JACOBIANS OF NEARLY COMPLETE AND THRESHOLD GRAPHS

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Abstract. The Jacobian of a graph, also known as the Picard Group, Sandpile Group, or Critical Group, is a discrete analogue of the Jacobian of an algebraic curve. It is known that the order of the Jacobian of a graph is equal to its number of spanning trees, but the exact structure is known for only a few classes of graphs. In this paper, we compute the Jacobian for graphs of the form $K_n \backslash E(H)$ where H is a subgraph of K_n on $n-1$ vertices that is either a cycle, or a union of two disjoint paths. We also offer a combinatorial proof of a result of Christianson and Reiner that describes the Jacobian for a subclass of threshold graphs.

1. Introduction

1.1. **Overview.** In this paper, we compute the Jacobian for several explicit classes of graphs as an application of a more combinatorial way to approach the algrebaic-geometric properties of graphs. The main new results are the proofs of two structural theorems on classes of graphs discussed by Lorenzini [9]. The first is that given a complete graph, K_n, with the edge set of a cycle incident with all but one of the vertices removed, its Jacobian is isomorphic to $\mathbb{Z}_k \times \mathbb{Z}_k$ if n is even and isomorphic to $\mathbb{Z}_k \times \mathbb{Z}_{k(n-4)}$ if n is odd, where k is such that the order of the group is the number of spanning trees of the graph. The second is that given a complete graph, K_n, with the edge set of two disjoint paths incident with all but one of the vertices removed, its Jacobian is cyclic if and only if the lengths of the paths are relatively prime.

Our interest in the Jacobians of these graphs is motivated by two facts. First, graphs whose Jacobians are cyclic are of interest because of applications in other fields, see for example [5]. Second, if G is obtained from K_n by deleting strictly fewer than $n-3$ edges, then $\mathrm{Jac}(G)$ is not cyclic, as pointed out in [9, Section 5].

The paper is structured as follows. In the first section, we introduce necessary definitions and prove several small theorems which will be required for the main results. The second section contains the statement and proof of the new results concerning nearly complete graphs. And the third and final section provides another application of these combinatorial techniques to present a simple combinatorial proof of a theorem of Christianson and Reiner.

1.2. **Motivation.** Determination of the Jacobian of graphs, as well as their use, dates back as early as 1970 in arithmetic geometry. Here, it is referred to as the *group of components*, the same name it takes in early graph theoretic contexts [11]. Lorenzini, for example, studied the group in 1991 in one such context, motivated by problems in arithmetic geometry [10]. In 1990, Dhar approached this group in the context of physics, referring to it as the *sandpile group* [7]. Seven years later, Bacher et al referred to it alternatively as the *Picard group* or *Jacobian group*, in the context of algebraic curves [1]. Most near our use, however, in 1999, Biggs approached the group using a different chip-firing game, calling it the *critical group* [4]. While these papers made use of the Jacobian of the graph in various ways, perhaps the most immediately relevant has been

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promoted by Biggs in [5]. In this paper, he discusses the uses of the critical group for cryptography, specifically for public key encryption.

The specific class of graphs discussed here, almost complete graphs, arise in Lorenzini [9]. He proves the first direction of Theorem 2.8, that in a complete graph with two vertex-disjoint paths, covering all but one of the vertices deleted, the Jacobian is cyclic if the two path lengths are relatively prime. He refers to incomplete work and numerical results in the other direction, that it is not cyclic when the path lengths are not relatively prime. He also suggests the structure of the Jacobian of a complete graph on n vertices with edges of a cycle of length $n-1$ deleted, but leaves the problem open. We settle it in Theorem 2.1.

This paper makes use almost entirely of the chip-firing game and the tools of harmonic morphisms, both essentially combinatorial, to approach the problem. The advantages to these methods are that we can provide more transparent and shorter proofs of previous results and that we can take better advantage of inherent symmetries in the problem. For example, the Jacobian of the wheel graph when the number of vertices is even was determined by Biggs in 1999 [4], but that of a complete graph delete a cycle, an almost identical question from our point of view, remained unknown until this paper. By making use of the explicit symmetries in the graph, we can avoid graph-specific computations and divine a wider range of information.

The work in threshold graphs is motivated similarly. Christianson et al in 2001 proved Theorem 3.1, but using linear algebra [6], referring to the class of threshold graphs with no bad sequences as *generic threshold graphs* and to the Jacobian of G as its critical group. Threshold graphs are themselves of value since, while they have a large amount of structure, determining their Jacobians may be useful in determining the Jacobians of more general graphs. Further, as per Christianson and Reiner, threshold graphs are extremal in a sense, possibly allowing results on threshold graphs to provide bounds on other types of graphs [6]. The purpose, then, of this aspect of the paper is to provide another example of a convenient use of the chip-firing game to determine Jacobians, even if we were not able here to move beyond the already established scope of the literature.

1.3. **Notation and terminology.** In this paper, we define a *graph* to be a finite, connected multigraph with no loop edges. A graph with no multiple edges will be called *simple*. For a graph G, $V(G)$ and $E(G)$ denote the vertex and edge set of the graph. A graph is said to be *k-connected* if for every $X \subseteq V(G)$ with $|X| < k$, the graph $G \backslash X$ is connected. Similarly, a graph is said to be *k-edge-connected* if for every $X \subseteq E(G)$, $|X| < k$, $G \backslash X$ is connected.

We define $g(G)$, the *genus* of G, to be $|E(G)| - |V(G)| + 1$, the number of independent cycles of the graph. While this differs from the traditional meaning of genus of a graph (the lowest genus among surfaces in which the graph can be embedded), it better highlights the analogue to algebraic notions of genus in the spirit of [2].

