TURÁN GRAPHS AND THE NUMBER OF COLORINGS.

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Abstract. We consider an old problem of Linial and Wilf to determine the structure of graphs which allow the maximum number of q-colorings among graphs with n vertices and m edges. We show that if r divides q then for all sufficiently large n the Turán graph $T_r(n)$ has more q-colorings than any other graph with the same number of vertices and edges. This partially confirms a conjecture of Lazebnik. Our proof builds on methods of Loh, Pikhurko and Sudakov, which reduce the problem to a quadratic program.

1. Introduction

Let $P_G(q)$ denote the number of proper q-colorings of a graph G. Birkhoff [2] was the first to consider this graph parameter. He has shown that $P_G(q)$ is a polynomial in q. This polynomial, called *the chromatic polynomial of* G, has been extensively investigated over the past century. In particular, Linial [5] and Wilf [1, 7] have independently posed the problem of describing graphs which for fixed q maximize $P_G(q)$ over the family of all graphs with n vertices and m edges. This problem for $q = 2$ has been solved by Lazebnik [3], but remains largely open in general. We refer the reader [6] for a more detailed discussion of the problem background.

In a recent breakthrough paper Loh, Pikhurko and Sudakov [6] have developed a new approach which allowed them to solve the problem asymptotically for many non-trivial ranges of parameters by reducing it to an optimization problem. In particular, they solved the original problem for $q = 3$ and a wide range of parameters m and n. They remark that "the remaining challenge is to find analytic arguments which solve the optimization problem for general q". In this note we present one such argument. We relax the optimization problem to a certain fractional version and solve some natural instances of this relaxation.

Our main result partially confirms the following conjecture of Lazebnik. Let $T_r(n)$ denote *the Turán graph* that is the complete r-partite graph on n vertices with partition sizes as close to being equal as possible. Lazebnik (see [4]) conjectured that for integers $k \geq 1$, $r \geq 2$, $n = rk$ and $m = \binom{r}{2}k^2$ the Turán graph $T_r(n)$ has more q-colorings than any other graph with the n vertices and m edges. This conjecture has been confirmed for $r = 2$ and $q = 3$ in [4], and for $r = q - 1$ and large n in [6]. We confirm this conjecture for r dividing q and large n.

Theorem 1.1. *Fix positive integers $q > r \geq 2$ such that r divides q. For all sufficiently large n the Turan graph $T_r(n)$ has more q-colorings than any other graph with the same number of vertices and edges.*

The remainder of the paper is organized as follows. In Section 2 we introduce results from [6] which reduce the asymptotic version of the original problem to a quadratic program. In Section 3 we solve relevant instances of this program, proving an approximate version of Theorem 1.1 and the following general bound on $P_G(q)$.

Theorem 1.2. *For any positive integer $q \geq 2$ and a positive real ϵ the following holds for all sufficiently large n. Let G be a graph on n vertices with m edges then*

$$P_G(q) \leq q^{(1+\epsilon)\, n} \left(1 - \frac{2m}{n^2}\right)^n$$

Finally, in Section 4 we derive Theorem 1.1 from the approximate version.

2. The optimization problem

In this section we present definitions and results from [6]. For the remainder of the paper we think of a positive integer q as being fixed. Let $[q] := \{1, 2, \ldots, q\}$.

It is shown in [6], that the asymptotic version of our original problem reduces to a quadratic program, which we will now define. For a vector $\boldsymbol{\alpha} = (\alpha_A)_{A \neq \emptyset, A \subseteq [q]}$, define

$$\mathrm{OBJ}(\boldsymbol{\alpha}) := \sum_{A \neq \emptyset} \alpha_A \log |A|; \qquad \mathrm{v}(\boldsymbol{\alpha}) := \sum_{A \neq \emptyset} \alpha_A, \qquad \mathrm{E}(\boldsymbol{\alpha}) := \sum_{A \cap B} \alpha_A \alpha_B.$$

