PFAFFIAN LABELINGS AND SIGNS OF EDGE COLORINGS

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Abstract. We relate signs of edge-colorings (as in classical Penrose's result) with "Pfaffian labelings", a generalization of Pfaffian orientations, whereby edges are labeled by elements of an Abelian group with an element of order two. In particular, we prove a conjecture of Goddyn that all k-edge-colorings of a k-regular Pfaffian graph G have the same sign. We characterize graphs that admit a Pfaffian labeling in terms of bricks and braces in their matching decomposition and in terms of their drawings in the projective plane.

1. Introduction

Graphs considered in this paper are finite and loopless, but not necessarily simple (parallel edges are allowed). A graph G is called *k-list-colorable* if for every set system $\{S_v : v \in V(G)\}$ such that $|S_v| = k$ there exists a proper vertex coloring c with $c(v) \in S_v$ for every $v \in V(G)$. Not every k-colorable graph is k-list colorable. A classic example is $K_{3,3}$ with bipartition (A, B) and $\{S_v : v \in A\} = \{S_v : v \in B\} = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$.

A graph is called *k-list-edge-colorable* if for every set system $\{S_e : e \in E(G)\}$ such that $|S_e| = k$ there exists a proper edge coloring c with $c(e) \in S_e$ for every $e \in E(G)$. The following famous list-edge-coloring conjecture was suggested independently by various researchers and first appeared in print in [3].

Conjecture 1.1. *Every k-edge-colorable graph is k-list-edge-colorable.*

In a k-regular graph G one can define an equivalence relation on k-edge colorings as follows. Let $c_1, c_2 : E(G) \to \{1, \ldots, k\}$ be two k-edge

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colorings of G. For $v \in V(G)$ let $\pi_v : \{1, \ldots, k\} \to \{1, \ldots, k\}$ be the permutation such that $\pi_v(c_1(e)) = c_2(e)$ for every $e \in E(G)$ incident with v, and let $c_1 \sim c_2$ if $\prod_{v \in V(G)} \mathrm{sgn}(\pi_v) = 1$. Obviously $\sim$ is an equivalence relation on the set of k-edge colorings of G and $\sim$ has at most two equivalence classes. We say that c_1 and c_2 *have the same sign* if $c_1 \sim c_2$ and we say that c_1 and c_2 *have opposite signs* otherwise.

A powerful algebraic technique developed by Alon and Tarsi [2] implies [1] that if in a k-edge-colorable k-regular graph G all k-edge colorings have the same sign then G is k-list-edge-colorable. In [6] Ellingham and Goddyn prove the following theorem.

Theorem 1.2. *In a k-regular planar graph all k-edge colorings have the same sign. Therefore every k-edge-colorable k-regular planar graph is k-list-edge-colorable.*

By The Four-Color Theorem Theorem 1.2 implies that every 2-connected 3-regular planar graph is 3–list-edge-colorable. This was proven independently by Jaeger and Tarsi. Penrose [16] was the first to prove that in a 3-regular planar graph all 3-edge colorings have the same sign.

In a directed graph we denote by uv an edge directed from u to v. Let D be a directed graph with vertex-set $\{1, 2, \ldots, 2n\}$ and let $M = \{u_1v_1, u_2v_2, \ldots, u_nv_n\}$ be a perfect matching of D. Define $\mathrm{sgn}_D(M)$, the *sign* of M, to be the sign of the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & \ldots & 2n-1 & 2n \\ u_1 & v_1 & u_2 & v_2 & \ldots & u_n & v_n \end{pmatrix}.$$

Note that the sign of a perfect matching is well-defined as it does not depend on the order in which the edges of M are listed. We say that an orientation D of a graph G with vertex-set $\{1, 2, \ldots, n\}$ is Pfaffian if the signs of all perfect matchings in D are positive. It is well-known and easy to verify that the existence of a Pfaffian orientation does not depend on the numbering of $V(G)$. Thus we say that a graph with an arbitrary vertex-set is *Pfaffian* if it is isomorphic to a graph with vertex-set $\{1, 2, \ldots, n\}$ that admits a Pfaffian orientation.

Pfaffian orientations have been introduced by Kasteleyn [8, 9, 10], who demonstrated that one can enumerate perfect matchings in a Pfaffian graph in polynomial time. A recent survey may be found in [19]. In [10] Kasteleyn proved the following theorem.