The *degree* of $v \in V(G)$, written $deg(v)$ is the number of edges in $E(G)$ incident with v. Similarly, given $A \subseteq V(G)$, $v \in A$, $outdeg_A(v)$, the *outdegree* of v relative to A, is the number of edges in $E(G), e = vw, w \notin A$. If $e \in E(G), v \in V(G), v \in e$ means that v is incident with e.

For other terms, see any introductory text in graph theory, for example [8].

1.4. **The chip-firing game.** As a mechanism for converting algebraic ideas into combinatorial ones, we introduce a *chip-firing game* played on the vertices of the graph, developed by Baker and Norine [3]. Informally, to each vertex we assign an integer number of points. A move in the game corresponds to a choice of a vertex and a choice of operation from two: push and absorb. If a vertex *absorbs*, it gains $deg(v)$ points and every adjacent vertex loses one point per edge joining the two. If a vertex *pushes*, it loses $deg(v)$ points and every adjacent vertex gains one point per edge joining the two. Define a *configuration on* G to be an assignment of points to the vertices of G. We say that two configurations, C_1 and C_2 are *equivalent* if there exists a sequence of moves starting from configuration C_1 and ending with C_2.

Formally, as in [3], we take $Div(G)$ to be the free abelian group of the vertices of G. We think of elements of $Div(G)$ as integer linear combinations of elements of $V(G)$ and write an element $C \in Div(G)$ as $\sum_{v \in V(G)} a_v v$ where each a_v is an integer. We refer to each element of $Div(G)$ as a *divisor* of G. For convenience, we refer to the coefficient of a vertex, v, in an element, C, as $C(v)$.

Given a graph G, a divisor, $C = \sum_{v \in V(G)} a_v(v)$, and a vertex u, we define $push_u(C) = (a_u - deg(u))(u) + \sum_{v \sim u}(a_v + 1)(v) + \sum_{v \not\sim u), v \neq u} a_v(v)$ and $absorb_u(C)$ to be the inverse operation. A *chip-firing move* is the application of one of these operations to a divisor. This then provides an equivalence relationship between divisors: two divisors $C \sim C'$ if there is a sequence of chip-firing moves, f, so that $f(C) = C'$.

Note several other properties of this game. First, there is no distinction in the order of these operations. Given a sequence of moves, f, if $f(C) = C'$, then any permutation of the moves applied to C will still result in C'. As a result, it is reasonable to discuss the notion of a set pushing or absorbing. Given $A \in V(G)$, applying a push operation to A is the same as applying a push operation to each vertex of A and similarly for absorbing. Since the order of the pushes or absorptions is irrelevant, these are well-defined operations and will be notated $push_A(C)$ and $absorb_A(C)$ for the push and absorb of a divisor C. With this new notation, we can see that an absorb from a set A is the same as a push from all vertices except those in A and vice versa. For a divisor $C \in Div(G)$, we define the degree of C to be $deg(C) = \sum_{v \in V(G)} C(v)$. It is worth noting that equivalent divisors have the same degree.

1.5. **The Jacobian.** This equivalence relation leads us to the definition of a natural group associated with a given graph, the Jacobian. Define $\mathrm{Jac}(G)$ to be the set of all divisors of G with degree 0, modulo the equivalence relation defined by the chip-firing game. As shown in [3], this definition of the Jacobian group of a graph is equivalent to other definitions of the Jacobian throughout the literature, most traditionally extracted from the Laplacian matrix, and is a discrete analogue of the Jacobian group for Riemann surfaces. Fixing an ordering on the vertices, $\{v_1, \ldots, v_n\}$, we define the *Laplacian matrix* associated with G to be the $n \times n$ matrix $Q = D - A$, where D is the diagonal matrix whose $(i,i)^{th}$ entry is the degree of v_i and A is the *adjacency matrix* of the graph, whose $(i,j)^{th}$ entry is the number of edges joining v_i and v_j. Then Lemma 4.3 in [3] gives that the chip-firing equivalence defined above is identical to the traditional equivalence defining the Jacobian (the $|V(G)|$-dimensional integer vectors modulo integer multiples of the graph Laplacian). This relation immediately gives us both the independence of the order of moves mentioned above and the fact that that the chip-firing equivalence is, in fact, a true equivalence relationship. Abel's Theorem for Graphs [1] relates this group to a number of interesting combinatorial properties of the graph in question, but for the purpose of this paper, it suffices that the understanding of the Jacobian of a graph is a useful endeavor without needing to know the particulars.

1.6. v_0**-reduced form.** In studying a particular equivalence class of divisors, it is often useful to have a canonical element to look at. Given $v_0 \in V(G)$, we say that a divisor C is in v_0*-reduced form* if $C(v) \geq 0$ for every $v \neq v_0$ and for any set $A \subset V(G), v_0 \notin A$, there is some $v \neq v_0$ such that $push_A(C)(v) < 0$.

