MINIMAL BRICKS

SERGUEI NORINE AND ROBIN THOMAS

Abstract. A brick is a 3-connected graph such that the graph obtained from it by deleting any two distinct vertices has a perfect matching. A brick is minimal if for every edge e the deletion of e results in a graph that is not a brick. We prove a generation theorem for minimal bricks and two corollaries: (1) for $n \geq 5$, every minimal brick on $2n$ vertices has at most $5n-7$ edges, and (2) every minimal brick has at least three vertices of degree three.

1. Introduction

All the graphs considered in this paper are finite and simple. A *brick* is a 3-connected graph such that the graph obtained from it by deleting any two distinct vertices has a perfect matching. The importance of bricks stems from the fact that they are building blocks of the matching decomposition procedure of Kotzig, and Lovász and Plummer [5]. In particular, many matching problems of interest (such as, for example, computing the dimension of the linear hull [2] or lattice [4] of incidence vectors of perfect matchings, or characterizing graphs that admit a "Pfaffian orientation" [7]) can be reduced to bricks.

In an earlier paper we proved a generation theorem for bricks. The precise statement requires a large number of definitions, and is given in Theorem 2.3 below. Let us describe the result informally first. Let G be a graph, and let v_0 be a vertex of G of degree two incident with the edges $e_1 = v_0v_1$ and $e_2 = v_0v_2$. Let H be obtained from G by contracting both e_1 and e_2 and deleting all resulting parallel edges. We say that H was obtained from G by *bicontracting* or *bicontracting the vertex* v_0, and write $H = G/v_0$. A subgraph J of a graph G is *central* if $G\backslash V(J)$ has a perfect matching. We say that a graph H is a *matching minor* of a graph G if H can be obtained from a central subgraph of G by repeatedly bicontracting vertices of degree two. We denote the fact that H is isomorphic to a matching minor of G by writing $H \hookrightarrow G$. Our generation theorem of [6] asserts that, except for a few well-described exceptions, if $H \hookrightarrow G$, then a graph isomorphic to H can be obtained from G by repeatedly applying a certain operation in such a way that all the intermediate graphs are bricks and no parallel edges are produced. The operation is as follows: first delete an edge, and for every vertex of degree two that results contract both edges incident with it. The theorem improves a recent result of de Carvalho, Lucchesi and Murty [1], but in this paper we seem to need our result.

We found our theorem useful for generating interesting examples of bricks and testing various conjectures, but even more useful was a variant for minimal bricks, which we prove in this paper. A brick G is *minimal* if $G\backslash e$ is not a brick for every edge $e \in E(G)$. (We

30 September 2005. Partially supported by NSF grants 0200595 and 0354742.

use \ for deletion.) The theorem asserts that every minimal brick other than the Petersen graph can be obtained from K_4 or the prism (the complement of a cycle of length six) by taking "strict extensions" in such a way that all the intermediate graphs are minimal bricks not isomorphic to the Petersen graph. The theorem is formally stated as Theorem 3.2. We postpone the definition of strict extensions until they are needed.

The paper is organized as follows. In the next section we introduce the results from [6] that we need. In Section 3 we state and prove our generation theorem for minimal bricks; we deduce it from the more general Theorem 3.1. In Section 4 we prove that, except for four graphs on at most eight vertices, every minimal brick on $2n$ vertices has at most $5n-7$ edges. Finally, in Section 5 we prove that every minimal brick has at least three vertices of degree three.

2. The tools

In this section we state the results of [6] that we need, but let us start with the following theorem of Lovász [3]; see also [5, Theorem 5.4.11].

Theorem 2.1. *Every brick has a matching minor isomorphic to K_4 or the prism.*

The theorem of de Carvalho, Lucchesi and Murty [1] mentioned in the introduction uses K_4 and the prism as the starting graphs of their generation procedure. We use a more restricted set of operations, and the price we pay for that is that the starting set has to be expanded. We now introduce the relevant classes of graphs.