Theorem 1.3. *Every planar graph is Pfaffian.*

Goddyn [7] conjectured that Theorem 1.2 generalizes to Pfaffian graphs. The main goal of this paper is to prove this conjecture. In fact, in Section 2 we prove that Theorem 1.2 extends to the larger class of graphs that admit a "Pfaffian labeling". Conversely, we prove that if a graph G does not admit a Pfaffian labeling, then some k-regular graph H obtained from G by replacing each edge by some non-negative number of parallel edges does not satisfy Theorem 1.2.

We also give two characterizations of graphs that admit a Pfaffian labeling. The first one in Section 3 characterizes graphs with a Pfaffian labeling in terms of bricks and braces in their tight cut decomposition. The relevant definitions are given in Section 3. The second characterization in Section 4 describes graphs with a Pfaffian labeling in terms of their drawings in the projective plane.

We propose the following conjecture. If true, it would generalize the Four-Color Theorem by Theorem 1.3.

Conjecture 1.4. *Every* 2*-connected* 3*-regular Pfaffian graph is* 3*-edge-colorable.*

2. Pfaffian Labelings and Signs of Edge Colorings

We generalize Pfaffian orientations to Pfaffian labelings and prove that Goddyn's conjecture holds for those graphs that admit a Pfaffian labeling. Let Γ be an Abelian multiplicative group, denote by 1 the identity of Γ and denote by -1 some element of order two in Γ. Let G be a graph with $V(G) = \{1, 2, \ldots, n\}$. For a perfect matching M of G we define $\text{sgn}(M)$ as $\text{sgn}_D(M)$, where D is the orientation of G that orients each edge from its lower numbered end to its higher numbered end. We say that $l : E(G) \to \Gamma$ is a *Pfaffian labeling* of G if for every perfect matching M of G, $\text{sgn}(M) = \prod_{e \in M} l(e)$. Clearly the existence of l does not depend on the numbering of vertices of G. We say that a graph H admits a *Pfaffian* Γ*-labeling* if it is isomorphic to a graph G with vertex-set $\{1, 2, \ldots, n\}$ that has a Pfaffian labeling $l : E(G) \to \Gamma$. We say that G admits a *Pfaffian labeling* if G admits a Pfaffian Γ-labeling for some Γ. It is easy to see that a graph G admits a Pfaffian $\mathbb{Z}_2$-labeling if and only if G is Pfaffian.

We need the following technical lemma.

Lemma 2.1. *Let X be a set and let $A_1, A_2, \ldots, A_n, B_1, B_2, \ldots, B_m \subseteq X$, such that $|A_i \cap B_j| = 1$ for every $1 \le i \le n, 1 \le j \le m$, and every $x \in X$ belongs to exactly two of the sets $A_1, A_2, \ldots, A_n$ and exactly two of the sets $B_1, B_2, \ldots, B_m$. For every $1 \le i \le n$ let*

$$S_i = \{\{x, y\} \subseteq X |\ x, y \in A_i, x \in B_{i_1} \cap B_{i_3}, y \in B_{i_2} \cap B_{i_4} \ \ for\ some \\ i_1 < i_2 < i_3 < i_4\}.$$

Symmetrically for every $1 \le j \le m$ let

$$T_j = \{\{x, y\} \subseteq X |\ x, y \in B_j, x \in A_{j_1} \cap A_{j_3}, y \in A_{j_2} \cap A_{j_4} \ \ for\ some \\ j_1 < j_2 < j_3 < j_4\}.$$

Then

$$\sum_{i=1}^{n} |S_i| = \sum_{j=1}^{m} |T_j|$$

modulo 2.

Proof. For $1 \le i \le n, 1 \le j \le m$ denote by x_{ij} the unique vertex of $A_i \cap B_j$. Let $Z = \{(a_1, b_1, a_2, b_2)|\ 1 \le a_1 < a_2 \le n, 1 \le b_2 < b_1 \le m, x_{a_1 b_1} \ne x_{a_2 b_2}\}$. Clearly $|Z| = n(n-1)m(m-1)/4 - |X|$ and $|X| = nm/4$. Moreover n and m are even, as $n = \sum_{i=1}^{n} |B_1 \cap A_i| = 2|B_1|$ and, similarly, $m = 2|A_1|$. Consequently $|Z|$ is even. For $\{u, v\} \subseteq X$ let $Z_{uv} = \{(a_1, b_1, a_2, b_2) \in Z|\ \{u, v\} = \{x_{a_1 b_1}, x_{a_2 b_2}\}\}$.

