ON TWO QUESTIONS ABOUT CIRCULAR CHOOSABILITY

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Abstract. We answer two questions of Zhu on circular choosability of graphs. We show that the circular list chromatic number of an even cycle is equal to 2 and give an example of a graph for which the infimum in the definition of the circular list chromatic number is not attained.

1. Introduction

We follow the definitions from [7]. Let $G = (V(G), E(G))$ be a graph and let p and q be positive integers (following [2] we relax the requirement $p \geq 2q$). A *(p,q)-coloring* of G is a function $c : V(G) \to \{0, 1, \dots, p-1\}$, such that for every edge $uv \in E(G)$ we have $q \leq |c(u) - c(v)| \leq p - q$. The *circular chromatic number* $\chi_c(G)$ of G is defined as

$$\chi_c(G) = \inf\{\frac{p}{q} \mid G \text{ admits a } (p,q)\text{-}coloring\}.$$

The circular chromatic number is a refinement of the chromatic number ($\chi(G) - 1 < \chi_c(G) \leq \chi(G)$ for any graph G [4, 6]) and as such might reflect more structural properties of the graph than the chromatic number. It has been extensively studied in recent years, see [8] for a survey of the subject. The circular choosability, considered in this paper, is a natural circular version of the list-chromatic number. We proceed with the definitions.

Let $t \geq 1$ be a real number. A *t-(p,q)-list assignment* L is a mapping which assigns to each $v \in V(G)$ a set $L(v) \subseteq \{0, 1, \dots, p-1\}$, such that $|L(v)| \geq tq$. An *L-(p,q)-coloring* of G is a (p,q)-coloring f of G such that for every $v \in V(G)$ we have $f(v) \in L(v)$. We say that G is *circular t-(p,q)-choosable* if for every t-(p,q)-list assignment L there exists an L-(p,q)-coloring of G. We say that G is *circular t-choosable* if

Date: September, 17, 2006, revised February, 11, 2008.

The first author was partially supported by NSF grants 0200595 and 0701033.

it is circular t-(p, q)-choosable for any positive integers p and q. Finally, the *circular list chromatic number* of G (or the *circular choosability* of G) is defined as

$$\chi_{c,l}(G) = \inf\{t \geq 1 \ : \ G \text{ is circular } t\text{-choosable}\}.$$

The concept of circular choosability has been recently introduced by Mohar [3] and Zhu [7] and many basic questions about it are still open. In this paper we answer two such questions.

In Section 2 we show that the circular choosability for even cycles $\chi_{c,l}(C_{2k})$ is equal to 2, answering a question of Zhu [8] and verifying a conjecture of Havet, Kang, Müller and Sereni [2]. The circular choosability for odd cycles was computed by Zhu in [7], where he shows that $\chi_{c,l}(C_{2k+1}) = 2 + \frac{1}{k}$.

In Section 3 we compute the circular choosability for the complete bipartite graph $K_{2,4}$. We show that $\chi_{c,l}(K_{2,4}) = 2$, while $K_{2,4}$ is not circular 2-choosable. This gives a negative answer to a question of Zhu [7], whether the infimum in the definition of circular choosability is always attained. By contrast, the infimum in the definition of circular chromatic number is always attained for a finite graph [4, 6].

The following important questions about circular choosability posed by Zhu [7] remain open.

(1) Is $\chi_{c,l}(G)$ always a rational number for a finite graph G? It is not known if there exists an algorithm that given a graph G and a rational number r tests whether $\chi_{c,l}(G) \leq r$.
(2) Does there exist a constant α such that $\chi_{c,l}(G) \leq \alpha\chi_l(G)$ for any finite graph G, where $\chi_l(G)$ denotes the list-chromatic number of a graph G? If it exists what is the smallest such α? In [5] a similar problem for a closely related concept of T_r-choosability is considered. By analogy with a conjecture stated in [5] one might expect $\chi_{c,l}(G) \leq 2\chi_l(G)$ for any graph G. By [7, Theorem 14] this bound, if correct, would be best possible.

We refer the reader to [2, 8] for a more comprehensive list of known results and open problems about circular choosability.