Let C_1 and C_2 be two vertex-disjoint cycles of length $n \geq 3$ with vertex-sets $\{u_1, u_2, \ldots, u_n\}$ and $\{v_1, v_2, \ldots, v_n\}$ (in order), respectively, and let G_1 be the graph obtained from the union of C_1 and C_2 by adding an edge joining u_i and v_i for each $i = 1, 2, \ldots, n$. We say that G_1 is a *planar ladder*. Let G_2 be the graph consisting of a cycle C with vertex-set $\{u_1, u_2, \ldots, u_{2n}\}$ (in order), where $n \geq 2$ is an integer, and n edges with ends u_i and u_{n+i} for $i = 1, 2, \ldots, n$. We say that G_2 is a *Möbius ladder*. A *ladder* is a planar ladder or a Möbius ladder. Let G_1 be a planar ladder as above on at least six vertices, and let G_3 be obtained from G_1 by deleting the edge u_1u_2 and contracting the edges u_1v_1 and u_2v_2. We say that G_3 is a *staircase*. Let $t \geq 2$ be an integer, and let P be a path with vertices $v_1, v_2, \ldots, v_t$ in order. Let G_4 be obtained from P by adding two distinct vertices x, y and edges xv_i and yv_j for $i = 1, t$ and all even $i \in \{1, 2, \ldots, t\}$ and $j = 1, t$ and all odd $j \in \{1, 2, \ldots, t\}$. Let G_5 be obtained from G_4 by adding the edge xy. We say that G_5 is an *upper prismoid*, and if $t \geq 4$, then we say that G_4 is a *lower prismoid*. A *prismoid* is a lower prismoid or an upper prismoid.

We need the following strengthening of Theorem 2.1, proved in [6, Theorem (1.8)].

Theorem 2.2. *Let G be a brick not isomorphic to K_4, the prism or the Petersen graph. Then G has a matching minor isomorphic to one of the following seven graphs: the graph obtained from the prism by adding an edge, the lower prismoid on eight vertices, the staircase on eight vertices, the staircase on ten vertices, the planar ladder on ten vertices, the wheel on six vertices, and the Möbius ladder on eight vertices.*

In the introduction we described our generation theorem by means of operations that reduce the larger graph G to its matching minor H. This version is easier to describe concisely, but for both the proof and the applications it is better to proceed the other way, namely to describe how to obtain G from H. Thus we reverse the process now and proceed in the other direction. Here are the relevant definitions.

Let $H, G, v_0, v_1, v_2, e_1, e_2$ be as in the definition of bicontraction. Assume that v_1, v_2 are not adjacent, that they both have degree at least three and that they have no common neighbors except v_0; then no parallel edges are produced during the contraction of e_1 and e_2. Let v be the new vertex that resulted from the contraction. We say that G was obtained from H by *bisplitting the vertex* v. We call v_0 the *new inner vertex* and v_1 and v_2 the *new outer vertices*. Let H be a graph. We wish to define a new graph H'' and two vertices of H''. Either $H'' = H$ and u, v are two nonadjacent vertices of H, or H'' is obtained from H by bisplitting a vertex, u is the new inner vertex of H'' and $v \in V(H'')$ is not adjacent to u, or H'' is obtained by bisplitting a vertex of a graph obtained from H by bisplitting a vertex, and u and v are the two new inner vertices of H''. Finally, let H' be obtained from H'' by adding an edge with ends u, v. We say that H' is a *linear extension* of H.

Since in the next theorem the graph H need not be a brick we need two more exceptional classes of graphs. Let C be an even cycle with vertex-set $v_1, v_2, \ldots, v_{2t}$ in order, where $t \geq 2$ is an integer and let G_6 be obtained from C by adding vertices v_{2t+1} and v_{2t+2} and edges joining v_{2t+1} to the vertices of C with odd indices and v_{2t+2} to the vertices of C with even indices. Let G_7 be obtained from G_6 by adding an edge $v_{2t+1}v_{2t+2}$. We say that G_7 is an *upper biwheel*, and if $t \geq 3$ we say that G_6 is a *lower biwheel*. A *biwheel* is a lower biwheel or an upper biwheel. Please note that biwheels are bipartite, and therefore are not bricks.

We are now ready to state a version of our generation theorem [6, Theorem (1.10)]. The version mentioned in the introduction follows easily, because a linear extension of a brick is a brick.

Theorem 2.3. *Let G be a brick other than the Petersen graph, and let H be a 3-connected matching minor of G. Assume that if H is a planar ladder, then there is no strictly larger planar ladder L with $H \hookrightarrow L \hookrightarrow G$, and similarly for Möbius ladders, wheels, lower biwheels, upper biwheels, staircases, lower prismoids and upper prismoids. If H is not isomorphic to G, then some matching minor of G is isomorphic to a linear extension of H.*

3. Generation Theorem for Minimal Bricks

In this section we prove a generation theorem for minimal bricks, Theorem 3.2 below. We derive it from the more general Theorem 3.1.