We claim that Z_{uv} is odd if and only if $\{u, v\}$ belongs to exactly one of $\triangle_{i=1}^{n} S_i$ and $\triangle_{j=1}^{m} T_j$. While a simple case analysis can be used to verify this claim, we would like to demonstrate another proof. Draw a blue straight line between the points $(0, i)$ and $(1, j)$ in $\mathbb{R}^2$ if $\{u\} = A_i \cap B_j$ and a red straight line if $\{v\} = A_i \cap B_j$. Then the resulting lines form blue and red closed curves, and as such they cross an even number of times. Note that $|Z_{uv}|$ is equal to the number of such crossings in $\mathbb{R}^2$ strictly between the lines $x = 0$ and $x = 1$; the number of times $\{u, v\}$ occurs in the sets $S_1, \ldots, S_n$ is equal the number of such crossings on the line $x = 0$ and the number of times $\{u, v\}$ occurs in the sets $T_1, \ldots, T_m$ is equal the number of crossings on the line $x = 1$. The claim follows.

From the claim, $\sum_{i=1}^{n} |S_i| + \sum_{j=1}^{m} |T_j| = \sum_{\{u,v\}\subseteq X} |Z_{uv}| = |Z| = 0$ modulo 2. $\square$

Corollary 2.2. *Let c_1 and c_2 be two k-edge-colorings of a k-regular graph G and let $V(G) = \{1, \ldots, 2n\}$. Then c_1 and c_2 have the same sign if and only if $\prod_{i=1}^{k} \operatorname{sgn}(c_1^{-1}(i)) = \prod_{i=1}^{k} \operatorname{sgn}(c_2^{-1}(i))$.*

Proof. Define for $1 \le i \le 2k$, $A_i = c_1^{-1}(i)$ for $1 \le i \le k$ and $A_i = c_2^{-1}(i-k)$ for $k+1 \le i \le 2k$. Let B_j be the set of all edges incident with the vertex j for $1 \le j \le 2n$. Note that the sets $A_1, A_2, \ldots, A_{2k}, B_1, B_2, \ldots, B_{2n}$ satisfy the conditions of Lemma 2.1. Let S_i and T_j be defined as in Lemma 2.1. Note that $\operatorname{sgn}(A_i)$ is equal to

$$(-1)^{|\{\{u,v\},\{u',v'\}\in A_i | u<u'<v<v'\}|} = (-1)^{|S_i|}.$$

On the other hand $\operatorname{sgn}(\pi_j) = (-1)^{|T_j|}$, where π_j is as in the definition of sign of edge-colorings. The colorings c_1 and c_2 have the same sign if and only if $\prod_{j=1}^{2n} \operatorname{sgn}(\pi_j) = 1$, but by Lemma 2.1

$$\prod_{j=1}^{2n} \operatorname{sgn}(\pi_j) = \prod_{i=1}^{2k} \operatorname{sgn}(A_i) = \prod_{i=1}^{k} \operatorname{sgn}(c_1^{-1}(i)) \prod_{i=1}^{k} \operatorname{sgn}(c_2^{-1}(i)). \quad \square$$

Theorem 2.3. *If a k-regular graph G admits a Pfaffian labeling then all k-edge-colorings of G have the same sign.*

Proof. We may assume that $V(G) = \{1, 2, \ldots, n\}$. Let c_1 and c_2 be two k-edge-colorings of G. By Corollary 2.2 c_1 and c_2 have the same sign if and only if

$$\prod_{i=1}^{k} \operatorname{sgn}(c_1^{-1}(i)) \prod_{i=1}^{k} \operatorname{sgn}(c_2^{-1}(i)) = 1.$$

Let $l : E(G) \to \Gamma$ be a Pfaffian labeling of G for some Abelian group Γ. Then

$$\prod_{i=1}^{k} \mathrm{sgn}(c_1^{-1}(i)) \prod_{i=1}^{k} \mathrm{sgn}(c_2^{-1}(i)) = \prod_{e \in E(G)} l(e) \times \prod_{e \in E(G)} l(e) =$$
$$= (\prod_{i=1}^{k} \mathrm{sgn}(c_1^{-1}(i))^2 = 1. \quad \square$$

By Theorem 2.1 in [6], as well as Corollary 3.9 in [1], a k-regular graph is k-list-edge-colorable if the sum of signs of all of its k-edge colorings is non-zero. Therefore the following corollary of Theorem 2.3 holds.