Logarithms above and in the rest of the paper are natural. Fix a positive real parameter γ. Let the *feasible set* of vectors $\mathrm{FEAS}(\gamma)$ be defined by $\boldsymbol{\alpha} \geq 0, \mathrm{v}(\boldsymbol{\alpha}) = 1, \mathrm{E}(\boldsymbol{\alpha}) \geq \gamma$. Let

$$\mathrm{OPT}(\gamma) := \max_{\boldsymbol{\alpha} \in \mathrm{FEAS}(\gamma)} \mathrm{OBJ}(\alpha).$$

As noted in [6], such a maximum exists by compactness. We say that $\boldsymbol{\alpha}$ *solves* $\mathrm{OPT}(\gamma)$ if $\boldsymbol{\alpha} \in \mathrm{FEAS}(\gamma)$ and $\mathrm{OBJ}(\boldsymbol{\alpha}) = \mathrm{OPT}(\gamma)$.

Given a vector $\boldsymbol{\alpha} \in \mathrm{FEAS}(\gamma)$ for some γ we construct a graph $G_{\boldsymbol{\alpha}}(n)$ on n vertices as follows. Partition the vertex set of $G_{\boldsymbol{\alpha}}(n)$ into clusters V_A such that $|V_A|$ differs from $\alpha_A n$ by less than 1, and for every pair $A, B \subseteq [q]$ with $A \cap B = \emptyset$ join every vertex in V_A to every vertex of V_B by an edge.

We are now ready to state theorems from [6].

Theorem 2.1. *For any $\epsilon > 0$ the following holds for all sufficiently large n. For every n-vertex graph G with $m \leq |E(T_q(n))|$ edges, we have*

$$P_G(q) \leq e^{(\mathrm{OPT}(m/n^2)+\epsilon)n}.$$

Given two graphs on the same number of vertices *their edit distance* is the minimum number of edges that need to be added or deleted from one graph to obtain a graph isomorphic to the other. We say that two graphs are *d-close* if their edit distance is at most d.

Theorem 2.2. *For any $\epsilon, \kappa > 0$ the following holds for all sufficiently large n. Let G be an n-vertex graph with $m \leq \kappa n^2$ edges, which has at least as many q-colorings as any other graph with the same number of vertices and edges. Then G is ϵn^2-close to a graph $G_{\boldsymbol{\alpha}}(n)$ for some $\boldsymbol{\alpha}$ which solves* $\mathrm{OPT}(\gamma)$ *for some $|\gamma - m/n^2| < \epsilon$ and $\gamma \leq \kappa$.*

Another auxiliary result from [6] will be used in Section 4.

Proposition 2.3. *The number of edges in any $G_{\boldsymbol{\alpha}}(n)$ differs from* $\mathrm{E}(\boldsymbol{\alpha})n^2$ *by less than $2^q n$. Also, for any other vector $\boldsymbol{\nu}$, the edit-distance between $G_{\boldsymbol{\alpha}}(n)$ and $G_{\boldsymbol{\nu}}(n)$ is at most $||\boldsymbol{\alpha} - \boldsymbol{\nu}||_1 n^2 + 2^q n$, where $|| \cdot ||_1$ is the L^1-norm.*

3. Asymptotic result

For a vector $\boldsymbol{\alpha} = (\alpha_A)_{A\neq\emptyset, A\subseteq[q]}$ define the *support* of $\boldsymbol{\alpha}$ as a collection of sets A such that $\alpha_A \neq 0$. We say that $\boldsymbol{\alpha}$ is *a balanced partition vector* if the support of $\boldsymbol{\alpha}$ is a partition of $[q]$ and all sets in the support have the same size.