If H is a graph, and $u, v \in V(H)$ are distinct nonadjacent vertices, then $H + (u, v)$ or $H + uv$ denotes the graph obtained from H by adding an edge with ends u and v. If u and v are adjacent or equal then $H + uv = H$. Now let $u, v \in V(H)$ be adjacent. By *bisubdividing* the edge uv we mean replacing the edge by a path of length three, say a path with vertices u, x, y, v, in order. Let H' be obtained from H by this operation. We say that x, y (in that order) are the *new vertices*. Thus y, x are the new vertices resulting from subdividing the edge vu (we are conveniently exploiting the notational asymmetry for edges). Now if $w \in V(H) - \{u\}$, then by $H + (w, uv)$ we mean the graph $H' + (w, x)$. Notice that the graphs $H + (w, uv)$ and $H + (w, vu)$ are different.

Let H be a graph, let $u, v \in V(H)$ be distinct, and let H' be obtained from $H + uv$ by bisubdividing uv, where the new vertices are x, y. Let $x' \in V(H) - \{u\}$ and $y' \in V(H) - \{v\}$ be not necessarily distinct vertices such that not both belong to $\{u, v\}$. In those circumstances we say that $H' + (x, x') + (y, y')$ is a *quasiquadratic extension of* H. We say that it is a *quadratic extension of* H if u and v are not adjacent in H. (Recall our convention that if u and v are adjacent in H, then $H + uv = H$.) We say that uv is *the base* of this quasiquadratic extension.

Now let u, v, H', x, y be as above, and let $a, b \in V(H)$ be distinct vertices such that $\{u, v\} \neq \{a, b\}$. Let H'' be obtained from $H' + ab$ by bisubdividing ab, and let x', y' be the new vertices. Then the graph $H'' + (x, x') + (y, y')$ is called a *quasiquartic extension of* H. It is a *quartic extension of* H if $uv, ab \in E(H)$. We say that uv, ab are *the bases* of the quasiquartic extension. Quadratic and quartic extensions were used in the proof of Theorem 2.3 in [6]; quasiquadratic and quasiquartic extensions are new.

We need to define two new types of extension. We say that a linear extension H' of a graph H is *strict* if $|V(H')| > |V(H)|$. Let u, v, w be pairwise distinct vertices of H, let H' be obtained from H by bisplitting u, and let u_0 be the new inner vertex and u_1 a new outer vertex. If $u_1 v \in E(H')$ and $vw \notin E(H)$ then the graph $H' + (u_0, vu_1) + (y, w)$, where x, y are the new vertices of $H' + (u_0, vu_1)$, is called a *bilinear* extension of H. If $uw \notin E(H)$ then the graph $H' + (u_0, u_1 u_0) + (b, w)$, where a, b are the new vertices of $H' + (u_0, u_1 u_0)$, is called a *pseudolinear* extension of H. See Figure 1.

Finally, we say that H' is a *strict* extension of H if H' is a quasiquadratic, quasiquartic, bilinear, pseudolinear or strict linear extension of H. It is not hard to see that a strict extension of a brick is a brick.

Theorem 3.1. *Let G be a brick other than the Petersen graph, and let H be a 3-connected matching minor of G such that $|V(H)| < |V(G)|$. Then some matching minor of G is isomorphic to a strict extension of H.*

Proof. Let a graph H' be chosen so that H is a spanning subgraph of H', $H' \hookrightarrow G$ and $|E(H')|$ is maximal.

Suppose first that H' is a planar ladder and there exists a planar ladder L with $H' \hookrightarrow L \hookrightarrow G$ and $|V(L)| > |V(H')|$. Then clearly $H' = H$, and if we choose L with $|V(L)|$ minimum, then L is a quartic extension of H and therefore the theorem holds. Therefore we can assume that if H' is a planar ladder, then there is no strictly larger planar ladder L with $H \hookrightarrow L \hookrightarrow G$, and similarly for Möbius ladders, wheels, lower biwheels, upper biwheels, staircases, lower prismoids and upper prismoids. By Theorem 2.3 and the choice of H' there exists a strict linear extension K of H' such that $K \hookrightarrow G$. We denote $E(H') - E(H)$ by E' and break the analysis into cases depending on the type of strict linear extension.