Corollary 2.4. *Every k-edge-colorable k-regular graph that admits a Pfaffian labeling is k-list-edge-colorable.*

Next we will prove a partial converse of Theorem 2.3. We need to precede it by another technical lemma.

Lemma 2.5. *Let m and n be positive integers. Let A be an integer matrix with m rows and n columns and let $\mathbf{b}$ be a rational column vector of length m. Then either there exists a rational vector $\mathbf{x}$ of length n such that $A\mathbf{x} - \mathbf{b}$ is an integer vector, or there exists an integer vector $\mathbf{c}$ such that $\mathbf{c}A = \mathbf{0}$ and $\mathbf{c} \cdot \mathbf{b}$ is not an integer.*

Proof. There exists a unimodular integer $m \times m$ matrix $U = (u_{ij})$ such that $H = UA$ is in the Hermitian normal form (see for example [17]): if $H = (h_{ij})$ then there exist $1 \leq k_1 < k_2 < \ldots < k_l \leq n$, such that

(1) $l \leq m$,
(2) $h_{ik_i} \neq 0$ for every $1 \leq i \leq l$,
(3) $h_{ij} = 0$ for every $1 \leq i \leq l, 1 \leq j < k_i$,
(4) $h_{ij} = 0$ for every $l < i \leq m, 1 \leq j \leq n$.

There exists $\mathbf{x} \in \mathbb{Q}^n$ such that first l coordinates of $H\mathbf{x} - U\mathbf{b}$ are zeros. Let $U\mathbf{b} = (d_j)_{1 \leq j \leq m}$. If $d_j \notin \mathbb{Z}$ for some $j > l$ then $\mathbf{c} = \{u_{j1}, u_{j2}, \ldots, u_{jm}\}$ is as required. If, on the other hand, $d_{l+1}, \ldots d_m \in \mathbb{Z}$ then $H\mathbf{x} - U\mathbf{b}$ is an integer vector and therefore so is $U^{-1}(H\mathbf{x} - U\mathbf{b}) = A\mathbf{x} - \mathbf{b}$. $\square$

Theorem 2.6. *If a graph G does not admit a Pfaffian labeling then there exist an integer k, a k-regular graph G' whose underlying simple graph is a spanning subgraph of G and two k-edge colorings of G' of different signs.*

Proof. We may assume that $V(G) = \{1, 2, \ldots, n\}$. Let $\mathcal{M}$ denote the set of all perfect matchings of G and let Γ be the additive group $\mathbb{Q}/\mathbb{Z}$. The identity of Γ is 0 and the only other element of order two is $1/2$. We will use the additive notation in this proof, instead of the multiplicative one we used before; in particular $\operatorname{sgn}(M) \in \{0, 1/2\}$ for $M \in \mathcal{M}$. The graph G does not admit a Pfaffian Γ-labeling; i.e., there exists no function $l : E(G) \to \mathbb{Q}/\mathbb{Z}$ such that $\sum_{e \in M} l(e) = \operatorname{sgn}(M)$ for every $M \in \mathcal{M}$. By Lemma 2.5 there exists a function $f : \mathcal{M} \to \mathbb{Z}$ such that $\sum_{M \ni e} f(M) = 0$ for every $e \in E(G)$ and $\sum_{M \in \mathcal{M}} f(M) \operatorname{sgn}(M) = 1/2$. For every edge $e \in E(G)$ let $m(e) = 1/2 \cdot \sum_{M \ni e} |f(M)|$; then $m(e)$ is an integer. Let G' be the graph constructed from G by duplicating every edge $m(e) - 1$ times (if $m(e) = 0$ we delete e). Then G' is k-regular, where $k = 1/2 \cdot \sum_{M \in \mathcal{M}} |f(M)|$. Moreover, there exist a k-edge coloring c_1 of G' such that a perfect matching M appears as a color class of c_1 if and only if $f(M)$ is positive, in which case it appears $f(M)$ times. Similarly, there exist a k-edge coloring c_2 of G' such that a perfect matching M appears as a color class of c_2 if and only if $f(M)$ is negative, in which case it appears $|f(M)|$ times. Note that $\sum_{i=1}^{k} c_1^{-1}(i) + \sum_{i=1}^{k} c_2^{-1}(i) = \sum_{M \in \mathcal{M}} |f(M)| \operatorname{sgn}(M) = \sum_{M \in \mathcal{M}} f(M) \operatorname{sgn}(M) = 1/2$. Therefore c_1 and c_2 have different signs by Corollary 2.2. □