Theorem 1.1. [3, Proposition 3.1] *Given a graph, G, a vertex $v_0 \in V(G)$, and a divisor of G, C, there exists a unique v_0-reduced divisor equivalent to C.*

1.7. **Harmonic Morphisms.** To augment the chip-firing game which provides a useful mechanism for the study of algebraic qualities of graphs, it is often useful to study maps between graphs which preserve, in a sense, the chip firing properties. To do so, we introduce the notion of a harmonic morphism [2]:

Definition. A *morphism* between two graphs, G and G', is a map $\phi : V(G) \cup E(G) \to V(G') \cup E(G')$ such that $\phi(V(G)) \subseteq V(G')$ and for every edge e with endpoints x and y either $\phi(e) \in E(G')$ and has endpoints $\phi(x)$ and $\phi(y)$ or $\phi(e) \in V(G')$ and $\phi(e) = \phi(x) = \phi(y)$.
A morphism is said to be *harmonic* if for all $x \in V(G)$, $y \in V(G')$ with $y = \phi(x)$, the quantity $|\{e \in E(G) | x \in e, \phi(e) = e'\}|$ is the same for all $e' \in E(G')$ such that y is incident with e'.

Our interest in discrete Jacobians arises, in part, from studing the anology between graphs and Riemann surfaces. In the case of surfaces, it is often important to study not only the surfaces themselves, but also the holomorphic maps between surfaces. Baker and Norine show in [2] that these harmonic morphisms between graphs are a natural analogue and we show here that their properties are useful in examining the Jacobians of graphs. For examples, see [2].

Making use of results from [2] that the number of preimages of an edge of G' is independent of the choice of edge, we define *deg(ϕ)* to be
$|\{e \in E(G) | \phi(e) = e'\}|$ for $e' \in E(G')$. These results give that this degree is independent of the choice of e', so is well-defined and also that ϕ is surjective if and only if $deg(\phi) \geq 0$. Given $e \in E(G)$ and $\phi : V(G) \cup E(G) \to V(G') \cup E(G')$, e is said to be a *vertical edge* if $\phi(e) \in V(G')$ and a *horizontal edge* otherwise. Given $v \in V(G)$, we define its *vertical multiplicity* to be the number of vertical edges incident with it and its *horizontal multiplicity* to be the number of horizontal edges incident with it that map to the same edge.

The relevance of harmonic morphisms for our purpose is for their useful properties in conjunction with Jacobians. Specifically, if $\phi : G \to G'$ is a harmonic morphism, then ϕ induces an injective map from $\mathrm{Jac}(G') \to \mathrm{Jac}(G)$ [2, Theorem 4.13]. Note that this implies that the order of $\mathrm{Jac}(G')$ divides the order of $\mathrm{Jac}(G)$. We can examine this relation in a little more depth: given a divisor C on G with a harmonic morphism ϕ, we define $\psi : Div(G) \to Div(G')$ to be

$$\psi(\sum_{v \in V(G)} a_v v) = \sum_{v' \in V(G')} (\sum_{v : \phi(v) = v'} a_v) v' \tag{1.2}$$

Note that if $C \sim C'$ in G, then $\psi(C) \sim \psi(C')$ in G'. This follows from noticing that for any vertex $v \in V(G)$, a push transfers from v to the preimages of u' in G exactly the horizontal multiplicity of v if $\phi(v)$ and u' are adjacent in G' or 0 otherwise. The image of the resulting divisor then is the same as pushing the horizontal multiplicity of v times from $\phi(v)$. This gives the desired result, then, that $\mathrm{Jac}(G')$ is a subgroup of $\mathrm{Jac}(G)$.

2. Explicit Determination of Jacobians

2.1. **Introduction.** In this section, we determine the exact structure of the Jacobian of two classes of graphs, those that are isomorphic to K_n with the edge sets of two disjoint paths with total length $n - 1$ removed and those isomorphic to K_n with the edge set of a cycle of length $n - 1$ removed. For the first case, when the lengths of the two paths are relatively prime, the result is known [9], although it was discovered using powerful algebraic methods. This section provides a purely combinatorial solution to both problems.

2.2. **Deleting a cycle from K_n.** We begin by examining the case of deleting the edge set of a cycle from K_n.

Theorem 2.1. *Let C be a cycle in K_n such that $|V(C)| = n-1$. Let $G = K_n \backslash E(C)$ and t be the number of spanning trees of G. Then the Jacobian of G is isomorphic to $\mathbb{Z}_{\sqrt{t}} \times \mathbb{Z}_{\sqrt{t}}$ if n is even and isomorphic to $\mathbb{Z}_{\sqrt{t/(n-4)}} \times \mathbb{Z}_{\sqrt{t(n-4)}}$ if n is odd.*

Proof. Note first that there are two natural generators of the Jacobian: Let v be the vertex with degree $n-1$ and u, w be distinct vertices, not adjacent to one another (so adjacent in C). Then the elements $u-v$ and $w-v$ form generators for the group: Take $w-v$, push once from w and absorb once from v. Then we have $(2-n)w+(n)v-u-x$, where x is the other vertex not adjacent to u. Adding $(n-2)(w-v)+(u-v)$ gives $v-x$, so we can generate $x-v$. Repeating this process, we generate every element of the form $t-v$, so generate the entire group.

To determine the exact orders of the subgroups, look at the generators above. Let X be the set of elements of $Jac(G)$ generated by $u-v$ and Y be the set of elements generated by $w-v$. Note that by symmetry, the orders of $u-v$ and $w-v$ are the same. We are interested, then, in $|X \cap Y|$, since we know that $|\operatorname{Jac}(G)| = \frac{|X||Y|}{|X \cap Y|}$. Note that if $\operatorname{Jac}(G)$ were cyclic, X and Y would be in $X \cap Y$ since both $u-v$ and $w-v$ have the same order and so would generate the same subgroup.

