Unions of perfect matchings in cubic graphs

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Abstract

We show that any cubic bridgeless graph with m edges contains two perfect matchings that cover at least $3m/5$ and three perfect matchings that cover at least $27m/35$ of its edges.

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1 Introduction

A well-known conjecture of Berge and Fulkerson states that every bridgeless cubic graph contains a family of six perfect matchings covering each edge exactly twice:

Conjecture 1.1 *Every cubic bridgeless graph G contains six perfect matching $M_1, \ldots, M_6$ such that each edge of G is contained in precisely two of the matchings.*

Conjecture 1.1 is attributed to Berge in [4], but it first appeared published in [3]. Cycle Double Conjecture is also closely related to this conjecture. Note also that Conjecture 1.1 trivially holds for cubic graphs G that are 3-edge-colorable.

The following weaker version of Conjecture 1.1 due to Berge is also open:

Conjecture 1.2 *Every cubic bridgeless graph G contains five perfect matchings $M_1, \ldots, M_5$ such that each edge of G is contained in at least one of the matchings.*

We remark that even if the number 5 in Conjecture 1.2 is replaced by any larger constant (independent of G), the statement is not known to be true. In this paper, we investigate the maximum possible size of the union of a given number of perfect matchings in a cubic bridgeless graph. More precisely, we study, for $k \in \{2, 3\}$, the numbers

$$m_k = \inf_G \max_{M_1,\ldots,M_k} \frac{|\bigcup_i M_i|}{|E(G)|},$$

where the infimum is taken over all bridgeless cubic graphs G, and $M_1, \ldots, M_k$ range over all perfect matchings of G. Note that Conjecture 1.2 asserts that $m_5 = 1$.

We determine the precise value of m_2 and provide a non-trivial lower bound on m_3. Let us begin by considering the upper bounds. The Petersen graph

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P_{10} has 15 edges and 6 distinct perfect matchings. It can be checked that any two perfect matchings of P_{10} have precisely one edge in common and that the intersection of any three perfect matchings is empty. Simple counting then shows that $m_2 \leq 3/5$ and $m_3 \leq 4/5$.

Our contribution can be summarized as follows:

Theorem 1.3 *The value of m_2 is $3/5$, and $0.771 \approx 27/35 \leq m_3 \leq 4/5$.*

Our main tool is the Perfect Matching Polytope Theorem of Edmonds [2] which we review in Section 2. Throughout the text, we use standard terminology and notation of graph theory as it can be found, e.g., in [1]. Supplementary information on Conjecture 1.1 can be found in [6], and a more detailed introduction to the theory of matching polytopes can be found in a recent monograph by Schrijver [5].

2 The perfect matching polytope

Let $G = (V, E)$ be a graph which may contain multiple edges. A *cut* in G is any set $C \subseteq E$ such that $G \setminus C$ has more components than G does, and C is inclusion-wise minimal with this property. A k-cut (where k is an integer) is a cut comprised of k edges. For a set $X \subseteq V$, we set ∂X to be the set of edges with precisely one end in X. Note that $\partial\emptyset = \partial V = \emptyset$.

Let w be a vector in $\mathbb{R}^E$. The entry of w corresponding to $e \in E$ is denoted by $w(e)$, and for $A \subseteq E$, we define the *weight* $w(A)$ of A as $\sum_{e \in A} w(e)$. The vector w is said to be a *fractional perfect matching* of G if it satisfies the following:

(i) $0 \leq w(e) \leq 1$ for each $e \in E$,

(ii) $w(\partial\{v\}) = 1$ for each vertex $v \in V$, and

(iii) $w(\partial X) \geq 1$ for each $X \subseteq V$ of odd cardinality.

$P(G)$ denotes the set of all fractional perfect matchings of G.