3. Pfaffian Labelings and Tight Cut Decomposition

The previous section established a relation between graphs that admit a Pfaffian labeling and k-regular graphs in which all k-edge colorings have the same sign. This motivates the study of graphs that admit a Pfaffian labeling. In this section we use the matching decomposition procedure developed by Kotzig, and Lovász and Plummer [12], which we briefly review.

We say that a graph is *matching-covered* if it is connected and every edge belongs to a perfect matching. Let G be a graph, and let $X \subseteq$

$V(G)$. We use $\delta(X)$ to denote the set of edges with one end in X and the other in $V(G) - X$. A *cut* in G is any set of the form $\delta(X)$ for some $X \subseteq V(G)$. A cut C is *tight* if $|C \cap M| = 1$ for every perfect matching M in G. If G has a perfect matching, then every cut of the form $\delta(\{v\})$ is tight; those are called *trivial*, and all other tight cuts are called *nontrivial*. Let $\delta(X)$ be a nontrivial tight cut in a graph G, let G_1 be obtained from G by identifying all vertices in X into a single vertex and deleting all resulting parallel edges, and let G_2 be defined analogously by identifying all vertices in $V(G) - X$. We say that G *decomposes* along C into G_1 and G_2. By repeating this procedure any matching-covered graph can be decomposed into graphs with no non-trivial tight cuts. This motivates the study of the graphs that have no non-trivial tight cuts.

The graphs with no non-trivial tight cuts were characterized in [5, 11]. A *brick* is a 3-connected bicritical graph, where a graph G is *bicritical* if $G\backslash u\backslash v$ has a perfect matching for every two distinct vertices $u, v \in V(G)$. A *brace* is a connected bipartite graph such that every matching of size at most two is contained in a perfect matching.

Theorem 3.1. [5, 11] *A matching covered graph has no non-trivial tight cuts if and only if it is either a brick or a brace.*

Thus every matching covered graph G can be decomposed into a multiset $\mathcal{J}'$ of bricks and braces. Let $\mathcal{J}$ consist of the underlying simple graphs of graphs in $\mathcal{J}'$. Lovász [11] proved that, up to isomorphism, the multiset $\mathcal{J}$ does not depend on the choice of tight cuts made in the course of the decomposition. We say that the members of $\mathcal{J}$ are the *bricks* and *braces of* G.

The following lemma reduces the study of graphs with Pfaffian labelings to bricks and braces. Its analogue for Pfaffian orientations is due to Vazirani and Yannakakis [20].

Lemma 3.2. *Let* Γ *be an Abelian group. A matching-covered graph* G *admits a Pfaffian* Γ*-labeling if and only if each of its bricks and braces admits a Pfaffian* Γ*-labeling.*

Proof. Let $C = \delta(X)$ be a tight cut in G and let G_1 and G_2 be obtained from G by identifying vertices in X and $V(G) - X$ respectively. It suffices to prove that G admits a Pfaffian Γ-labeling if and only if both G_1 and G_2 admit a Pfaffian Γ-labeling. Without loss of generality, we assume that $V(G) = \{1, 2, \dots, 2n\}, X = \{1, 2, \dots 2k+1\}$ and that G_1 and G_2 inherit the order on vertices from G; in particular, the vertex produced by identifying vertices of $V(G) - X$ has number $2k+2$ in G_1, the vertex produced by identifying vertices of X has number 1 in G_2. For every perfect matching M of G the sets of edges $M \cap E(G_1)$ and $M \cap E(G_2)$ are perfect matchings of G_1 and G_2 respectively. Moreover, $\mathrm{sgn}(M) = \mathrm{sgn}(M \cap E(G_1))\,\mathrm{sgn}(M \cap E(G_2))$.