2. The circular list chromatic number of an even cycle

In this section we make use of the combinatorial Nullstellensatz, which we restate here for convenience.

Theorem 2.1. [1, Theorem 1.2] *Let F be an arbitrary field, and let $f = f(x_1, \ldots, x_n)$ be a polynomial in $F[x_1, \ldots, x_n]$. Suppose the degree $deg(f)$ of f is $\sum_{j=1}^{n} t_j$, where each t_j is a non-negative integer, and suppose the coefficient of $\prod_{j=1}^{n} x_j^{t_j}$ in f is non-zero. Then if $S_1, \ldots, S_n$ are subsets of F with $|S_j| > t_j$, there are $s_1 \in S_1, \ldots, s_n \in S_n$ so that*

$$f(s_1, \ldots, s_n) \neq 0.$$

Theorem 2.2. *Every even cycle has circular choosability 2.*

Proof. For every graph with G with at least one edge, one trivially has $\chi_{c,l}(G) \geq 2$. It remains to show that for any positive integers p, q and n and a 2-(p, q)-list assignment L there exists an L-(p, q)-coloring of the even cycle C_{2n} with $2n$ vertices. Let $v_1 = v_{2n+1}, v_2, \ldots, v_{2n}$ be the vertices of C_{2n}, in order. Let i denote a complex square root of -1.

Consider the polynomial $f \in \mathbb{C}[x_1, x_2, \ldots, x_{2n}]$:

$$f(x_1, x_2, \ldots, x_{2n}) = \prod_{j=1}^{2n} \prod_{k=-q+1}^{q-1} \left(x_j - e^{2\pi ik/p} x_{j+1}\right),$$

where $x_{2n+1} = x_1$, by convention. Then $\deg(f) = 2n(2q - 1)$. We consider the coefficient of $\prod_{j=1}^{2n} x_j^{2q-1}$ in f. It is equal to $\sum_{l=0}^{2q-1} a_l^{2n}$, where a_l is the coefficient of $x_j^{2q-1-l} x_{j+1}^{l}$ in $\prod_{k=-q+1}^{q-1}(x_j - e^{2\pi ik/p} x_{j+1})$. Clearly,

$$a_l = \sum_{\substack{J \subseteq \{-q+1, \ldots, q-1\} \\ |J| = l}} \prod_{s \in J} (-e^{2\pi is/p}).$$

In particular, for every $0 \leq l \leq 2q-1$ the number a_l is equal to its own complex conjugate and therefore a_l is real. Further note that $a_0 = 1$. It follows that the coefficient in question is a positive real number.

Let $\phi : \{0, 1, \ldots, p-1\} \to \mathbb{C}$ be defined by $\phi(k) = e^{2\pi ik/p}$, and let $S_j = \phi(L(v_j))$ for $1 \leq j \leq 2n$. Then ϕ is an injection and $|S_j| > 2q-1$. By Theorem 2.1 there exist $s_1 \in S_1, s_2 \in S_2 \ldots, s_{2n} \in S_{2n}$ such that $f(s_1, \ldots, s_{2n}) \neq 0$. Define $c(v_j) = \phi^{-1}(s_j)$ for $j \in \{1, 2, \ldots, 2n\}$. We claim that c is a valid L-(p, q)-coloring of C_{2n}. By definition $c(v_j) \in \phi^{-1}(\phi(L(v_j))) = L(v_j)$ for every $1 \leq j \leq 2n$. Moreover, $q \leq |c(v_j) - c(v_{j+1})| \leq p-q$, as $s_j - e^{2\pi ik/p} s_{j+1} \neq 0$ for all k, such that $0 \leq |k| \leq q-1$ or $p - q + 1 \leq |k| \leq p - 1$. Thus the claim holds. $\square$

3. Circular choosability of $K_{2,4}$

As noted in [7] if a graph G is not k-list-colorable for an integer k then it is not circular k-choosable. Therefore the complete bipartite graph $K_{2,4}$ is not circular 2-choosable, as $\chi_l(K_{2,4}) = 3$. In this section we show that $K_{2,4}$ is circular t-choosable for every $t > 2$. It follows that $\chi_{c,l}(K_{2,4}) = 2$, thereby providing a negative answer to a question of Zhu, whether the infimum in the definition of the circular list-chromatic number is always attained.