Lemma 3.1. *For every $0 \le \gamma \le \frac{q-1}{2q}$ we have $\mathrm{OPT}(\gamma) \le \log(q(1-2\gamma))$ with the equality holding if and only if $\gamma = \frac{r-1}{2r}$ for some integer r dividing γ. Moreover, if the equality holds for γ and $\boldsymbol{\alpha}$ solves $\mathrm{OPT}(\gamma)$ then $\mathrm{E}(\boldsymbol{\alpha}) = \gamma$ and $\boldsymbol{\alpha}$ is a balanced partition vector.*

Proof. It is easy to verify that if $\gamma = \frac{r-1}{2r}$ for some integer r dividing γ and $\boldsymbol{\alpha}$ is a balanced partition vector corresponding to a partition of $[q]$ into r equal parts then $\boldsymbol{\alpha} \in \mathrm{FEAS}(\gamma)$ and $\mathrm{OBJ}(\boldsymbol{\alpha}) = \log(q(1-2\gamma))$. It remains to verify the only if part of the lemma statement.

Let S denote the set of vectors $\boldsymbol{w} = (w_1, w_2, \dots, w_q)$ such that $w_i \ge 0$ for every $i \in [q]$ and $\sum_{i=1}^q w_i = 1$. For a vector $\boldsymbol{\alpha} = (\alpha_A)_{A\neq\emptyset, A\subseteq[q]}$ and a vector $\boldsymbol{w} \in S$ define

$$f(\boldsymbol{\alpha}, \boldsymbol{w}) := \sum_A \alpha_A \log\left(\sum_{i\in A} w_i\right).$$

We say that $(\boldsymbol{\alpha}, \boldsymbol{w})$ is *a weighted balanced partition vector* if the sets in the support of $\boldsymbol{\alpha}$ are pairwise disjoint, the value $\sum_{i\in A} w_i$ is the same for every A in the support of $\boldsymbol{\alpha}$, and every $i \in [q]$, such that $w_i > 0$, belongs to some set in the support of $\boldsymbol{\alpha}$. As $\mathrm{OBJ}(\boldsymbol{\alpha}) = f(\boldsymbol{\alpha}, \boldsymbol{w}_0) + \log q$, where $\boldsymbol{w}_0 := (1/q, 1/q, \dots, 1/q)$, the lemma is implied by the following more general claim.

Claim 3.2. *Let $\boldsymbol{\alpha} \in \mathrm{FEAS}(\gamma)$ and $\boldsymbol{w} \in S$ be such that $\sum_{i\in A} w_i > 0$ for all A in the support of $\boldsymbol{\alpha}$. Then $f(\boldsymbol{\alpha}, \boldsymbol{w}) \le \log(1-2\gamma)$. Further, if the equality holds then $\mathrm{E}(\boldsymbol{\alpha}) = \gamma$ and $(\boldsymbol{\alpha}, \boldsymbol{w})$ is a weighted balanced partition vector.*

Suppose that for fixed $\boldsymbol{\alpha}$ the vector $\boldsymbol{w} \in S$ is chosen to maximize $f(\boldsymbol{\alpha}, \boldsymbol{w})$. Without loss of generality, we ignore zero-valued w_i and assume that $w_i > 0$ for every i. By the choice of $\boldsymbol{w}$ the value

$$\frac{\partial f}{\partial w_k} = \sum_{\substack{A\subseteq[q]\\ k\in A}} \frac{\alpha_A}{\sum_{i\in A} w_i}$$

is the same for every $k \in [q]$. Consider a random variable X on $2^{[q]} - \{\emptyset\}$ such that $\Pr[X = A] = \alpha_A$ for every $A \subseteq [q], A \neq \emptyset$. Consider further a random variable Y on $[q]$, dependent on X, such that for every $k \in [q]$ we have

$$\Pr[Y = k \mid X = A] = \begin{cases} \frac{w_k}{\sum_{i\in A} w_i} & \text{if } k \in A, \\ 0 & \text{if } k \notin A. \end{cases}$$

