DRAWING PFAFFIAN GRAPHS

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Abstract. We prove that a graph is Pfaffian if and only if it can be drawn in the plane (possibly with crossings) so that every perfect matching intersects itself an even number of times.

1. Introduction

In this paper we prove a theorem that connects Pfaffian orientations with the parity of crossings in planar drawings. The proof is elementary, but it has other consequences and raises interesting questions. Before we can state the theorem we need some definitions.

All graphs considered in this paper are finite and have no loops or multiple edges. For a graph G we denote its edge set by $E(G)$. A *labeled graph* is a graph with vertex-set $\{1, 2, \ldots, n\}$ for some n. If u and v are vertices in a graph G, then uv denotes the edge joining u and v and directed from u to v if G is directed. A *perfect matching* is a set of edges in a graph that covers all vertices exactly once.

Let G be a directed labeled graph and let $M = \{u_1v_1, u_2v_2, \ldots, u_kv_k\}$ be a perfect matching of G. Define the *sign* of M to be the sign of the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & \ldots & 2k-1 & 2k \\ u_1 & v_1 & u_2 & v_2 & \ldots & u_k & v_k \end{pmatrix}.$$

Note that the sign of a perfect matching is well-defined as it does not depend on the order in which the edges are written. We say that a labeled graph G is *Pfaffian* if there exists an orientation D of G such that the signs of all perfect matchings in D are positive, in which case we say that D is a *Pfaffian orientation* of G. An unlabeled graph G is *Pfaffian* if it is isomorphic to a labeled Pfaffian graph. It is well-known and also follows from Theorem 1.1 below that in that case every labeling of G is Pfaffian. The importance of Pfaffian graphs will be discussed in the next section.

By a *drawing* Γ of a graph G we mean an immersion of G in the plane such that edges are represented by homeomorphic images of $[0, 1]$, not containing vertices in their interiors. Edges are permitted to intersect, but there are only finitely many intersections and each intersection is a crossing. For edges e, f of a drawing Γ let $cr(e, f)$ denote the

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number of times the edges e and f cross. For a perfect matching M let $cr_\Gamma(M)$, or $cr(M)$ if the drawing is understood from context, denote $\sum cr(e,f)$, where the sum is taken over all unordered pairs of distinct edges $e, f \in M$.

The following theorem is the main result of this paper. The proof will be presented in Section 4.

Theorem 1.1. *A graph G is Pfaffian if and only if there exists a drawing of G in the plane such that $cr(M)$ is even for every perfect matching M of G.*

The "if" part of this theorem as well as the "if" part of its generalization (Theorem 5.1) was known to Kasteleyn [5] and was proved by Tesler [16]; however our proof of this part is different. The "only if" part is new.

2. Pfaffian Graphs

Pfaffian orientations have been introduced by Kasteleyn [3, 4, 5], who demonstrated that one can enumerate perfect matchings in a Pfaffian graph in polynomial time.

We say that an $n \times n$ matrix $A(D) = (a_{ij})$ is a *skew adjacency matrix* of a directed labeled graph D with n vertices if

$$a_{ij} = \begin{cases} 1 & \text{if } ij \in E(D), \\ -1 & \text{if } ji \in E(D), \\ 0 & \text{otherwise.} \end{cases}$$

Let A be a skew-symmetric $2n \times 2n$ matrix. For each partition $P = \{\{i_1, j_1\}, \{i_1, j_1\}, \dots, \{i_n, j_n\}\}$ of the set $\{1, 2, \dots, 2n\}$ into pairs, define

$$a_P = \begin{pmatrix} 1 & 2 & \dots & 2n-1 & 2n \\ i_1 & j_1 & \dots & i_n & j_n \end{pmatrix} a_{i_1 j_1} \dots a_{i_n j_n}.$$

Note that a_P is well defined as it does not depend on the order of the pairs in the partitions nor on the order in which pairs are listed. The *Pfaffian* of the matrix A is defined by

$$Pf(A) = \sum_P a_P,$$

where the sum is taken over all partitions P of the set $\{1, 2, \dots, 2n\}$ into pairs. Note that if D is a Pfaffian orientation of a labeled graph G then $Pf(A(D))$ is equal to the number of perfect matchings in G. One can evaluate the Pfaffian efficiently using the following identity from linear algebra: for a skew-symmetric matrix A

$$det(A) = (Pf(A))^2.$$

Thus the number of perfect matchings, and more generally the generating function of perfect matchings of a Pfaffian graph can be computed in polynomial time.

The problem of recognizing Pfaffian bipartite graphs is equivalent to many problems of interest outside graph theory, eg. the Pólya permanent problem [13], the even circuit problem for directed graphs [18], or the problem of determining which real square matrices are sign non-singular [6], where the latter has applications in economics [15].