Take an element generated by $u-v$, and take the divisor $C = au - av$. Let the vertices of C be numbered $v_1, v_2, ..., v_{n-1}$ in order with $u = v_1$ and $w = v_2$. We say that a divisor, D, has symmetry across the line $v_i v$ if $D(v_{i+j}) = D(v_{i-j})$ for all j where the index arithmetic is performed modulo $n-1$. Note that the v-reduced form of any element generated by $u-v$ has symmetry across the line through u and v since the v-reduced form of the element is unique and its reflection about the line through u and v would also be v-reduced.

So the v-reduced divisor of an element generated by $u-v$ has symmetry across the line through u and v. Similarly, any v-reduced divisor of an element generated by $w-v$ must have symmetry across the line between w and v, so any element in the intersection must have both symmetries. Note that this means that the Jacobian of G is not cyclic since neither X nor Y is in $X \cap Y$.

If $n-1$ is odd, then the group generated by these two symmetries has two orbits on $V(G)$: $\{v\}$ and $V(G) - \{v\}$, so the intersection of the two sets is trivial, which gives the desired order of the Jacobian. If $n-1$ is even, the orbits are $\{v\}$, A, and B, where, if $n = 2m+1$, $A = \{v_2, v_4, ..., v_{2m}\}$ and $B = \{v_1, v_3, ..., v_{2m-1}\}$.

Note that each vertex in A is adjacent to each vertex in B except for two and vice-versa. So

$$\begin{aligned}\sum_{s\in A}(n-5)s-(n-5)mv &\sim \sum_{s\in A}(n-m-4)s+\sum_{s\in B}(m-2)s-m(2m-5)v \\ &\sim \sum_{s\in A}(m-3)s+\sum_{s\in B}(m-2)s-m(2m-5)v \\ &\sim \sum_{s\in B}(s-v)\end{aligned}$$

The first step comes from pushing from each vertex in A and the last from absorbing $m-3$ times from the center, v. Therefore, $\sum_{x\in A} x - mv$ is a generator for the intersection group, so to complete the proof, we need to show that its order is exactly $n-4$.

Let $C_A = \sum_{x\in A} x - mv$ and let $C_B = \sum_{x\in B} x - mv$. Then $(n-4)C_A \sim (n-5)C_A + C_A \sim C_B + C_A \sim 0$ by absorbing from B, so the order of C_A is at most $n-4$. Since the outdegree of each vertex of A is $m-1$, for $p \leq m-2$, pC_A is v-reduced. For $m-2 < p < n-4$, $pC_A \sim (n-4-p)C_B$ and $(n-4-p)C_B$ is v-reduced since the outdegree of each vertex in B is $m-1$. Therefore the order of C_A is exactly $n-4$.

Let s be the order of $u-v$ and t be the number of spanning trees of G. Then $s^2/(n-4) = t$, so $s = \sqrt{(n-4)t}$. Consider the subgroup of $\operatorname{Jac}(G)$ generated by $u+w-2v$. The v-reduced forms of the elements of this subgroup must have symmetry about the line between u and w through v. So the intersection of the subgroup generated by $u - v$ and $u + w - 2v$ is trivial. Note that $u - v$ and $u + w - 2v$ generate the group, so these divisors have the appropriate orders and the give the required Jacobian for G.

□

Theorem 2.2. *(Even portion originally by Biggs)*[4]*Let G be a wheel graph on n vertices and t be the number of spanning trees of G. Then the Jacobian of G is isomorphic to $\mathbb{Z}_{\sqrt{t}} \times \mathbb{Z}_{\sqrt{t}}$ if n is even and isomorphic to $\mathbb{Z}_{\sqrt{t/(n-4)}} \times \mathbb{Z}_{\sqrt{t(n-4)}}$ if n is odd.*

Proof. The same proof as above holds. □

2.3. K_N **with two paths removed.** We turn now to the case of K_N with two paths removed:

Definition. Let $G = K_N$ and let P_1 and P_2 be disjoint paths, such that $|V(P_1) \cup V(P_2)| = N - 1$. Then $G' = G \backslash E(P_1) \cup E(P_2)$ is said to be *nearly complete*, P_1 and P_2 are its *deleted paths*, and the sole vertex of $V(G) \backslash V(P_1) \cup V(P_2)$ is the *source*.

We introduce an auxiliary function that is often useful.

Definition. Let $a, m \in \mathbb{Z}$. Then we define $\text{res}_m(a)$ to be

$$\text{res}_m(a) = \begin{cases} a \bmod m, & \text{if } a < m \bmod 2m \\ -(a+1) \bmod m, & \text{if } a \geq m \bmod 2m \end{cases}$$

Note that $\text{res}_m(a)$ is periodic with period $2m$, that $\text{res}_m(a) = \text{res}_m(2m - (a+1))$, and that $\text{res}_m(a+m) = m - 1 - \text{res}_m(a)$.

On nearly complete graphs, let a be one end of P_1 and b be one end of P_2. Let $C = b - a$. Let u, v be vertices of G. Define a relation $\sim$ on $V(G)$ such that $u \sim v$ if the divisor $D = v - u$ is equivalent to 0 modulo C (or, equivalently, if there exists $n \in \mathbb{Z}$ such that $nC + D \sim 0$). Then $\sim$ is clearly an equivalence relation on $V(G)$.

We note several properties of this equivalence relationship.

Claim 1. *Let G be a nearly complete graph with deleted paths P and Q, with vertices u_0 and v_0 on P, u_0 adjacent to v_0 in P and $u_0 \sim v_0$. Take m to be the number of vertices of $P \backslash u_0$ in the component containing v_0 and n to be the number of vertices of $P \backslash v_0$ in the component containing u_0 and let u_i be the vertex at a distance of i from u_0 in the component of $P \backslash v_0$ containing u_0 and similarly for v_i. If $m \leq n$ and $r < n$, then $v_r \sim u_{\text{res}_m(r)}$.*

Proof. Let $|V(G)| = N$.