Then,

$$\Pr[Y = k] = \sum_{\substack{A\subseteq[q],\\ A\neq\emptyset}} \Pr[Y = k \mid X = A] \Pr[X = A] = w_k \sum_{\substack{A\subseteq[q]\\ k\in A}} \frac{\alpha_A}{\sum_{i\in A} w_i}$$

for $k \in [q]$. It follows from the choice of $\boldsymbol{w}$ that the value $w_k^{-1} \Pr[Y = k]$ is the same for every k. We conclude that $\Pr[Y = k] = w_k$. For given A we have

$$\sum_{B\cap A=\emptyset} \alpha_B = \Pr[X \cap A = \emptyset] \le \Pr[Y \notin A] = \sum_{i\notin A} w_i. \tag{3.3}$$

Therefore

$$2\gamma \leq \sum_A \alpha_A \left(\sum_{B \cap A = \emptyset} \alpha_B \right) \leq \sum_A \alpha_A \left(\sum_{i \notin A} w_i \right), \tag{3.4}$$

and

$$\sum_A \sum_{i \in A} \alpha_A w_i \leq 1 - 2\gamma.$$

Finally, by concavity of log we have

$$f(\boldsymbol{\alpha}, \boldsymbol{w}) = \sum_A \alpha_A \log \left(\sum_{i \in A} w_i \right) \leq \log \left(\sum_A \sum_{i \in A} \alpha_A w_i \right) \leq \log (1 - 2\gamma). \tag{3.5}$$

If $f(\boldsymbol{\alpha}, \boldsymbol{w}) = \log(1-2\gamma)$ then equality holds in (3.3) for every A with $\alpha_A \neq 0$, in (3.4) and (3.5). Therefore, from (3.3) we have $A \cap B = \emptyset$ for all A, B in the support of $\boldsymbol{\alpha}$, $A \neq B$. From (3.4) we deduce $E(\boldsymbol{\alpha}) = \gamma$. Further, every $k \in [q]$ with $w_k > 0$ belongs to some set in the support of $\boldsymbol{\alpha}$, as

$$\sum_{A \ni k} \alpha_k \geq \Pr[Y = k] = w_k > 0.$$

Finally, the equality in (3.5) implies that $\sum_{i \in A} w_i$ is the same for all A in the support of $\boldsymbol{\alpha}$. Thus $(\boldsymbol{\alpha}, \boldsymbol{w})$ is a weighted balanced partition vector. This finishes the proof of the claim and the lemma. □

Proof of Theorem 1.2. If $m > |E(T_q(n))|$ then $P_G(q) = 0$ by Turán's theorem. Otherwise, the theorem follows immediately by substituting the bound on $\mathrm{OPT}(m/n^2)$ from Lemma 3.1 into the bound on $P_G(q)$ from Theorem 2.1. □

4. Exact result

In this section we prove Theorem 1.1. In addition to the tools and results presented earlier we will use the following general bound on $P_G(q)$ from [3].

Theorem 4.1. *Let G be a graph on n vertices and m edges and let $k \geq 2$ be an integer. Then*

$$P_G(k) \leq \left(1 - \frac{1}{k}\right)^{\lceil (\sqrt{1+8m}-1)/2 \rceil} k^n \leq \left(1 - \frac{1}{k}\right)^{\sqrt{m}} k^n.$$

The following lemma is the main result of this section. It ensures that Theorem 1.1 holds locally. Together with the asymptotic and stability results of the previous section it allows us to deduce that Theorem 1.1 holds in general. Although the proof of the lemma is technical and relatively long, it represents a fairly standard stability argument, similar to the ones employed in [6]. While it seems that solving optimization problem from Section 2 for general q requires new ideas, once the solution is known refinement of the asymptotic results into precise solution of the original problem is likely to be possible using existing techniques.