Suppose first that $l : E(G) \to \Gamma$ is a Pfaffian labeling of G. For every $e \in C$ fix a perfect matching $M_2(e)$ of G_2 containing e. Define $l_1(e) = \mathrm{sgn}(M_2(e)) \prod_{f \in M_2(e)} l(f)$ for every $e \in C$ and define $l_1(e) = l(e)$ for every $e \in E(G_1) \setminus C$. For a perfect matching M of G_1 let $e \in C \cap M$. We have

$$\prod_{f \in M} l_1(f) = \prod_{f \in M \setminus \{e\}} l(f) \prod_{f \in M_2(e)} l(f)\ \mathrm{sgn}(M_2(e)) =$$
$$= \mathrm{sgn}(M \cup M_2(e))\,\mathrm{sgn}(M_2(e)) = \mathrm{sgn}(M).$$

Therefore $l_1 : E(G_1) \to \Gamma$ is a Pfaffian labeling of G_1.

Suppose now that $l_i : E(G_i) \to \Gamma$ is a Pfaffian labeling of G_i for $i \in \{1, 2\}$. Define $l(e) = l_i(e)$ for every $e \in E(G_i) \setminus C$ and define $l(e) = l_1(e) l_2(e)$ for every $e \in C$. It is easy to see that $l : E(G) \to \Gamma$ is a Pfaffian labeling of G. □

For our analysis of Pfaffian labelings of bricks and braces we will need two theorems. The first of them is proved in [4] for bricks and in [12] for braces. It also follows from the results of [13].

Theorem 3.3. *Let G be a brick or brace different from K_2, C_4, K_4, the prism and the Petersen graph. Then there exists $e \in E(G)$ such that $G \setminus e$ is a matching covered graph with at most one brick in its tight cut decomposition and this brick is not the Petersen graph.*

For a graph G let the *matching lattice*, $\mathrm{lat}(G)$, be the set of all linear combinations with integer coefficients of the incidence vectors of perfect matchings of G. The next theorem of Lovász [11] gives a description of the matching lattice. It can be deduced from Theorem 3.3.

Theorem 3.4. *If a matching covered graph G has no brick isomorphic to the Petersen graph, then*

$$lat(G) = \{x \in \mathbb{Z}^{E(G)} \mid x(C) = x(D) \text{ for any two tight cuts } C \text{ and } D\}.$$

Lemma 3.5. *A brace or a brick not isomorphic to the Petersen graph admits a Pfaffian labeling if and only if it is Pfaffian.*

Proof. By induction on $|E(G)|$. The base holds for K_2, C_4, K_4 and the prism as all those graphs are Pfaffian. We may assume that $V(G) = \{1, 2, \ldots, n\}$.

For the induction step let $e \in E(G)$ be as in Theorem 3.3 and denote $G \backslash e$ by G'. The bricks and braces of G' satisfy the induction hypothesis and therefore by Lemma 3.2 either G' admits a Pfaffian orientation or G' does not admit a Pfaffian labeling. If G' does not admit a Pfaffian labeling then neither does G.

Therefore we can assume that G' admits a Pfaffian labeling $l : E(G') \to \mathbb{Z}_2$. It will be convenient to use additive notation for the group operation. Suppose l does not extend to a Pfaffian labeling of G. Then there exist perfect matchings M_1 and M_2 in G such that $e \in M_1 \cap M_2$ and $\sum_{f \in M_1 \backslash \{e\}} l(f) - \sum_{f \in M_2 \backslash \{e\}} l(f) \neq \mathrm{sgn}(M_1) - \mathrm{sgn}(M_2)$.

We claim that $|M_1 \cap C| = |M_2 \cap C|$ for any tight cut $C = \delta(X)$ in G'. Indeed, G' has at most one brick in its decomposition. Therefore we can assume that the graph G'' obtained from G' by identifying vertices in X is bipartite. It follows that $|M_1 \cap E(G'')| = |M_2 \cap E(G'')|$ and, consequently, that $|M_1 \cap C| = |M_2 \cap C|$.