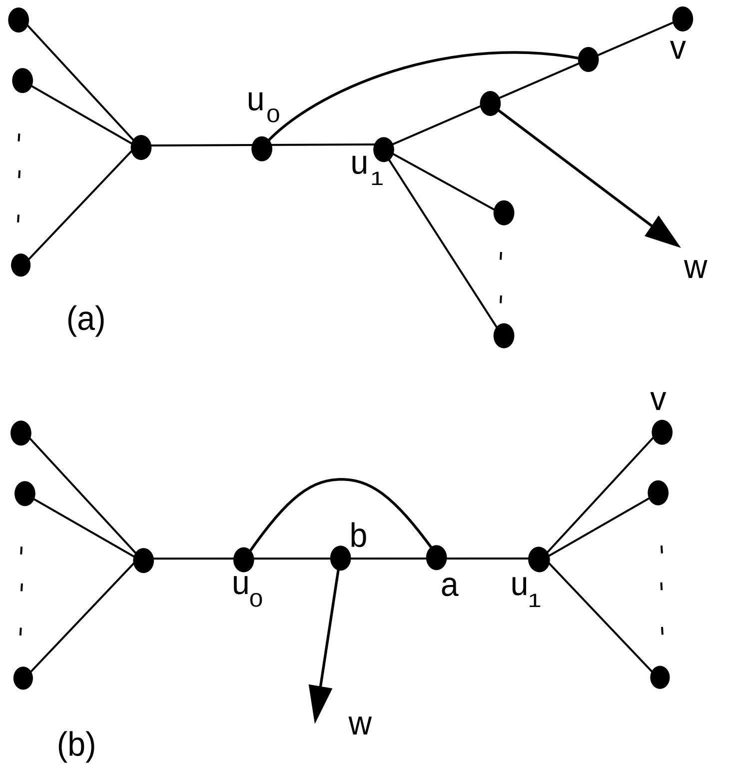

FIGURE 1. (a) Bilinear extension, (b) Pseudolinear extension

Suppose first that $K = K' + uv$, where K' is obtained from H' by bisplitting a vertex, v is the new inner vertex of K' and $u \in V(H')$. Let v_1 and v_2 be the new outer vertices. We have $E(H') \subseteq E(K')$, in the natural way. For $i = 1, 2$ let d_i be the number of edges of $E(H)$ that are incident with v_i in K' (or K). We assume without loss of generality that $d_1 \geq d_2$. Note that $d_1 + d_2 \geq 3$, because v has degree at least three in H.

If $d_2 \geq 2$ then $K \setminus E'$ is a strict linear extension of H. If $d_2 = 1$ let $f \in E'$ be an edge incident with v_2; then $K \setminus (E' - \{f\})$ is a quadratic extension of H. Finally, if $d_2 = 0$ and $f_1, f_2 \in E'$ are incident with v_2 then $K \setminus (E' - \{f_1, f_2\})$ is a quasiquadratic extension of H.

Now suppose $K = K' + u_1u_2$, where K' is obtained by bisplitting a vertex of a graph obtained from H' by bisplitting a vertex, and u_1 and u_2 are the two new inner vertices of K'. Let v_1, v_2 and v_3, v_4, respectively, be the corresponding new outer vertices. Let d_1, d_2, d_3 and d_4 be defined analogously as above. We start by assuming that v_1, v_2, v_3 and v_4 are pairwise distinct and without loss of generality assume $d_1 \geq d_2$, $d_3 \geq d_4 \geq d_2$.

If $d_2 \geq 2$ then $K \setminus E'$ is a strict linear extension of H. If $d_2 = 1, d_4 \geq 2$ then $K \setminus E'/v_2$ is isomorphic to a strict linear extension of H unless the edge of H incident with v_2 is incident also with one of the vertices v_3 and v_4. In this case $K \setminus (E' - \{f\})$ is a bilinear extension of H, for every $f \in E'$ incident with v_2. If $d_2 = d_4 = 1$ for $i \in \{1, 2\}$ let e_i denote the unique edge in $E(H)$ incident with d_{2i} and let f_i denote some edge in E' incident with d_{2i}. If $e_1 = e_2$ then $K \setminus (E' - \{f_1, f_2\})$ is a quasiquartic extension of H. Otherwise, without loss of generality we assume that e_2 is not incident with v_1 and deduce that $K \setminus (E' - \{f_1\})/v_4$ is a quadratic extension of H with base e_1.