Lemma 4.2. *There exists $\delta = \delta(q)$ such that the following holds for sufficiently large n. Let $2 \leq r < q$ be an integer dividing q. Let G be an n-vertex graph such that $|E(G)| \geq |E(T_r(n))|$ and G is δn^2-close to $T_r(n)$. Then G has at most as many q-colorings as $T_r(n)$ with the equality holding if and only if G is isomorphic to $T_r(n)$.*

Proof. Throughout the proof we will be making a series of claims which hold for positive δ, sufficiently small as a function of q, and for n, sufficiently large as a function of q and δ. The eventual choice of δ and n will be implicitly made so that all of these claims are valid.

Let the graph G be as in the lemma statement. Suppose that G has at least as many q-colorings as $T_r(n)$. Given a partition $\mathcal{A} = (A_1, A_2, \dots, A_r)$ of $V(G)$ denote by $E_{\mathcal{A}}(G)$ the set of edges of G joining vertices in different parts of $\mathcal{A}$. By the choice of G, there exists a partition $\mathcal{A}$ of $V(G)$ into r parts such that $|E_{\mathcal{A}}(G)| \geq |E(T_r(n))| - \delta n^2$. Assume that $\mathcal{A}$ is chosen to maximize $|E_{\mathcal{A}}(G)|$. Let $\delta' := n^{-1} \max_i ||A_i| - \frac{n}{r}|$. It is easy to verify that

$$|E_{\mathcal{A}}(G)| \leq \binom{r}{2}\left(\frac{n}{r}\right)^2 - \frac{(\delta' n)^2}{r^4} \leq |E(T_r(n))| - \frac{(\delta' n)^2}{r^4} + o(n^2)$$

The above inequality allows us to assume that $||A_i| - \frac{n}{r}| \leq \delta n$ for all $i \in [r]$, by modifying the choice of δ.

Let $\epsilon := \delta^{1/3}$. We say that a vertex $v \in V(G)$ is *regular* if $d_{\mathcal{A},i}(v) \geq (1-\epsilon)|A_i|$ for every i such that $v \notin A_i$, and *irregular*, otherwise. Let Z denote the set of irregular vertices of G and let $T_{\mathcal{A}}$ denote the complete multipartite graph with parts determined by $\mathcal{A}$. Then, as G is δn^2-close to $T_r(n)$, we have

$$\delta n^2 \geq |E(T_{\mathcal{A}}) \setminus E_{\mathcal{A}}(G)| \geq \epsilon\left(\frac{n}{r} - \delta n\right)|Z|,$$

and $|Z| \leq \epsilon^{-1}\delta(\frac{1}{r} - \delta)n \leq \sqrt{\delta}n$ for sufficiently small δ.

Let $f : V(G) \to [q]$ be a q-coloring of G. For $i \in [r]$, let $R_f(i) := \{c \in [q] \mid |f^{-1}(c) \cap A_i| > \epsilon|A_i|\}$, that is R_i is the set of colors which occur relatively frequently in A_i. Clearly, for every $c \in R_f(i)$ we have $f^{-1}(c) \subseteq A_i \cup Z$. For sufficiently small δ we have $|Z| < \epsilon|A_i|$ for every $i \in [q]$, and therefore $R_f(i) \cap R_f(j) = \emptyset$ for $i \neq j$. Let $\boldsymbol{R}_f := (R_f(1), \dots, R_f(r))$. Given a vector $\boldsymbol{R} = (R_1, R_2, \dots, R_r)$, such that components of $\boldsymbol{R}$ are disjoint subsets of $[q]$, we will bound the number $P_G(\boldsymbol{R})$ of colorings f of G such that $\boldsymbol{R}_f = \boldsymbol{R}$.