We proceed by induction on r. For $r = 0$, this is true by assumption, since $\text{res}_m(0) = 0$ and $v_0 \sim u_0$.

Assume $r > 0$ and that the claim is true for all integers strictly less than r. Note that in some of the following calculaltions, we might result in v_{-1} or u_{-1}. In those cases, we let $v_{-1} = u_0$ and $u_{-1} = v_0$ and note that $\text{res}_m -1 = 0$, so the claim is true for v_{-1}.

Suppose first that $\text{res}_m(r-1) = m - 1$, then take the divisor $u_{m-1} - v_{r-1}$ (noting that by the induction hypothesis $u_{m-1} \sim v_{r-1}$ and that u_{m-1} is the last vertex on the path). Pushing once from u_{m-1} and absorbing once from v_{r-1} gives $(3-N)u_{m-1} - u_{m-2} + v_{r-2} + v_r + (N-4)v_{r-1}$. We can reduce this by $(4-N)(u_{m-1} - v_{r-1})$ to get $-u_{m-1} - u_{m-2} + v_{r-2} + v_r$. Since $\text{res}_m(r-1) = m-1$, either $r \equiv m \mod 2m$ or $r \equiv m+1 \mod 2m$. If $r \equiv m \mod 2m$, then $\text{res}_m(r-2) = m-2$, in which case $-u_{m-1} - u_{m-2} + v_{r-2} + v_r \sim v_r - u_{m-1}$. So $v_r \sim u_{m-1} = u_{\text{res}_m(r)}$. On the other hand, If

$r \equiv m+1 \mod 2m$, then $\mathrm{res}_m(r-2) = m-1$ in which case $-u_{m-1} - u_{m-2} + v_{r-2} + v_r \sim v_r - u_{m-2}$. So $v_r \sim u_{m-2} = u_{\mathrm{res}_m(r)}$.

In the other case, when $\mathrm{res}_m(r-1) \neq m-1$, let $k = \mathrm{res}_m(r-1)$. Take $u_k - v_{r-1}$, absorb from v_{r-1} and push from u_k to get $(4-N)u_k - u_{k+1} - u_{k-1} + v_r + v_{r-2} + (N-4)v_{r-1}$. Reducing this by $(N-4)(v_{r-1} - u_k)$ gives $-u_{k+1} - u_{k-1} + v_r + v_{r-2}$. Then either $\mathrm{res}_m(r-2) = k+1$ and $\mathrm{res}_m(r) = k-1$ in which case this divisor gives us $-u_{k-1} + v_r$ or $\mathrm{res}_m(r-2) = k-1$ and $\mathrm{res}_m(r) = k+1$ in which case we get $-u_{k+1} + v_r$, which completes the proof. □

Corollary 2.3. *If G is a nearly complete graph with deleted paths P and Q with vertices x, y, z consecutive in P and pairwise equivalent, then for any vertex $u \in V(P)$, $x \sim u$.*

Proof. Let $V_d \subset V(P)$ be the vertices that have distance d to y in P. So $V_1 = \{x, z\}$. Then we proceed by induction on d. If $d = 2$, take $u \in V_2$. Without loss of generality, assume u is not adjacent to x in P (so it is adjacent to z). By Claim 1, $u \sim x$, since they are equidistant from the pair y, z with $y \sim z$. Then $u \sim x \sim y \sim z$, so all are pairwise equivalent.

Now assume $d > 2$ and assume the claim for all integers strictly less than d. Let $u \in V_d$. Then there are three vertices, x', y', z' consecutive with z' adjacent to u, which by the induction hypothesis are all equivalent to x. Then by the argument in the previous paragraph, $x' \sim u$, so $u \sim x$. □

Claim 2. *Let G be a nearly complete graph with deleted paths P and Q, with a_0 the first vertex in P and b_0 the first vertex in Q. Let $|V(P)| = n$, $|V(Q)| = m$ and $m \leq n$. Take $r < n$. Then $a_r \sim b_{\mathrm{res}_m(r)}$.*

Proof. Let $|V(G)| = n + m + 1 = N$.

We prove the claim only for the case where $r \leq m+1$ and note that if $a_{m+1} \sim b_m \sim a_m$, we can then apply Claim 1 to complete the proof.

We proceed by induction on r. For $r = 0$, this is true by assumption, since $\mathrm{res}_m(0) = 0$ and $a_0 \sim b_0$.

Assume $m \geq r > 0$ and that the claim is true for all integers strictly less than r. In this case note that $\mathrm{res}_m(r) = r$. Then we have $a_{r-1} \sim b_{r-1}$. Take the divisor $a_{r-1} - b_{r-1}$, push once from a_{r-1} and absorb once from b_{r-1} to get $(4-N)a_{r-1} - a_r - a_{r-2} + b_r + b_{r-2} + (N-4)b_{r-1}$. By the induction hypothesis, we know that $a_{r-2} \sim b_{r-2}$, so we can reduce this to get $b_r - a_r$, so $a_r \sim b_r$.

Finally, let $r = m+1$. Then $\mathrm{res}_m(r) = m+1$. Take $a_m - b_m$ and push once from a_m and absorb once from b_m to get $(4-N)a_m - a_{m-1} - a_{m+1} + b_{m-1} + (N-3)b_m$. Reducing this by $(4-N)(a_m - b_m)$ and noting that $a_{m-1} \sim b_{m-1}$, we get $b_m - a_{m+1}$.