If M is a perfect matching, then the characteristic vector $\chi^M \in \mathbb{R}^E$ of M is contained in $P(G)$. Furthermore, if $w_1, \ldots, w_n \in P(G)$, then any convex combination $\sum_{i=1}^n \alpha_i w_i$ of $w_1, \ldots, w_n$ (where $\alpha_1, \ldots, \alpha_n \geq 0$ are positive reals summing up to 1) also belongs to $P(G)$. It follows that $P(G)$ contains the convex hull of all the vectors χ^M where M is a perfect matching of G. The Perfect Matching Polytope Theorem asserts that the converse inclusion also holds:

Theorem 2.1 (Edmonds [2]) *For any graph G, the set $P(G)$ coincides with* *the convex hull of the characteristic vectors of perfect matchings of G.*

Naturally, $P(G)$ is called the *perfect matching polytope* of a graph G. Another fact that will be useful in our considerations is the following:

Lemma 2.2 *If w is a fractional perfect matching in a graph $G = (V, E)$ and $c \in \mathbb{R}^E$, then G has a perfect matching M such that*

$$c \cdot \chi^M \geq c \cdot w,$$

where $\cdot$ denotes the scalar product. Moreover, there exists such a perfect matching M that contains exactly one edge of each cut C with $w(C) = 1$.

3 Sketch of proof of Theorem 1.3

In this section, we sketch the proof of Theorem 1.3. By our discussion in Section 1, it suffices to show that $m_2 \geq 3/5$ and $m_3 \geq 4/5$:

Proof. [Proof of Theorem 1.3] Fix a cubic bridgeless graph G. Define $w_1 \in \mathbb{R}^E$ to have the value $1/3$ on all edges $e \in E$. It is easy to verify that w_1 is a fractional perfect matching of G. Moreover, $w_1(C) = 1$ for each 3-cut C of G. Hence, by Lemma 2.2, there is a perfect matching M_1 intersecting each 3-cut in a single edge (the existence of M_1 can be also shown by induction on the size of G).

We now use M_1 to define the following vector $w_2 \in \mathbb{R}^E$:

$$w_2(e) = \begin{cases} 1/5 \text{ if } e \in M_1, \\ 2/5 \text{ otherwise.} \end{cases}$$

Again, it can be verified that w_2 is a fractional perfect matching of G (it is important that M_1 contains exactly one edge of each 3-cut of G). For each $e \in E$, set $c_2(e) = 1 - \chi^{M_1}(e)$. By Lemma 2.2, there exists a perfect matching M_2 such that

$$c_2 \cdot \chi^{M_2} \geq c_2 \cdot w_2 = \frac{2}{5} \cdot \frac{2}{3} |E| = \frac{4}{15} |E| \ .$$

Since $c_2 \cdot \chi^{M_2}$ is just $|M_2 \setminus M_1|$, it follows that

$$|M_1 \cup M_2| = (\frac{1}{3} + \frac{4}{15}) \cdot |E| = \frac{3}{5} |E| \, .$$

We conclude that $m_2 = 3/5$.

It remains to establish a lower bound on m_3. It can be shown, using Lemma 2.2 that if M_1 contains a 5-cut C, then $|C \cap M_2| = 1$. Therefore, the following vector $w_3 \in \mathbb{R}^E$ is a fractional perfect matching of G:

$$w_3(e) = \begin{cases} 1/7 \text{ if } e \in M_1 \cap M_2, \\ 2/7 \text{ if } e \in (M_1 \cup M_2) \setminus (M_1 \cap M_2), \\ 3/7 \text{ otherwise.} \end{cases}$$

Similarly as above, we can argue that G contains a perfect matching M_3 such that

$$|M_3 \setminus (M_1 \cup M_2)| \geq \frac{3}{7} \cdot |E \setminus (M_1 \cup M_2)| \,.$$

Consequently,

$$\begin{aligned} |M_1 \cup M_2 \cup M_3| &= |M_1 \cup M_2| + |M_3 \setminus (M_1 \cup M_2)| \\ &\geq \frac{3}{5} |E| + \frac{3}{7} \cdot \frac{2}{5} |E| = \frac{27}{35} |E| \,. \end{aligned}$$

We infer that $m_3 \geq 27/35$. □

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