Let $k := q/r$. Suppose first that $|R_i| \neq k$ for some $i \in [r]$. Then

$$\begin{aligned}
P_G(\boldsymbol{R}) &\leq q^{|Z|}\left(\prod_{i=1}^{r}|R_i|^{|A_i|}\right)\left(\prod_{i=1}^{r}2^{q\epsilon|A_i|}\binom{|A_i|}{\lfloor\epsilon|A_i|\rfloor}^q\right) \\
&\leq q^{\sqrt{\delta}n}\left(\prod_{i=1}^{r}|R_i|\right)^{n/r+\delta n}\left(\prod_{i=1}^{r}\left(\frac{2e}{\epsilon}\right)^{q\epsilon|A_i|}\right) \\
&\leq k^n\left(1 - \frac{1}{k^2}\right)^{n/r}\exp\left((\sqrt{\delta}\log q + \delta r\log k + \epsilon\log(2e\epsilon^{-1})q)n\right) \\
&< \frac{k^{n-1}}{r^q},
\end{aligned}$$

for δ sufficiently small. In the first line of the above sequence of inequalities we estimate $P_G(\boldsymbol{R})$ by allowing vertices of Z to be colored arbitrarily, by allowing $|R_i|$ choices of colors for vertices in A_i, and, finally, by accounting for possible choices of at most $\epsilon|A_i|$ vertices in A_i which could be colored in any color, not necessarily a color in R_i. In the second line we use bounds on $|Z|$ and $|A_i|$, as well as an upper bound on the binomial coefficient. In the third line, we use that by the choice of $\boldsymbol{R}$ we have $\prod_{i=1}^{r}|R_i| \leq (k^2-1)k^{r-2}$. Finally, the last inequality holds for δ sufficiently small as the second term in the third line is of the form e^{cn} for some c depending only on q, while the coefficient of n in the exponent in the last term goes to zero as δ approaches zero. It follows that $\sum_{\boldsymbol{R}} P_G(\boldsymbol{R}) \leq k^{n-1}$, where summation is taken over all $\boldsymbol{R}$ such that $|R_i| \neq k$ for some $i \in [r]$.

It remains to bound $P_G(\boldsymbol{R})$ when $\boldsymbol{R}$ corresponds to a partition of $[q]$ into r parts each of size k. Remember that for such $\boldsymbol{R}$, for every coloring f with $\boldsymbol{R}_f = \boldsymbol{R}$ and every $i \in [r]$, all the vertices in $R_i - Z$ receive colors from $R_f(i)$. Suppose first that there exists a vertex $v \in V(G)$ such that $d_{\mathcal{A},i}(v) \geq \delta^{2/5}|A_i|$ for every $i \in [q]$. If $\boldsymbol{R}_f = \boldsymbol{R}$ and $f(v) \in R_f(i)$ then all the neighbors of v in $A_i - Z$ receive one of $k-1$ colors in $R_f(i) - f(v)$. It follows that

$$P_G(\boldsymbol{R}) \leq rk^n \left(\frac{k-1}{k}\right)^{\delta^{2/5}(1-\delta)n/r} q^{\delta^{1/2}n} < \frac{k^n}{r^q},$$

for sufficiently small δ. Combining this with the preceding calculations we obtain $P_G(q) < k^n$, which is less than the number of q-colorings of $T_r(n)$, a contradiction. Therefore a vertex v as above does not exist. It follows from the choice of $\mathcal{A}$ that for every $i \in [r]$ the subgraph $G[A_i]$ of G induced by A_i has maximum degree at most $\delta^{2/5}n$. Let e_i denote the number of edges of G with both ends in $A_i - Z$. As $|E(G)| \geq |E(T_{\mathcal{A}})|$, we have

$$\sum_{i=1}^{r}(e_i + \delta^{2/5}n|Z \cap A_i|) \geq \sum_{i=1}^{r}|E(G[A_i])| \geq \epsilon(1-\delta)\frac{n}{r}|Z|.$$