By Theorem 3.4 we have $\chi_{M_1} - \chi_{M_2} = \sum_{M \in \mathcal{M}} c_M \chi_M$, where $\mathcal{M}$ denotes the set of perfect matchings of G' and c_M is an integer for every $M \in \mathcal{M}$. Therefore for every Pfaffian labeling $l' : E(G') \to \Gamma$ of G'

$$\sum_{M \in \mathcal{M}} c_M \,\mathrm{sgn}(M) = \sum_{M \in \mathcal{M}} (c_M \sum_{f \in M} l'(f)) = \sum_{f \in M_1 \backslash \{e\}} l'(f) - \sum_{f \in M_2 \backslash \{e\}} l'(f).$$

But for $l' = l$ this expression is not congruent to $\mathrm{sgn}(M_1) - \mathrm{sgn}(M_2)$ modulo 2. It follows that $\sum_{f \in M_1 \backslash \{e\}} l'(f) - \sum_{f \in M_2 \backslash \{e\}} l'(f) \neq \mathrm{sgn}(M_1) -$

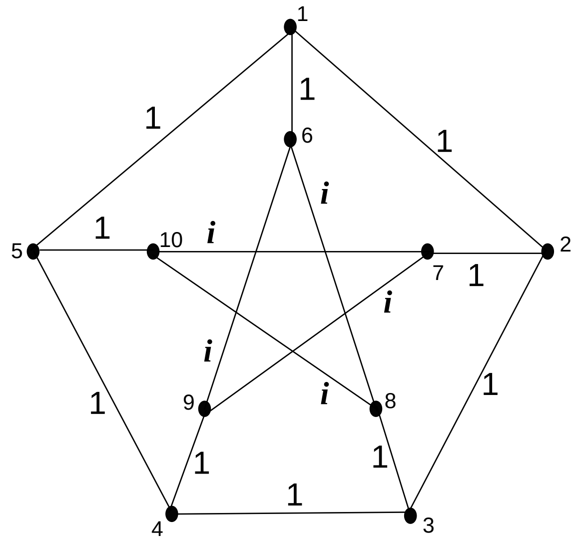

FIGURE 1. A μ_4-labeling of the Petersen graph

$\mathrm{sgn}(M_2)$ for every Pfaffian labeling $l' : E(G') \to \Gamma$. Therefore no Pfaffian labeling of G' extends to a Pfaffian labeling of G, i.e. G does not admit a Pfaffian labeling. □

Note that the Petersen graph admits a Pfaffian μ_4-labeling, where μ_n is the multiplicative group of nth roots of unity. Figure 1 shows an example of such labeling. Note that while the letter i was used for indexing above, from this point on it is used to denote a square root of -1.

The next theorem constitutes the main result of this section. It follows immediately from the observation above and Lemmas 3.2 and 3.5.

Theorem 3.6. *A graph G admits a Pfaffian labeling if and only if every brick and brace in its tight cut decomposition is either Pfaffian or isomorphic to the Petersen graph. If G admits a Pfaffian Γ-labeling for some Abelian group Γ then G admits a Pfaffian μ_4-labeling.*

4. Drawing Graphs with Pfaffian Labelings

By a *drawing* Φ of a graph G on a surface S we mean an immersion of G in S such that edges are represented by homeomorphic images of

$[0,1]$, not containing vertices in their interiors. Edges are permitted to intersect, but there are only finitely many intersections and each intersection is a crossing. For edges e, f of a graph G drawn on a surface S let $cr(e,f)$ denote the number of times the edges e and f cross. For a set $M \subseteq E(G)$ let $cr_\Phi(M)$, or $cr(M)$ if the drawing is understood from context, denote $\sum cr(e,f)$, where the sum is taken over all unordered pairs of distinct edges $e, f \in M$. The next lemma follows from the results of each of the papers [14, 15, 18].

Lemma 4.1. *Let D be an orientation of a graph G and let $V(G) = \{1, 2, \ldots, 2n\}$. Then there exists a drawing Φ of G in the plane such that $\text{sgn}_D(M) = (-1)^{cr_\Phi(M)}$ for every perfect matching M of G. Moreover, for any $S \subseteq E(G)$ the drawing Φ can be chosen in such a way that there exists a point in the plane that belongs to the image of each edge in S and does not belong to the image of any other edge or vertex of G.*

Conversely, for any drawing Φ of G in the plane there exists an orientation D of G such that $\text{sgn}_D(M) = (-1)^{cr_\Phi(M)}$ for every perfect matching M of G.