Let $R_p = \{0, 1, 2, \ldots, p-1\}$ considered with the natural circular order and metric on it. For $x, y \in R_p$ denote by $d(x \to y)$ the remainder of $y - x$ modulo p and by $d(x, y)$ the distance between x and y, i.e. $\min(d(x \to y), d(y \to x))$. For $x, y \in R_p$ and a positive integer q denote by $B_q(x)$ the interval $\{z \in R_p \mid d(x, z) < q\}$, and denote $B_q(x) \cup B_q(y)$ by $B_q(x, y)$. We omit the index q when it is clear from context. We proceed by proving two technical lemmas.

Lemma 3.1. *Let $x_1, y_1, x_2, y_2 \in R_p$ (not necessarily distinct) appear in R_p in circular order. Suppose that $p \geq 4q-3$ and $d(y_1 \to x_2) + d(y_2 \to x_1) \geq 2q-1$. Then*

$$|B_q(x_1, y_1) \cap B_q(x_2, y_2)| \leq 2q - 1.$$

Proof. Note that $B(x_1) \cap B(x_2) \subseteq B(y_1) \cup B(y_2)$. Therefore, we have $B(x_1) \cap B(x_2) \subseteq (B(y_1) \cap B(x_1) \cap B(x_2)) \cup (B(y_2) \cap B(x_1) \cap B(x_2))$ and hence $B(x_1) \cap B(x_2) \subseteq (B(y_1) \cap B(x_2)) \cup (B(x_1) \cap B(y_2))$. Similarly $B(y_1) \cap B(y_2) \subseteq (B(y_1) \cap B(x_2)) \cup (B(x_1) \cap B(y_2))$. It follows that

$$B(x_1, y_1) \cap B(x_2, y_2) \subseteq (B(y_1) \cap B(x_2)) \cup (B(x_1) \cap B(y_2)).$$

For any $x, y \in R_p$ the set $B(x) \cap B(y)$ is an interval in R_p, as otherwise $B(x) \cup B(y)$ must cover R_p, while we have $p = |R_p| > |B(x)| + |B(y)| - 2 = 4q - 4$. It follows, in particular, that

$$|B(x) \cap B(y)| = \max(2q - 1 - d(x, y), 0).$$

If $d(y_1 \to x_2) = d(y_1, x_2)$ and $d(y_2 \to x_1) = d(y_2, x_1)$ then by the above we have

$$\begin{aligned}
&|(B(y_1) \cap B(x_2)) \cup (B(x_1) \cap B(y_2))| \\
&\qquad \leq |(B(y_1) \cap B(x_2))| + |(B(x_1) \cap B(y_2))| \\
&\qquad \leq \max(2q-1, 4q-2-d(y_1 \to x_2) - d(y_2 \to x_1)) \\
&\qquad = 2q-1.
\end{aligned}$$

If, on the other hand, the above condition does not hold, then without loss of generality, we assume $d(x_2, y_1) = d(x_2 \to y_1)$. In this case, we have $B(y_1) \cap B(x_2) \subseteq B(x_1) \cap B(y_2)$ and

$$|(B(y_1) \cap B(x_2)) \cup (B(x_1) \cap B(y_2))| \leq |B(x_1) \cap B(y_2)| \leq 2q-1.$$

□

By $[x, y]$ we denote the interval of R_p with ends x and y, formally defined as follows

$$[x, y] = \{z \in R_p \mid d(x \to z) \leq d(x \to y)\}.$$

Note that this notation is asymmetric, i.e. if $x \neq y$ then $[x, y] \neq [y, x]$.

Lemma 3.2. *Let $x_1, x_2, ..., x_{q+1}, y_1, y_2, ..., y_{q+1} \in R_p$ be distinct and appear in circular order. Suppose that $d(x_i, y_i) \geq 2q+1$ for every i such that $1 \leq i \leq q+1$. Then*

$$|\bigcap_{i=1}^{q+1} B_q(x_i, y_i)| \leq 2q-2.$$

Proof. We prove the lemma by induction on q. The base case $q = 1$ is trivial.