It remains to consider the subcase when $d_2 = 0$. Let $f, f' \in E'$ be incident with v_2 such that f has no end in $\{v_3, v_4\}$. If $d_4 \geq 2$ then $K \setminus (E' - \{f\}) \setminus u_1v_1/u_1$ is a strict linear extension of H. If $d_4 = 1$ let e denote the unique edge in $E(H)$ incident with v_4. If e is not incident with v_1 then $K \setminus (E' - \{f, f'\})/v_4$ is a quasiquadratic extension of H if f' is not incident with v_4 and $K \setminus (E' - \{f, f'\})$ is a quasiquartic extension of H if f' is incident with v_4. If on the other hand e is incident with v_1 then $K \setminus (E' - \{f, f''\}) \setminus u_1v_1/u_1$ is a quadratic extension of H, where f'' is any edge in E' incident with v_4. Finally, if $d_4 = 0$ let $f^* \in E'$ be incident with v_4 and have no end in $\{v_1, v_2\}$. Then $K \setminus (E' - \{f, f', f^*\}) \setminus u_2v_3/u_2$ is a quasiquadratic extension of H. This completes the case when v_1, v_2, v_3 and v_4 are pairwise distinct.

We now assume without loss of generality that $v_1 = v_4$. Then v_1, v_2 and v_3 are pairwise distinct and we assume $d_2 \geq d_3$, again without loss of generality. Suppose first $d_1 = 0$. If $d_3 \geq 2$ then $K \setminus (E' - \{g\})$ is a pseudolinear extension of H, where $g \in E'$ is incident with v_1; if $d_3 = 1$ then $K \setminus (E' - \{g\})/v_3$ is a quadratic extension of H and if $d_3 = 0$ then $K \setminus (E' - \{f, g\})/v_3$ is a quasiquadratic extension of H, where f is an edge in E' incident with v_3 and not adjacent to g. Therefore we may assume $d_1 \geq 1$. If $d_2 \geq 2$ and $d_3 \geq 1$ then $K \setminus E'$ or $K \setminus E'/v_3$ is a strict linear extension of H. If $d_2 \geq 2$ and $d_3 = 0$ then $K \setminus (E' \setminus f)/v_3$ is a quadratic extension of H, where f is as above. If, finally, $d_2 \leq 1$ then let E'' be obtained from E' by deleting $2 - d_2$ edges of E' incident with v_2 and $1 - d_3$ edges incident with v_3; in that case $K \setminus E'' \setminus v_1u_2/u_2$ is a quasiquadratic extension of H.

This completes the case analysis. □

Theorem 3.1 implies the following generation theorem for minimal bricks.

Theorem 3.2. *Let G be a minimal brick other than the Petersen graph. Then G can be obtained from K_4 or the prism by taking strict extensions, in such a way that all the intermediate graphs are minimal bricks not isomorphic to the Petersen graph.*

Proof. Suppose the statement of the theorem is false and let G be a counterexample with $|V(G)|$ minimum.

By Theorem 2.1 we may choose a minimal brick $H \hookrightarrow G$ such that H can be obtained from K_4 or the prism by taking strict extensions and, subject to that, $|V(H)|$ is maximum. If $|V(H)| = |V(G)|$ then H is isomorphic to G by the minimality of G. If, on the other hand, $|V(H)| < |V(G)|$, then by Theorem 3.1 there exists a strict extension $H' \hookrightarrow G$ of H. Let $H'' \hookrightarrow H'$ be a minimal brick with $|V(H'')| = |V(H')|$; then $H'' \hookrightarrow G$. It follows that H'' is not isomorphic to G, for otherwise so is H', contrary to our assumption that G is a counterexample to the theorem. By the minimality of G the graph H'' can be obtained from K_4 or the prism by taking strict extensions, contrary to the choice of H. □

Note that there exist bricks obtained from K_4 or the prism by a sequence of strict extensions, that are not minimal. A simple example follows.

Let G be the prism, $V(G) = \{v_1, v_2, v_3, u_1, u_2, u_3\}$, the vertices v_1,v_2,v_3 are pairwise adjacent and so are the vertices u_1, u_2, u_3, and u_i is adjacent to v_i for $i \in \{1, 2, 3\}$. Let $G' = G + u_1v_2$ and let $G'' = G' + (u_2, u_1v_2) + v_1y$, where x, y are the new vertices of $G' + (u_2, u_1v_2)$. Then G'' is a quasiquadratic extension of G and $G'' \setminus u_1v_1$ is a brick, which can be obtained from a prism by a quadratic extension or a sequence of two linear extensions.