The complete bipartite graph $K_{3,3}$ is not Pfaffian. Each edge of $K_{3,3}$ belongs to exactly two perfect matchings and therefore changing an orientation of any edge does not change the parity of the number of perfect matchings with negative sign. One can easily verify that for some (and therefore for every) orientation of $K_{3,3}$ this number is odd.

In fact, Little [8] proved that a bipartite graph is Pfaffian if and only if it does not contain an "even subdivision" H of $K_{3,3}$ such that $G \setminus V(H)$ has a perfect matching.

A structural characterization of Pfaffian bipartite graphs was given by Robertson, Seymour and Thomas [14] and independently by McCuaig [9]. They proved that a bipartite graph is Pfaffian if and only if it can be obtained from planar graphs and one specific non-planar graph (the Heawood graph) by repeated application of certain composition operations. This structural theorem implies a polynomial time algorithm for recognition of Pfaffian bipartite graphs.

No satisfactory characterization is known for general Pfaffian graphs. The result of this paper was obtained while attempting to find such a description.

3. The parity of the number of crossings

The parity of the number of crossings in drawings is frequently investigated in relation with the theory of crossing numbers. In this section we survey some results in this area.

Perhaps the most well-known result is the following theorem due to Hanani [1] and Tutte [17]:

Theorem 3.1. *Let Γ be a drawing of a graph G in the plane such that $cr(e, f)$ is even for every two distinct non-adjacent edges $e, f \in E(G)$. Then G is planar.*

We say that an edge $e \in E(G)$ is *even* in a drawing Γ of a graph G if $cr(e, f)$ is even for every $f \in E(G)$. A generalization of Theorem 3.1 was proved by Pach and Tóth [12]:

Theorem 3.2. *For a drawing Γ of a graph G let F denote the set of all even edges. Then G can be drawn in such a way that the edges of F are not involved in any crossing.*

Another related result is due to Kleitman [7]:

Theorem 3.3. *Let $G = K_{2j+1}$ or $G = K_{2j+1,2k+1}$ for some positive integers j and k. Then the parity of the total number of crossings of non-adjacent edges is independent of the choice of a drawing of G.*

Theorem 1.1 connects the theory of Pfaffian graphs and the theory of crossing numbers. In [10] I was able to generalize the methods used in the proof of Theorem 1.1 to prove a result on the numbers of crossings in "T-joins" in different drawings of a fixed graph. Theorems 3.1 and 3.3 follow from that result.

4. Main Theorem

Let Γ be a drawing of a graph G in the plane. We say that $S \subseteq E(G)$ is a *marking* of Γ if $cr(M)$ and $|M \cap S|$ have the same parity for every perfect matching M of G.

Theorem 1.1 follows from the following more general result.

Theorem 4.1. *For a graph G the following are equivalent:*
(a) G is Pfaffian;
(b) some drawing of G in the plane has a marking;
(c) every drawing of G in the plane has a marking;
(d) there exists a drawing of G in the plane such that $cr(M)$ is even for every perfect matching M of G.

We say that Γ is *a standard drawing* of a labeled graph G if the vertices of Γ are arranged on a circle in order and every edge of Γ is drawn as a straight line.

The equivalence of conditions (a), (b) and (c) of Theorem 4.1 immediately follows from the next two lemmas.

Lemma 4.2. *Let Γ be a standard drawing of a labeled graph G. Then G is Pfaffian if and only if Γ has a marking.*

Proof. Let D be an orientation of G. Let $M = \{u_1v_1, u_2v_2, \ldots, u_kv_k\}$ be a perfect matching of D. The sign of M is the sign of the permutation

$$P = \begin{pmatrix} 1 & 2 & 3 & 4 & \ldots & 2k-1 & 2k \\ u_1 & v_1 & u_2 & v_2 & \ldots & u_k & v_k \end{pmatrix}.$$

Let $i(P)$ denote the number of inversions in P, then

$$\begin{aligned} sgn(M) = sgn(P) = (-1)^{i(P)} = \prod_{1 \le i < j \le 2k} sgn(P(j) - P(i)) = \\ = \prod_{1 \le i < j \le k} sgn((u_j - u_i)(v_j - u_i)(u_j - v_i)(v_j - v_i)) \times \\ \times \prod_{1 \le i \le k} sgn(v_i - u_i). \end{aligned} \tag{1}$$

In Γ edges u_iv_i and u_jv_j cross if and only if each of the two arcs of the circle containing the vertices of Γ with the ends u_i and v_i contains one of the vertices u_j and v_j, in other words if and only if

$$sgn((u_j - u_i)(v_j - u_i)(u_j - v_i)(v_j - v_i)) = -1.$$

Define $S_D = \{uv \in E(D) | u > v\}$. Note that for every $S \subseteq E(G)$ there exists (unique) orientation D such that $S = S_D$. From (1) we deduce that

$$sgn(M) = (-1)^{cr(M)} \times (-1)^{|M \cap S|}.$$

Therefore M has a positive sign if and only if $cr(M)$ and $|M \cap S|$ have the same parity. It follows that D is a Pfaffian orientation of G if and only if S_D is a marking of the standard drawing of G. □

Notice that we have in fact shown that there exists a one-to-one correspondence between Pfaffian orientations of a labeled graph and markings of its standard drawing.