For the induction step, consider $x_1, x_2, ..., x_{q+1}, y_1, y_2, ..., y_{q+1} \in R_p$ satisfying the conditions of the lemma with $|\bigcap_{i=1}^{q+1} B_q(x_i, y_i)|$ maximal and subject to that with $d(x_1 \to x_{q+1}) + d(y_1 \to y_{q+1})$ minimal. Let $Z = \{x_1, x_2, ..., x_{q+1},\ y_1, y_2, ..., y_{q+1}\}$. We claim that

Claim 1. *there exists j such that $1 \leq j \leq q+1$ and*

$$\{x_j + 1, y_j + 1\} \cap Z = \emptyset.$$

Proof. Suppose not. Let us choose k minimal so that

$$|\{x_k + 1, y_k + 1\} \cap Z| < 2.$$

Then $k < q+1$, as otherwise Z is contained in an interval of length $2q+2$ and in particular $d(x_1, y_1) = q+1 < 2q+1$. Without loss of generality we assume $y_k + 1 = y_{k+1}$. Let $x'_i = x_i + 1$ for $1 \le i \le k$ and let $x'_i = x_i$ for $k+1 \le i \le q+1$. We have

$$d(x'_1 \to x'_{q+1}) + d(y_1 \to y_{q+1}) < d(x_1 \to x_{q+1}) + d(y_1 \to y_{q+1}).$$

For $i \le k$, we have

$$\begin{aligned} d(x'_i \to y_i) &= d(x'_i \to (y_{k+1} - k - 1 + i)) \\ &= d((x'_i - i + k + 1) \to y_{k+1}) \ge d(x_{k+1} \to y_{k+1}), \end{aligned}$$

and therefore

$$\begin{aligned} d(x'_i, y_i) &= \min(d(x'_i \to y_i), d(y_i \to x'_i)) \\ &\ge \min(d(x_{k+1} \to y_{k+1}), d(y_i \to x_i) + 1)) \ge 2q + 1. \end{aligned}$$

It remains to show that $|\bigcap_{i=1}^{k+1} B(x'_i, y_i)| \ge |\bigcap_{i=1}^{k+1} B(x_i, y_i)|$ to verify that the set $\{x'_1, x'_2, ..., x'_{q+1}, y_1, y_2, ..., y_{q+1}\}$ gives a contradiction with the choice of Z and therefore to prove the claim. Note that

$$\begin{aligned} \bigcap_{i=1}^{q+1} B(x_i, y_i) - \bigcap_{i=1}^{q+1} B(x'_i, y_i) &\subseteq \bigcup_{i=1}^{k} (B(x_i) - B(x'_i)) \\ &= \{x_i - q + 1 : i = 1, 2, \ldots, k\}. \end{aligned}$$

For any $1 \le i \le k$ we have

$$\begin{aligned} d(y_{i+1} \to (x_i - q + 1)) &= d((y_i + 1) \to (x_i - q + 1)) \\ &= d(y_i \to x_i) - q \ge q + 1. \end{aligned}$$

It follows that $x_i - q + 1 \notin B(x_{i+1}, y_{i+1})$ and consequently

$$\{x_i - q + 1 : i = 1, 2, \ldots, k\} \cap (\bigcap_{i=1}^{q+1} B(x_i, y_i)) = \emptyset.$$

Therefore $\bigcap_{i=1}^{q+1} B(x'_i, y_i) \supseteq \bigcap_{i=1}^{q+1} B(x_i, y_i)$ and the claim holds. $\square$

Denote by $\mathcal{I}(a, b)$ the set of all intervals in R_a of cardinality b. Then by considering the bijection $x \leftrightarrow B_q(x)$ between R_p and $\mathcal{I}(p, 2q-1)$ one can see that

$$|\bigcap_{i=1}^{q+1} B(x_i, y_i)| = |\{I \in \mathcal{I}(p, 2q-1) \mid |I \cap Z| = q+1\}|.$$