It follows that $\sum_{i=1}^{r} e_i \geq \delta^{2/5}|Z|n$, for sufficiently small δ. Using Theorem 4.1 we obtain

$$P_G(\boldsymbol{R}) \leq q^{|Z|}k^n \prod_{i=1}^{r}\left(1-\frac{1}{k}\right)^{\sqrt{e_i}} \leq k^n \exp\left(\log q|Z| - (\log k - \log(k-1))\sqrt{\delta^{2/5}|Z|n/r}\right) \tag{4.3}$$

$$\leq k^n \exp\left((\log q - \delta^{-1/20}r^{-1/2}(\log k - \log(k-1)))|Z|\right).$$

If $Z \neq \emptyset$ then $P_G(\boldsymbol{R})$ once again becomes negligible compared to k^n, as δ approaches zero. It follows that $Z = \emptyset$. Let t denote the number of ordered partitions of $[q]$ into r parts of size k. Then the number of colorings of $T_r(n)$ is at least tk^n. If $\sum_{i=1}^{r} e_i \neq 0$ then the first inequality of (4.3) together with our bound on the number of colorings corresponding to unbalanced partitions of $[q]$ implies $P_G(q) \leq tk^{n-1}(k-1) + k^{n-1} < tk^n$. It follows that G is a subgraph of $T_{\mathcal{A}}$. As $|E(G)| \geq |E(T_r(n))|$, we conclude that G is isomorphic to $T_r(n)$, as desired. □

Proof of Theorem 1.1. Suppose for a contradiction that there exists an increasing sequence of positive integers $\{n_i\}_{i=1}^{\infty}$ and a sequence of graphs $\{G_i\}_{i=1}^{\infty}$, such that $|V(G_i)| = n_i$, $|E(G_i)| = |E(T_r(n_i))|$, G_i is not isomorphic to $T_r(n_i)$ and has at least as many q-colorings as any other graph with the same number of vertices and edges. We apply Theorem 2.2 for $\kappa = \frac{r-1}{r}$ and a sequence of positive real $\{\epsilon_i\}_{i=1}^{\infty}$ with $\lim_{i\to\infty}\epsilon_i = 0$. By possibly restricting $\{n_i\}$ to a subsequence, we obtain a sequence $\{\boldsymbol{\alpha}_i\}_{i=1}^{\infty}$ such that G_i is $\epsilon_i n_i^2$-close to $G_{\boldsymbol{\alpha}_i}(n_i)$, $\boldsymbol{\alpha}_i$ solves $\mathrm{OPT}(\gamma_i)$ for some $\gamma_i \leq \kappa$, and $\lim_{i\to\infty}\gamma_i = \kappa$. By further restricting our sequence we assume that $\{\boldsymbol{\alpha}_i\}$ converges in L^1-norm to a vector $\boldsymbol{\alpha}$ with $\mathrm{E}(\boldsymbol{\alpha}) = \gamma$.

By Lemma 3.1 and monotonicity of $\mathrm{OBJ}(\gamma)$, we have $\log(q(1-2\kappa)) \leq \mathrm{OPT}(\gamma_i) \leq \log(q(1-2\gamma_i))$. Therefore

$$\log(q(1-2\,\mathrm{E}(\boldsymbol{\alpha})) - \mathrm{OBJ}(\boldsymbol{\alpha}) = \lim_{i\to\infty}(\log(q(1-2\,\mathrm{E}(\boldsymbol{\alpha}_i)) - \mathrm{OBJ}(\boldsymbol{\alpha}_i)) = 0.$$

By Lemma 3.1, $\boldsymbol{\alpha}$ is a balanced partition vector, and so $G_{\boldsymbol{\alpha}}(n) = T_r(n)$ for every n. By Proposition 2.3, $G_{\boldsymbol{\alpha}_i}(n_i)$ is $\delta n_i^2/2$-close to $T_r(n_i)$ for sufficiently large i. Consequently, G_i is δn_i^2-close to $T_r(n_i)$ for sufficiently large i. This contradicts Lemma 4.2, finishing the proof of the theorem. □

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