For a point p and a drawing Φ of a graph G in the plane, such that Φ maps no vertex of G to p, let $cr_{p,\Phi}(e,f)$ denote the number of times the edges e and f cross at points other than p. For a perfect matching M of G let $cr_{p,\Phi}(M)$ denote $\sum cr_{p,\Phi}(e,f)$, where the sum is taken over all unordered pairs of distinct edges $e, f \in M$.

Lemma 4.2. *For a graph G the following are equivalent.*

(1) *G admits a Pfaffian labeling,*
(2) *There exist a point p and a drawing Φ of a graph G in the plane, such that Φ maps no vertex of G to p and $|M \cap S|$ and $cr_{p,\Phi}(M)$ are even for every perfect matching M of G, where $S \subseteq E(G)$ denotes the set of edges whose images contain p.*

Proof. We may assume that $V(G) = \{1, 2, \ldots, n\}$.

(1) $\Rightarrow$ (2). By Theorem 3.6 there exists a Pfaffian μ_4-labeling $l : E(G) \to \{\pm 1, \pm i\}$ of G. Let D be the orientation of G such that $uv \in E(D)$ if and only if $u < v$ and $l(uv) \in \{1, i\}$, or $u > v$ and

$l(uv) \in \{-1, -i\}$. Let $S = \{e \in E(G) \mid l(e) = \pm i\}$ and let $S' = \{e \in E(G) \mid l(e) \in \{-1, -i\}\}$. Note that $\operatorname{sgn}_D(M) = (-1)^{|M \cap S'|} \operatorname{sgn}(M)$ and $\prod_{e \in M} l(e) = (-1)^{|M \cap S'|} i^{|M \cap S|}$ for every perfect matching M of G.

By Lemma 4.1 there exist a point p and a drawing Φ of the graph G in the plane such that $\operatorname{sgn}_D(M) = (-1)^{cr_\Phi(M)}$ for every perfect matching M of G, Φ maps no vertex of G to p, the images of the edges in S contain p and images of other edges do not contain p. Note that $\prod_{e \in M} l(e) \in \mathbb{R}$ for every perfect matching M and therefore $|M \cap S|$ is even. Denote $|M \cap S|/2$ by $z(M)$. We have $cr_{p,\Phi}(M) = cr_\Phi(M) + z(M)(2z(M) - 1)$. It follows that

$$(-1)^{cr_{p,\Phi}(M)} = \operatorname{sgn}_D(M)(-1)^{z(M)} = (-1)^{|M \cap S'|} i^{|M \cap S|} \operatorname{sgn}(M) = \\ = \prod_{e \in M} l(e) \operatorname{sgn}(M) = 1.$$

Therefore $cr_{p,\Phi}(M)$ is even.

(2) $\Rightarrow$ (1). By Lemma 4.1 there exists an orientation D of G such that $\operatorname{sgn}_D(M) = (-1)^{cr_\Phi(M)}$ for every perfect matching M of G. For $uv \in E(G)$ with $u < v$ let $l_1(e) = 1$ if $uv \in E(D)$ and let $l_1(e) = -1$ otherwise; let $l_2(e) = i$ if $uv \in S$ and let $l_2(e) = 1$ otherwise. Finally, let $l(e) = l_1(e) l_2(e)$. One can verify that $l : E(G) \to \{\pm 1, \pm i\}$ is a Pfaffian labeling of G by reversing the argument used above. □

We say that a region C of the projective plane is *a crosscap* if its boundary is a simple closed curve and its complement is a disc. We say that a drawing Φ of a graph G in the projective plane is *proper with respect to the crosscap* C if no vertex of G is mapped to C and for every $e \in E(G)$ such that the image of e intersects C and every crosscap $C' \subseteq C$ the image of e intersects C'.

Now we can reformulate Lemma 4.2 in terms of drawings in the projective plane.

Theorem 4.3. *For a graph G the following are equivalent.*

1. *G admits a Pfaffian labeling,*
2. *There exist a crosscap C in the projective plane and a proper drawing Φ of G with respect to C, such that $|M \cap S|$ and $cr_\Phi(M)$*

are even for every perfect matching M of G, where $S \subseteq E(G)$ denotes the set of edges whose images intersect C.

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