda} = (\mathcal{C}_S)^\lambda = \mathcal{C}_T^S$, the latter category being the category of all $(C, \alpha : C \longrightarrow TC, \beta : SC \longrightarrow C)$, subject to the condition that*

$$\begin{tikzcd} SC \arrow[r, "S\alpha"] \arrow[d, "\beta"'] & STC \arrow[r, "\lambda"] & TSC \arrow[d, "T\beta"] \\ C \arrow[rr, "\alpha"'] & & TC \end{tikzcd}$$

*commute.*

PROOF. If $(S, \beta)$ is an $S$-algebra, define $T^\lambda(C, \beta) = (TC, T\beta.\lambda C)$. A map $\alpha : (C, \beta) \longrightarrow T^\lambda(C, \beta)$ is a morphism in the category of $\mathcal{C}^S$, which is a morphism $C \longrightarrow TC$ making the square of the statement commute. Moreover, if $(C, \alpha)$ is a $T$-coalgebra structure, we can define $S_\lambda(C, \alpha) = (SC, \lambda C.S\alpha)$. The dual argument shows the equivalence of this with the commutativity in the statement. ∎

2.4. LINEAR DISTRIBUTIVITY. The question that motivated this note was whether linear distributivity is an instance of a distributive law. If $A$ and $C$ are objects of $\mathcal{C}$ and we let $S(B) = A \otimes B$ and $T(B) = B \oplus C$, then a distributive law $ST \longrightarrow TS$ is just a natural map $A \otimes (- \oplus C) \longrightarrow (A \otimes -) \oplus C$, which is just a linear distributivity. Incidentally, it seems clear for various reasons that $S$ is a functor you might be interested in algebras for and $T$ is a functor you would more likely be interested in coalgebras. Of course you would first be interested in algebras for the functor $\lambda A, A \otimes A$ and coalgebras for the functor $\lambda A, A \oplus A$.

## 3. Where do distributive laws come from?

The clue to answering the question that heads this section is to realize that a distributive law actually has a wider scope. Not only does it allow the lifting of $T$ to $S$-algebra (as well as an the lifting of $S$ to $T$-coalgebras), but it allows a more general lifting. There is a category whose objects consist of 3-tuples $(C, D, \beta : SC \longrightarrow D)$, with the obvious definition of morphism, and a distributive law allows $T$ to lifted to those algebras as well. Define $T(C, D, \beta) = (TC, TD, T\beta.\lambda)$. Another way to put this is that a distributive law gives rise to a natural transformation $\mathrm{Hom}(S-, -) \longrightarrow \mathrm{Hom}(ST-, T-)$ as functors $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \longrightarrow \mathcal{S}et$. The converse is true too.

3.1. Theorem. *There is a one-one correspondence between distributive laws* $\lambda : ST \longrightarrow TS$ *and natural transformations* $\mathrm{Hom}(S-, -) \longrightarrow \mathrm{Hom}(ST-, T-)$.

Proof. Given a distributive law $\lambda : ST \longrightarrow TS$, the map $\theta C : \mathrm{Hom}(SC, D) \longrightarrow \mathrm{Hom}(STC, TD)$ is defined by $\theta(f) = T(f).\lambda C$. Naturality is the commutativity, for any $f : C' \longrightarrow C$ and $g : D \longrightarrow D'$ of the outer square of

$$\begin{tikzcd}
\mathrm{Hom}(SC, D) \arrow[r, "T"] \arrow[d, "\mathrm{Hom}(Sf,g)"'] & \mathrm{Hom}(TSC, TD) \arrow[r, "\mathrm{Hom}(\lambda C,TD)"] \arrow[d, "\mathrm{Hom}(TSf,Tg)"] & \mathrm{Hom}(STC, TD) \arrow[d, "\mathrm{Hom}(STf,Tg)"] \\
\mathrm{Hom}(SC', D') \arrow[r, "T"'] & \mathrm{Hom}(TSC', D') \arrow[r, "\mathrm{Hom}(\lambda C',TD')"'] & \mathrm{Hom}(STC', D')
\end{tikzcd}$$

The left hand square commutes because $T$ is a functor and the right hand square does because of the naturality of $\lambda$.

To go the other way, suppose $\theta : \mathrm{Hom}(S-, -) \longrightarrow \mathrm{Hom}(ST-, T-)$ is a natural transformation. Then we define $\lambda C = \theta(C, SC)(\mathrm{id}) : STC \longrightarrow TSC$. The naturality of $\theta$ implies the commutativity, for any $f : SC \longrightarrow D$, of

$$\begin{tikzcd}
\mathrm{Hom}(SC, SC) \arrow[r, "\theta(C,SC)"] \arrow[d, "\mathrm{Hom}(SC,f)"'] & \mathrm{Hom}(STC, TSC) \arrow[d, "\mathrm{Hom}(STC,Tf)"] \\
\mathrm{Hom}(SC) \arrow[r, "\theta(C,D)"'] & \mathrm{Hom}(STC, TD)
\end{tikzcd}$$

Applied to the identity of $SC$, the lower left path gives $\theta(C, D).\mathrm{Hom}(SC, f)(\mathrm{id}) = \theta(C, D)(f)$ and the upper right gives $\mathrm{Hom}(STC, Tf).\theta(C, SC)(\mathrm{id}) = \mathrm{Hom}(STC, Tf)(\lambda) = \lambda.Tf$. ∎

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