We now apply Claim 1 with $u_i = a_{m-i-1}$ and $v_i = a_{m+i}$. Take $a_r = v_{r-m}$. By Claim 1, $a_r \sim u_{\mathrm{res}_m(r-m)} = a_{m-1-\mathrm{res}_m(r-m)}$. By the properties of the res function, $m - 1 - \mathrm{res}_m(r-m) = \mathrm{res}_m(r)$, so we have $a_r \sim a_{\mathrm{res}_m(r)} \sim b_{\mathrm{res}_m(r)}$ which completes the proof. □

Corollary 2.4. *Let G be a nearly complete graph with deleted paths P and Q, $|V(P)| = n$, $V(Q) = m$, and $n > m$. Then there exist $u, v \in V(P)$ with u and v adjacent in P and $u \sim v$.*

Proof. Let $V(P) = \{a_0, a_1, ..., a_{n-1}\}$. Then by Claim 2, $a_m \sim b_m$ and $a_{m+1} \sim b_m$. Since $m < n$, a_{m+1} is well-defined, so $a_m \sim a_{m+1}$. □

Corollary 2.5. *Let G be a nearly complete graph with deleted paths P and Q, with a_0 the first vertex in P and b_0 the first vertex in Q. Let $|V(P)| = n$, $|V(Q)| = m$ and $m \leq n$. Assume for some i, $a_i \sim a_{i+1}$. Then either $m | i$ or there exists $j \leq i$ with $b_j \sim b_{j+1}$.*

Proof. By Claim 2, $a_i \sim b_{\mathrm{res}_m(i)}$ and $a_{i+1} \sim b_{\mathrm{res}_m(i+1)}$. If $m|i$, then $\mathrm{res}_m(i) = \mathrm{res}_m(i+1)$, otherwise $|\mathrm{res}_m(i) - \mathrm{res}_m(i+1)| = 1$, so $b_{\mathrm{res}_m(i)}$ and $b_{\mathrm{res}_m(i+1)}$ are adjacent. Since $\mathrm{res}_m(i) \leq i$, this completes the proof. □

Corollary 2.6. *Let G be a nearly complete graph with deleted paths P and Q, with a_0 the first vertex in P and b_0 the first vertex in Q. Assume $a_i \sim a_j$ for all $i, j < |V(P)|$. Then* $\mathrm{Jac}(G)$ *is cyclic.*

Proof. By Claim 2, each b_k is equivalent to some a_r. Since all the a_r are pairwise equivalent, all the b_k must be pairwise equivalent as well and also pairwise equivalent with each a_r. So, since all the vertices are equivalent, any divisor must be a multiple of $a_0 - b_0$, so G is cyclic. □

To prove the main result, we first need the following lemma:

Lemma 2.7. *Given a nearly complete graph G, with deleted paths P and Q and vertices u, v adjacent in P with $u \sim v$, if $|V(P)|$ and $|V(Q)|$ are relatively prime, then* $\mathrm{Jac}(G)$ *is cyclic.*

Proof. Let m be the number of vertices of $P \backslash u$ in the component containing v and n the number of vertices of $P \backslash v$ in the component containing u

We may assume without loss of generality that $m \leq n$. We proceed by induction on m.

Let a_0 be the first vertex in P and b_0 the first vertex in Q with a_i and b_i the i^{th} vertices in P and Q respectively.

If $m = 1$, look at $a_1 - a_0$. Pushing from a_1 and absorbing from a gives $a_2 - a_0$, so a_2 and a_1 are equivalent. By Corollary 2.3, all the vertices on P are pairwise equivalent, so by Corollary 2.6, $\mathrm{Jac}(G)$ is cyclic.

Assume $m > 1$ and that the lemma holds for integers strictly less than m.

By Corollary 2.5, we may assume that m does not divide the length of P, since if it did, we would have an analogous situation in Q and m does not divide the length of Q since the lengths of Q and P are relatively prime.

By Claim 1 and the peridocity of the res function, P breaks up into subpaths of length $2m$ with $a_i \sim a_{2m+i}$. Since m does not divide the length of P, there are some leftover vertices. If the number of leftover vertices is greater than m, let x and y be the two middle vertices (the m^{th} and $m + 1^{st}$), so that $x \sim y$. Otherwise, let y be the first of these vertices and x be the last of the previous set. Then $x \sim y$. Let $|V(P)| = pm + c$ where c is the number of vertices in the component of $G \backslash x$ containing y, by our choice of x and y, and with $c < m$. The value c cannot be 0 since m does not divide $|V(P)|$. Then x and y are equivalent vertices a distance $c < m$ from the end, so by the inductive hypothesis, $\mathrm{Jac}(G)$ is cyclic. □

Theorem 2.8. *Let G be a nearly complete graph. Then* $\mathrm{Jac}(G)$ *is cyclic if and only if $|V(P_1)|$ and $|V(P_2)|$ are relatively prime.*

Proof. Let $n = |V(P_1)|$ and $m = |V(P_2)|$.

We begin by proving that if $|V(P_1)|$ and $|V(P_2)|$ are relatively prime then $\mathrm{Jac}(G)$ is cyclic. So assume n and m are relatively prime.

We may assume without loss of generality that $n > m$. Then by Corollary 2.4, P_1 has two equivalent adjacent vertices. Applying Lemma 2.7 completes the proof.