Consider now a bijective map $\tau : R_p - \{x_j, x_j+1, y_j, y_j+1\} \to R_{p-4}$ that preserves the circular order. Note that $\tau(Z)$ satisfies the conditions of the lemma with q replaced by $q-1$ and therefore by the induction hypothesis and the observation above we have

$$|\{I \in \mathcal{I}(p-4, 2q-3) \mid |I \cap \tau(Z)| = q\}| \leq 2q-4.$$

Note that τ maps elements of $\{I \in \mathcal{I}(p, 2q-1) \mid |I \cap Z| = q+1\}$ to elements of $\mathcal{I}(p-4, 2q-2) \cup \mathcal{I}(p-4, 2q-3)$ and is injective. Furthermore, if $I \in \mathcal{I}(p, 2q-1)$ and $|I \cap Z| = q+1$ then $|\tau(I) \cap \tau(Z)| = q$. We have

$$\begin{aligned}
&|\{I \in \mathcal{I}(p, 2q-1) \mid |I \cap Z| = q+1\}| \\
\leq\ & |\{I \in \mathcal{I}(p-4, 2q-3) \mid |I \cap \tau(Z)| = q\}| \\
&\quad + |\{I \in \mathcal{I}(p, 2q-1) \mid |I \cap Z| = q+1,\ |\tau(I)| = 2q-2\}| \\
\leq\ & 2q-4 \\
&\quad + |\{I \in \mathcal{I}(p, 2q-1) \mid |I \cap Z| = q+1,\ |\tau(I)| = 2q-2\}|.
\end{aligned}$$

We finish the proof of the lemma by showing that

$$|\{I \in \mathcal{I}(p, 2q-1) \mid |I \cap Z| = q+1,\ |\tau(I)| = 2q-2\}| \leq 2.$$

The set $\{I \in \mathcal{I}(p, 2q-1) \mid |\tau(I)| = 2q-2\}$ consists of four intervals, namely $[x_j - 2q - 2, x_j]$, $[x_j + 1, x_j + 2q - 1]$, $[y_j - 2q - 2, y_j]$ and $[y_j + 1, y_j + 2q - 1]$. The intervals $[x_j + 1, x_j + 2q - 1]$ and $[y_j + 1, y_j + 2q - 1]$ contain neither x_j nor y_j and therefore each of them contains less than q elements of Z. The required inequality follows. □

Theorem 3.3. *For every $t > 2$ the graph $K_{2,4}$ is circular t-choosable.*

Proof. Let $\{u_1, u_2\}, \{v_1, v_2, v_3, v_4\}$ be the parts of $K_{2,4}$. Let L be a t-(p, q)-list assignment for some $t > 2$ and positive integers p and q. Note that $|L(w)| \geq 2q + 1$ for every $w \in V(K_{2,4})$.

For $x_1 \in L(u_1), x_2 \in L(u_2)$ we may assume that there exists $i \in \{1, 2, 3, 4\}$ such that $L(v_i) \subseteq B_q(x_1, x_2)$, as otherwise there exists an L-(p, q)-coloring c of $K_{2,4}$, defined as follows. Let $c(u_j) = x_j$ for $j \in \{1, 2\}$ and choose $c(v_i) \in L(v_i) - B_q(x_1, x_2)$ for all $i \in \{1, 2, 3, 4\}$. Let $f(x_1, x_2) = f(x_2, x_1)$ one index i such that $L(v_i) \subseteq B_q(x_1, x_2)$.

In particular, it follows that $L(u_1) \cap L(u_2) = \emptyset$ and $p \geq 4q+2$. We say that an interval $[x, y]$ in R_p is *clean* if for some $\{i, j\} = \{1, 2\}$ we have $x, y \in L(u_i)$ and $[x, y] \cap L(u_j) = \emptyset$. Let $[x_0, y_0], [x_1, y_1], \ldots, [x_{2k-1}, y_{2k-1}]$ for some $k \geq 1$ be all the maximal clean intervals in R_p. Then these intervals are disjoint and we assume without loss of generality that $x_0, y_0, x_1, y_1, \ldots, x_{2k-1}, y_{2k-1}$ appear in R_p in the clockwise order, and that $x_i, y_i \in L(u_j)$ for $i \in \{0, 1, \dots, 2k-1\}$ and $j \in \{1, 2\}$ if and only if i equals to j modulo 2.