4. Edge Bound for Minimal Bricks

The following theorem is [5, Corollary 5.4.16].

Theorem 4.1. *If G is a minimal bicritical graph with $n \geq 6$ vertices, then $|E(G)| \leq 5(n-2)/2$.*

We use Theorem 3.1 to prove a similar bound for minimal bricks.

Theorem 4.2. *Let G be a minimal brick on $2n$ vertices. Then $|E(G)| \leq 5n - 7$, unless G is the prism or the wheel on four, six or eight vertices.*

Proof. The theorem holds for the Petersen graph, so from now on we assume that G is not the Petersen graph, the prism or the wheel on six or eight vertices. Denote the last three graphs by R_6, W_6 and W_8, respectively.

Note that a strict linear extension increases the number of vertices in a graph by 2 or 4 and the number of edges by 3 or 5, respectively. Similarly, a quasiquadratic extension increases the number of vertices by 2 and the number of edges by at most 5, while quasiquartic, bilinear and pseudolinear extensions increase the number of vertices by 4 and the number of edges by at most 8.

We say that a brick H is *sparse* if $|E(H)| \leq \frac{5}{2}|V(H)| - 7$ and we say that H is *dense* otherwise. We claim that any minimal brick that contains a sparse matching minor is sparse. Suppose G_1 and G_2 are bricks, $G_1 \hookrightarrow G_2$, G_1 is sparse and G_2 is minimal. Let a sparse brick $H \hookrightarrow G_2$ be chosen with $|V(H)|$ maximum. From Theorem 3.1 we deduce that either $|V(H)| = |V(G_2)|$ or some strict extension H' of H is a matching minor of G_2. In the latter case, by the calculations above, H' is sparse in contradiction with the choice of H. Therefore $|V(H)| = |V(G_2)|$ and G_2 is isomorphic to H by the minimality of G_2. The claim follows.

Suppose G is dense. By Theorem 2.2 G has a matching minor isomorphic to one of the seven graph mentioned therein, and hence G has a matching minor isomorphic to one of the following four graphs: R_6, W_6, the staircase on eight vertices, and the Möbius ladder on eight vertices. Among these graphs only two are dense: R_6 and W_6.

Assume first that G contains R_6 as a matching minor. By Theorem 3.1 there exists a strict extension H of the prism such that $H \hookrightarrow G$. By the calculations above H is sparse, unless H is a quadratic extension of $R_6 + uv$ with base uv, where $uv \notin E(R_6)$. We will show that there exists $e \in E(H)$ such that $H \setminus e$ is a brick. Note that $H \setminus e$ is sparse. Therefore it follows that any minimal brick containing the prism as a matching minor and not equal to it is sparse. We prove the existence of e by listing all possible quasiquadratic extensions of R_6 with 14 edges in Figure 2. An edge e that satisfies the conditions above is indicated by