For the other direction, assume $|V(P_1)|$ and $|V(P_2)|$ are not relatively prime. Assume for the sake of contradiction that the Jacobian of G is cyclic. Let v be the source vertex and $V(P_1) = \{a_0, a_1, ..., a_{n-1}\}$ and $V(P_2) = \{b_0, b_1, ..., b_{m-1}\}$.

Let $p = \gcd(n, m)$, $n = kp$, $m = lp$. Let $f : V(G) - \{v\} \to \mathbb{Z}$ such that $f(a_s) = \mathrm{res}_p(s)$ and $f(b_s) = \mathrm{res}_p(s)$

Define a graph H on $p + 1$ vertices as follows. Holding one vertex back as a source, arrange the p vertices in a path, for any vertices that would be adjacent on the path, insert $k + l - 1$ edges between them and between any other vertices add $k + l$ edges. Then add the source with $k + l$ edges to each vertex.

Let the p vertices along the path be h_0 to h_{p-1} and define $\phi : V(G) \cup E(G) \to V(H) \cup E(H)$ to take $v \in V(G)$ to $h_{f(v)}$, the source of G to the source of H, and any edge between vertices with different values under f to an edge between the images of those vertices under ϕ and such that for any vertex in $V(G)$, no two edges incident with it map to the same edge under ϕ. Then this map is a morphism since every edge is mapped appropriately, and is harmonic since for every $x \in V(G)$ and e' incident with $\phi(x)$, there is exactly one e incident with x such that $\phi(e) = e'$. Note that the Jacobian of H is cyclic, since the element $s - h_0$ where s is the source is a generator (pushing once from h_0 and absorbing from s gives $s - h_1$ and repeating generates the rest of the elements).

Since $\mathrm{Jac}(G)$ and $\mathrm{Jac}(H)$ are both cyclic, the kernel of any surjective group homomorphism from $\mathrm{Jac}(G)$ to $\mathrm{Jac}(H)$ is exactly those elements x such that $x^{\frac{|\mathrm{Jac}(G)|}{|\mathrm{Jac}(H)|}} = 1$, since the kernel is a cyclic subgroup of G of that order. Therefore, the kernel of the map is independent of our choice of map. Specifically, $a_0 - b_0$ maps to 0 under ϕ, so is in the kernel of ϕ.

Note that not both of k and l are even since they are relatively prime. Let k be odd. Then define $f'(a_s) = p - 1 - res_p(s)$ and $f'(b_s) = res_p(s)$ and let ϕ' be the map between G and H induced by f'. Then $\phi'(a_0 - b_0) = h_{p-1} - h_0 \not\sim 0$, so $a_0 - b_0$ is not in the kernel of ϕ' which is a contradiction, so $\mathrm{Jac}(G)$ must not be cyclic. □

3. Threshold Graphs

3.1. Threshold Graph.

Definition. A *threshold graph* is any graph constructed via the following process. Starting with the empty graph, at every stage we add a vertex to the graph and mark it as either *heavy* or *light*. If v is heavy, we add edges from v to every other vertex and if v is light, we add no edges.

For convenience, we define the convention that for $u, v \in V(G)$, u is to the right of v if u was added before v and to the left if u was added after v. Similarly, for the ease of discussion, we number the vertices as they are added into the graph, so that v_1 is the first vertex added and v_n is the final vertex.

3.2. Jacobians on Threshold Graphs.

To explicitly characterize the Jacobians of threshold graphs, we require first an auxiliary function. For $v \in V(G)$, let $\phi(v)$ be $\deg(v)$ if v is a light vertex or $\deg(v)+1$ if v is a heavy vertex. Order the vertices, excluding the first and last, as above by when they were added to the graph and add an edge between any two vertices v_k and v_{k+1} if $\phi(u)$ and $\phi(v)$ are unequal and not relatively prime. Note that this generates a collection of disjoint paths. To each path we assign the value N_k to be the product of $\phi(v)$ for all v in the path. Then we define the group $A(G)$ to be $\oplus\mathbb{Z}_{N_k}$. Christianson and Reiner's conjecture [6] proposes that for all threshold graphs, $A(G) \cong \mathrm{Jac}(G)$. For example, consider the graph obtained from K_4 by removing the edge sets of a path on 2 vertices and a path on 1 vertex (so K_4 delete an edge). Note that this is the same as the threshold graph built by taking two light vertices and then two heavy vertices.

By Theorem 2.8, the Jacobian of this graph is isomorphic to $\mathbb{Z}_8$, rather than $\mathbb{Z}_4 \times \mathbb{Z}_2$. Since 2 and 4, the numbers obtained from ϕ, are unequal and not relatively prime, the conjectured Jacobian is also $\mathbb{Z}_8$.

Define a bad sequence of vertices to be consecutive vertices that are heavy-light-heavy or light-heavy-light. Like Christianson, in the case of a graph with no bad sequences, we can prove that $A(G) = \mathrm{Jac}(G)$, but Christianson and Reiner's conjecture, that this statement is true for all threshold graphs remains open.

Theorem 3.1. *(Originally in Christianson and Reiner)*[6] *For a connected threshold graph G with no bad sequence, $A(G) \cong \mathrm{Jac}(G)$.*

Proof. We procede by induction on $n = |V(G)|$ and let the vertices of G be labeled $v_1, \dots, v_n$ in order that they appear in the construction of the threshold graph. The vertex v_n is heavy as G is connected. If v_{n-1} is light then the theorem trivially holds by induction as deleting it changes neither $A(G)$ nor $\mathrm{Jac}(G)$. Thus we assume that v_{n-1} is also heavy.