Lemma 4.3. *Let Γ_1 and Γ_2 be two drawings of a labeled graph G in the plane. Then Γ_1 has a marking if and only if Γ_2 has one.*

Proof. For any n and any two sequences $(a_1, a_2, .., a_n)$ and $(b_1, b_2, .., b_n)$ of pairwise distinct points in the plane, there clearly exists a homeomorphic transformation of the plane that takes a_i to b_i for all $1 \leq i \leq n$. Therefore without loss of generality we assume that the vertices of G are represented by the same points in the plane in both Γ_1 and Γ_2.

It suffices to prove the statement of the lemma for drawings Γ_1 and Γ_2 that differ only in the position of a single edge $e = uv$. Let e_1 and e_2 denote the images of e in Γ_1 and Γ_2 correspondingly. Define $C = e_1 \cup e_2$. The closed curve C separates its complement into two sets P_1 and P_2 with the property that every simple curve with the ends $a \in P_i$ and $b \in P_j$ crosses C even number of times if and only if $i = j$.

Clearly if $e \notin M$ we have

$$cr_{\Gamma_1}(M) = cr_{\Gamma_2}(M). \tag{2}$$

Let $c = 0$ if both P_1 and P_2 contain an even number of vertices of G and let $c = 1$ otherwise. For two curves C_1 and C_2, let $cr(C_1, C_2)$ denote the total number of times C_1 crosses C_2. For any perfect matching M of G, such that $e \in M$, the following identity holds modulo 2:

$$\begin{aligned} cr_{\Gamma_1}(M) + cr_{\Gamma_2}(M) = 2 \sum_{\{f,g\} \subseteq M \setminus \{e\}} cr(f, g) + \sum_{f \in M \setminus \{e\}} (cr(f, e_1) + cr(f, e_1)) \\ = \sum_{f \in M \setminus \{e\}} cr(f, C) = c. \end{aligned} \tag{3}$$

Suppose S is a marking of Γ_1. Identities (2) and (3) imply that S is a marking of Γ_2 if $c = 0$, and that $S \triangle \{e\}$ is a marking of Γ_2 if $c = 1$. □

Since clearly (d) implies (b), to finish the proof of Theorem 4.1 it remains to show that (b) implies (d). Suppose G satisfies (b) and consider a drawing of G in the plane with a marking S. Suppose there exists $e \in S$. We change the way e is drawn, so that the closed curve C which is composed from the old and the new drawing of e separates one vertex of G from the rest. From the proof of Lemma 4.3 it follows that

$S\backslash\{e\}$ is a marking in the new drawing. By repeating the procedure we produce a drawing of G such that the empty set is a marking, therefore demonstrating that G satisfies condition (d) of Theorem 4.1.

5. Concluding Remarks

1. The following generalization of Theorem 1.1 follows from the proof in previous section.

Theorem 5.1. *Let G be a graph and let $\mathcal{M}$ be the set of all perfect matchings of G. Let $s : \mathcal{M} \to \{-1, 1\}$. Then the following are equivalent:*
(1) there exists an orientation D of G such that for every $M \in \mathcal{M}$ its sign in the corresponding directed graph is equal to $s(M)$;
(2) there exists a drawing of G in the plane such that for every $M \in \mathcal{M}$

$$s(M) = (-1)^{cr(M)}.$$

2. For a labeled graph G, an orientation D of G and a perfect matching M of G, denote the sign of M in the directed graph corresponding to D by $D(M)$. We say that a graph G is *k-Pfaffian* if there exist a labeling of G, orientations $D_1, D_2, \dots, D_k$ of G and real numbers $\alpha_1, \alpha_2, \dots, \alpha_k$, such that for every perfect matching M of G

$$\sum_{i=1}^{k} \alpha_i D_i(M) = 1.$$

For surfaces of higher genus the following result was mentioned by Kasteleyn [4] and proved by Galluccio and Loebl [2] and independently by Tesler [16].

Theorem 5.2. *Every graph that can be embedded on a surface of genus g is 4^g-Pfaffian.*

I was able to prove the following analogue of Theorem 1.1 for the torus [11].

Theorem 5.3. *Every 3-Pfaffian graph is Pfaffian. A graph G is 4-Pfaffian if and only if there exists a drawing of G on the torus such that $cr(M)$ is even for every perfect matching M of G.*

Theorems 5.2 and 5.3 suggest several questions. For which $k \geq 5$ do there exist graphs that are k-Pfaffian, but not $(k-1)$-Pfaffian? Is it true that a graph G is 4^g-Pfaffian if and only if there exists a drawing of G on a surface of genus g such that $cr(M)$ is even for every perfect matching M of G?

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