Let $x_{2k} = x_0$. Note that for distinct $i, j \in \{0, 1, \dots, 2k-1\}$ one has $[x_{i+1}, y_j] \cup [x_{j+1}, y_i] \supseteq L(u_1) \cup L(u_2)$ and therefore $d(x_{i+1} \to y_j) + d(x_{j+1} \to y_i) \geq |L(u_1)| + |L(u_2)| - 2 \geq 4q$. Therefore $B(y_i, x_{i+1}) \cap B(y_j, x_{j+1}) < 2q + 1$ by Lemma 3.1 and consequently $f(y_i, x_{i+1}) \neq f(y_j, x_{j+1})$. It follows by the pigeon-hole principle that $k \leq 2$. We consider the cases $k = 2$ and $k = 1$ separately.

Suppose first that $k = 2$. Without loss of generality we assume $|[x_0, y_0] \cap L(u_2)| \geq q + 1$ and $|[x_1, y_1] \cap L(u_1)| \geq q + 1$. Therefore there exist $z_1 \in [x_1, y_1] \cap L(u_1)$ such that $[x_1, z_1] \cap L(u_1) = q + 1$ and $z_2 \in [x_0, y_0] \cap L(u_2)$ such that $[z_2, y_0] \cap L(u_2) = q + 1$. It follows that $d(x_1 \to z_1) + d(z_2 \to y_0) \geq 2q$ and from Lemma 3.1 we have $f(y_0, x_1) \neq f(z_1, z_2)$. Additionally, it is easy to see that $d(z_1 \to y_i) + d(x_{i+1} \to z_2) \geq 2q$ for every $i \in \{1, 2, 3\}$, as $[z_1, y_i] \cup [x_{i+1}, z_2]$ contains $q+1$ elements of $L(u_1)$ and $q + 1$ elements of $L(u_2)$. Therefore $f(y_i, x_{i+1}) \neq f(z_1, z_2)$ for every $i \in \{1, 2, 3\}$ and combining this with the result in the previous paragraph we obtain a contradiction.

It remains to consider the case $k = 1$. Assume for convenience and without loss of generality that $f(y_0, x_1) = 3$ and $f(y_1, x_0) = 4$. Let $x_1 = s_1, s_2, \dots, s_{|L(u_1)|} = y_1$ be all the elements of $L(u_1)$ numbered in the clockwise order and let $x_0 = t_1, t_2, \dots, t_{|L(u_2)|} = y_0$ be all the elements of $L(u_2)$ numbered in the clockwise order. Note that $d(s_i, t_i) \geq 2q+1$. For every $i \in \{1, 2, \dots 2q+1\}$ we have $[x_1, s_i] \cup [t_i, y_0] \supseteq \{s_1, \dots s_i, t_i, t_{i+1} \dots, t_{2q+1}\}$ and therefore $d(x_1 \to s_i) + d(t_i \to y_0) \geq 2q$. Similarly, $d(s_i \to y_1) + d(x_0 \to t_i) \geq 2q$.

Thus another application of Lemma 3.1 shows that $f(s_i, t_i) \in \{1, 2\}$ for every $i \in \{1, 2, \dots 2q + 1\}$. Therefore there exists $Z \subseteq \{1, 2, \dots 2q + 1\}$ and $j \in \{1, 2\}$ such that $|Z| \geq q + 1$ and $L(v_j) \subseteq B(s_i, t_i)$ for every $i \in Z$. However, by Lemma 3.2 we have $|\bigcap_{i \in Z} B(s_i, t_i)| \leq 2q - 2$ and consequently $L(v_j) \not\subseteq \bigcap_{i \in Z} B(s_i, t_i)$. This contradiction finishes the proof. $\square$

Acknowledgments. I would like to thank Daniel Kral for introducing me to the problem and Robin Thomas for valuable discussions and comments on the draft.

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