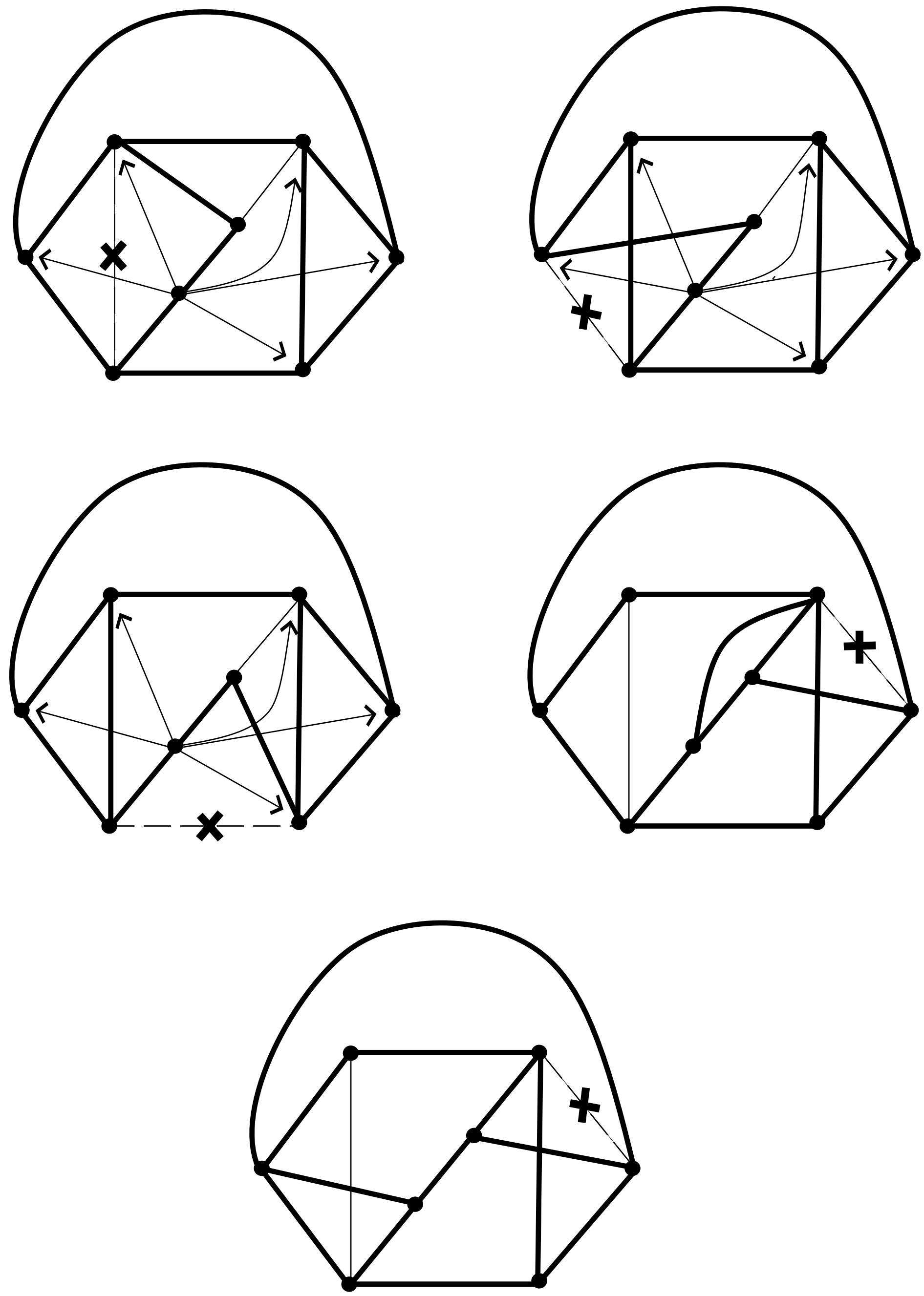

FIGURE 2. Quasiquadratic extensions of the prism with 14 edges

a cross. A spanning bisubdivision or bisplit of R_6 or W_6 in $H \setminus e$ is indicated by bold lines and allows the reader to easily verify that the claim holds in each of the cases.

Therefore we may assume that G contains W_6 as a matching minor and does not contain R_6. By Theorem 2.3 G is a wheel or G contains a linear extension of a wheel as a matching minor. All the wheels on at least ten vertices and all strict linear extensions of W_6 and W_8 are sparse and therefore G must contain a graph obtained from W_6 or W_8 by an edge addition. Every graph obtained from W_8 by adding an edge has a matching minor isomorphic to a graph obtained from W_6 by adding an edge. The latter graph is unique up to isomorphism and contains R_6 as a spanning subgraph, in contradiction with our assumptions. $\square$

The bound given in Theorem 4.2 is tight for every $n \geq 4$. An example of a minimal brick G_n on $2n+4$ vertices with $5n+3$ edges for $n \geq 2$ follows. Let $V(G_n) = \{x, y, z, t, v_1, u_1, v_2, u_2, \ldots, v_n, u_n\}$. For every $i \in \{1, 2, \ldots, n\}$ let $xt, yt, zt, xu_i, yu_i, yv_i, zv_i$ and $u_i v_i$ be the edges of G_n. Then for every $e \in E(G_n)$ the graph $G_n \setminus e$ contains a vertex of degree two, and hence is not a brick. It remains to show that G_n is a brick for every n. Note that G_k is a quasiquadratic extension of G_{k-1} for every $k > 2$. Therefore it suffices to show that G_2 is a brick. The graph $G_2 \setminus u_1 y \setminus v_1 y$ is isomorphic to the prism with one of its edges bisubdivided and consequently G_2 can be obtained from the prism by a quadratic extension.

5. Three Cubic Vertices

De Carvalho, Lucchesi and Murty [1] proved that every minimal brick has a vertex of degree three. According to them (private communication) it had been conjectured by Lovász. We prove the following strengthening.