We provide the generators and then prove that they generate the Jacobian and that their orders match with those conjectured. We define $\phi(v)$ as above and let $P_1, ..., P_m$ be the subpaths specified in the conjecture. Let p_i be the vertex of P_i added last and let q_i be the vertex of G added immediately before the first vertex of P_i. Then if $|V(P_i)| = 1$ and one of p_i and q_i is light and the other is heavy, then let $C_i = \phi(q_i)p_i - \phi(q_i)q_i$. Otherwise, let $C_i = p_i - q_i$. Note that since there are no bad sequences in G, the longest such path has at most 2 vertices.

We prove that the C_i are generators of $\mathrm{Jac}(G)$. We prove first that the order of C_i is the product of $\phi(v)$ for $v \in V(P_i)$.

If $|V(P_i)| = 1$ and both p_i and q_i are heavy, then $C_i = p_i - q_i$. Taking $\phi(p_i)C_i$ and pushing out once from p_i and absorbing once from q_i gives exactly 0. Taking kC_i with $k < \phi(p_i) - 1$ is q_i-reduced, and if $k = \phi(p_i) - 1$, pushing once from p_i gives a q_i-reduced element that is nonzero. Similarly, if $|V(P_i)| = 1$ and both p_i and q_i are light, then $C_i = p_i - q_i$. Taking $\phi(p_i)C_i$ and pushing out once from p_i and absorbing once from q_i gives exactly 0. Again, kC_i with $k < \phi(p_i)$ is q_i-reduced and nonzero.

Next, if $|V(P_i)| = 1$ and p_i is heavy and q_i is light, then $C_i = \phi(q_i)p_i - \phi(q_i)q_i$. Let $q_i = v_s, p_i = v_{s+1}$ and $D = \sum_{t<s} v_t$. Then taking $\phi(p_i)C_i$ and pushing out $\phi(q_i)$ times from p_i gives $\phi(q_i)D$. Pushing out from D and absorbing $\phi(q_i)$ times from q_i gives exactly 0. Taking kC_i with $k < \phi(p_i)$ gives (possibly after pushing out from p_i some number of times) a q_i-reduced divisor. Note that a similar argument holds if $|V(P_i)| = 1$ and p_i is light and q_i is heavy.

Finally, if $|V(P_i)| = 2$, then one of p_i and q_i is heavy and the other light. Then $C_i = p_i - q_i$. If $q_i = v_s, p_i = v_{s+2}$, then the order of C_i is, by symmetry, exactly the same as the order of $p_i - v_{s+1}$ which we determined in the previous paragraph was $\phi(q_i)\phi(p_i)$, so C_i has the appropriate order.

All that remains to be shown is that these divisors generate $\mathrm{Jac}(G)$.

Claim 3. *For $t \leq m$, let the last vertex added in P_t be v_k. Then $C_1, ...C_t$ generate all degree zero divisors on $v_1, ..., v_k$.*

Proof. For $t = 1$, if $|V(P_1)| = 1$, then $C_1 = v_2 - v_1$, which satisfies the claim. Otherwise, $C_1 = v_3 - v_1$, in which case, taking $\phi(v_3)C_1$, pushing once from v_3 and absorbing once from v_1 gives either $v_2 - v_1$ or $v_1 - v_2$ (depending on whether v_3 was light or heavy).

Assume $t > 0$ and assume the claim is true for integers strictly less than t. If $|V(P_1)| = 1$, then by induction we can generate any zero degree divisor on $v_1, ..., v_{k-1}$. If v_k and v_{k-1} are both heavy or both light, then $C_t = v_k - v_{k-1}$, so $C_1, ...C_t$ clearly generate all degree zero divisors on $v_1, ..., v_k$.

If $|V(P_1)| = 1$ and one of v_k and v_{k-1} is light and the other heavy, then $C_t = \phi(v_{k-1})v_k - \phi(v_{k-1})v_{k-1}$. Without C_t, we can generate any divisor with degree zero on $v_1, ..., v_k$ with the coefficient of v_k a multiple of $\phi(v_k)$, since taking the zero divisor, absorbing from v_k and then pushing from v_{k-1} gives a divisor with $\phi(v_k)$ as the coefficient of v_k and the coefficient of $v_r, r > k$

is 0. Since $\phi(v_k)$ and $\phi(v_{k-1})$ are relatively prime, by applying the Euclidean algorithm, there is some combination of $C_1, \ldots C_t$ that generates zero degree divisors with arbitrary coefficients for v_k. Since by induction, we can generate arbitrary zero degree divisors on $v_1, ..., v_{k-1}$, we can add these together to generate any degree zero divisors on $v_1, ..., v_k$.

If $|V(P_1)| = 2$, then $C_t = v_k - v_{k-2}$. So we can easily generate all degree zero divisors on $v_1, ..., v_{k-2}, v_k$. Taking $\phi(v_k)C_t$, pushing once from v_k and absorbing once from v_{k-2} gives a divisor with coefficient 1 or -1 (depending on whether v_k is light or heavy) and coefficient 0 for $v_r, r > k$. Adding this divisor to the ones we can already generate allows us to generate any degree zero divisor. □

By the claim, then, $C_1, ..., C_m$ generate all degree zero divisors on $v_1, \ldots, v_{n-1}$. Pushing once from v_{n-1} gives a divisor with coefficient 1 on v_n, so we can, in fact, generate all degree zero divisors on G with $C_1, ..., C_m$.

By [4], we know the total order of the Jacobian, so these must be exactly the generators, so we have the required explicit characterization of the Jacobian. □

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