Theorem 5.1. *Every minimal brick has at least three vertices of degree three.*

Proof. Let a minimal brick G that has at most two vertices of degree three be chosen with $|V(G)|$ minimal. By Theorem 3.2 there exists a minimal brick $H \hookrightarrow G$ with at least three vertices of degree three, such that G is isomorphic to a strict extension of H.

Note that if a strict linear extension is used to obtain G from H then the degree of at most one vertex of H increased and at least one vertex in $V(G) - V(H)$ has degree three. If a quasiquartic, bilinear or pseudolinear extension is used to obtain G then $V(G) - V(H)$ contains at least three vertices of degree three. Therefore G is isomorphic to a quasiquadratic extension of H that is not quadratic.

We assume without loss of generality that $V(G) - V(H) = \{u_1, u_2\}$ and there exist $v_1, v_2, v_3, v_4 \in V(H)$ such that $E(G) - E(H) = \{u_1 v_1,\ u_1 v_2,\ u_2 v_3,\ u_2 v_4, u_1 u_2\}$, at least three of the vertices v_1, v_2, v_3, v_4 are distinct, $v_1 \neq v_2$ and $v_3 \neq v_4$. Note that the vertices of degree three in H must form a subset of $\{v_1, v_2, v_3, v_4\}$ and that $v_1 v_3, v_2 v_3, v_2 v_4, v_1 v_4 \notin E(H)$, for the deletion of such an edge from G results in a quadratic extension of H, contrary to the fact that G is a minimal brick.

Since H is a brick, it is not a biwheel. By Theorem 2.3 either H is a ladder, wheel, staircase or prismoid or H is a linear extension of a brick. If H is a ladder, wheel, staircase or prismoid distinct from K_4 then H has at least 5 vertices of degree three, and consequently G has at least three vertices of degree three. If $H = K_4$ then G is not minimal, by an observation in the previous paragraph.

Therefore, H is a linear extension of a brick, and hence there exists $e \in E(H)$ such that $H \setminus e$ becomes a brick after possible bicontractions of vertices of degree two in such a way that no parallel edges are created by these bicontractions. Note that H is minimal and therefore at least one end of e is a vertex of degree three in H. Assume first that exactly one end of e has degree three in H. Without loss of generality this end is v_1. The graph $G \setminus e$ is a brick, because it can be obtained by a linear extension (first bisplit to produce $H \setminus e$, then add the edge v_1v_3) followed by a quadratic extension with base v_1v_3, a contradiction. Recall that v_1 is not adjacent to v_3 in H.

It remains to consider the case when both of the ends of e have degree three in H. Without loss of generality we assume that $e = v_1v_2$, and hence v_1, v_2, v_3 and v_4 are pairwise distinct. It follows that $G \setminus e$ is a strict linear extension of $H + v_1v_3 + v_1v_4$ and is again a brick. This completes the case analysis. □

We conjecture the following strengthening of Theorem 5.1.

Conjecture 5.2. *There exists $\alpha > 0$ such that every minimal brick G has at least $\alpha|V(G)|$ vertices of degree three.*

Even a much weaker strengthening, namely, the conjecture that every brick has at least four vertices of degree three, seems to require new ideas or a substantial refinement of our techniques.

References

[1] M. H. de Carvalho, C. L. Lucchesi and U. S. R. Murty, How to build a brick, to appear in *Discrete Math.*

[2] J. Edmonds, L. Lovász and W. R. Pulleyblank, Brick decompositions and matching rank of graphs, *Combinatorica* **2** (1982), 247–274.

[3] L. Lovász, Ear-decompositions of matching-covered graphs, *Combinatorica* **2** (1983), 395–407.

[4] L. Lovász, Matching structure and the matching lattice, *J. Combin. Theory Ser. B* **43** (1987), 187–222.

[5] L. Lovász and M. Plummer, Matching theory, *Annals of Discrete Math.* **29**, North-Holland, Amsterdam (1986).

[6] S. Norine and R. Thomas, Generating bricks, submitted. Available from `http://www.math.gatech.edu/~thomas/PAP/brick.pdf`

[7] V. V. Vazirani and M. Yannakakis, Pfaffian orientations, 0-1 permanents, and even cycles in directed graphs, *Discrete Appl. Math.* **25** (1989), 179–190.

This material is based upon work supported by the National Science Foundation under Grants No. 0200595 and 0354742. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.

School of Mathematics, Georgia Tech, Atlanta, GA 30332-0160