DRAWING 4-PFAFFIAN GRAPHS ON THE TORUS

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Abstract. We say that a graph G is k-Pfaffian if the generating function of its perfect matchings can be expressed as a linear combination of Pfaffians of k matrices corresponding to orientations of G. We prove that 3-Pfaffian graphs are 1-Pfaffian, 5-Pfaffian graphs are 4-Pfaffian and that a graph is 4-Pfaffian if and only if it can be drawn on the torus (possibly with crossings) so that every perfect matching intersects itself an even number of times. We state conjectures and prove partial results for $k > 5$.

1. Introduction

All graphs considered in this paper are finite and have no loops or multiple edges. For a graph G we denote its edge set by $E(G)$. A *labeled graph* is a graph with vertex-set $\{1, 2, \dots, n\}$ for some n. If u and v are vertices in a graph G, then uv denotes the edge joining u and v and directed from u to v if G is directed. A *perfect matching* is a set of edges in a graph that covers each vertex exactly once. We denote the symmetric difference of sets X and Y by $X \triangle Y$.

Let G be a labeled graph, let D be an orientation of G and let $M = \{u_1v_1, u_2v_2, \dots, u_kv_k\}$ be a perfect matching of D. Define the *sign* of a perfect matching M in D, denoted by $D(M)$, to be the sign of the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & \dots & 2k-1 & 2k \\ u_1 & v_1 & u_2 & v_2 & \dots & u_k & v_k \end{pmatrix}.$$

Note that the sign of a perfect matching is well-defined as it does not depend on the order in which the edges are written. We say that a labeled graph G is *k-Pfaffian* if there exist orientations $D_1, D_2, \dots, D_k$ of G and real numbers $\alpha_1, \alpha_2, \dots, \alpha_k$, such that for every perfect matching M of G

$$\sum_{i=1}^{k} \alpha_i D_i(M) = 1.$$

The following result was mentioned by Kasteleyn [3] and proved by Galluccio and Loebl [1] and independently by Tesler [8].

Theorem 1.1. *Every graph that can be embedded on an orientable surface of genus g is 4^g-Pfaffian.*

We say that a graph is *Pfaffian* if it is 1-Pfaffian. Pfaffian graphs have been introduced by Kasteleyn [2, 3, 4] and have been extensively studied since.

By *a drawing* Γ of a graph G on a surface S we mean an immersion of G in S such that edges are represented by locally homeomorphic images of $[0, 1]$, not containing vertices in their interiors. Edges are permitted to intersect, but there are only finitely many intersections and each intersection is a crossing. For edges e, f of a drawing Γ let $cr_\Gamma(e,f)$ denote the number of times the edges e and f cross. For a perfect matching M let $cr_\Gamma(M)$, or $cr(M)$ if the drawing is understood from context, denote $\sum cr_\Gamma(e,f)$, where the sum is taken over all unordered pairs of distinct edges $e, f \in M$.

The following characterization of Pfaffian graphs was given by the author in [5, 6].

Theorem 1.2. *A graph G is Pfaffian if and only if there exists a drawing of G in the plane such that $cr(M)$ is even for every perfect matching M of G.*

The main results of this paper are a characterization of 4-Pfaffian graphs, similar to the above characterization of Pfaffian graphs, and a generalization of Theorem 1.1 to drawings with crossings.

Theorem 1.3. *A graph G is 4-Pfaffian if and only if there exists a drawing of G on the torus such that $cr(M)$ is even for every perfect matching M of G.*

Theorem 1.4. *Let G be a graph. If there exists a drawing of G on an orientable surface of genus g such that $cr(M)$ is even for every perfect matching M of G then G is 4^g-Pfaffian.*

In the next section we examine sequences of signs of perfect matchings in orientations of a k-Pfaffian graph. We prove that 3-Pfaffian graphs are Pfaffian and that 5-Pfaffian graphs are 4-Pfaffian. Section 3 contains proofs of Theorems 1.3 and 1.4.

2. Admissible sets of sign sequences

Let $k > 1$ be an integer. We say that a set $\mathcal{M}$ of $(1,-1)$-vectors of length k is *realizable* if there exists a labeled graph G that is k-Pfaffian, but not $(k-1)$-Pfaffian, orientations $D_1, D_2, \ldots, D_k$ of G and real numbers $\alpha_1, \alpha_2, \ldots, \alpha_k$ such that

$$\mathcal{M} = \{(D_1(M), D_2(M), \ldots, D_k(M)) \mid M \text{ is a perfect matching of } G\}$$

and for every perfect matching M of G

$$\sum_{i=1}^{k} \alpha_i D_i(M) = 1.$$

We say that G *realizes* $\mathcal{M}$. For a vector V of length k denote its i-th coordinate by $V(i)$ and for $S \subseteq \{1, 2, \ldots k\}$ denote $\prod_{i \in S} V(i)$ by $V(S)$.

We establish some conditions, which every realizable set $\mathcal{M}$ has to satisfy. The validity of the first such condition below follows trivially from the definition.

A1: There exist real non-zero numbers $\alpha_1, \alpha_2, \ldots, \alpha_k$ such that

$$\sum_{i=1}^{k} \alpha_i V(i) = 1$$

for every $V \in \mathcal{M}$.

To verify the next condition we first need to prove the following lemma.

Lemma 2.1. *Let G be a labeled graph, let k be an odd integer and let $D_1, D_2, \dots, D_k$ be orientations of G. Then there exists an orientation D of G such that for every perfect matching M of G we have*

$$D(M) = D_1(M)D_2(M)\dots D_k(M).$$

Proof. Define the orientation D of G as follows. For every edge $uv \in E(G)$, let $uv \in E(D)$ if $|\{i \,|\, 1 \le i \le k, uv \in D_i\}|$ is odd and let $vu \in E(D)$ otherwise. Denote by S_i the set of edges on which D differs from D_i. We have

$$D_i(M) = (-1)^{|M \cap S_i|} D(M).$$

It follows that

$$D_1(M)D_2(M)\dots D_k(M) = (-1)^{|M\cap S_1|+|M\cap S_2|+\dots+|M\cap S_k|} D(M).$$

It remains to note that by definition of D

$$|E \cap S_1| + |E \cap S_2| + \dots + |E \cap S_k|$$

is even for every $E \subseteq E(G)$. □

It follows from Lemma 2.1 that every realizable set $\mathcal{M}$ satisfies the following condition.

A2: For every odd $S_1, S_2, \dots, S_{k-1} \subseteq \{1, 2, \dots k\}$ and real numbers $\alpha_1, \alpha_2, \dots, \alpha_{k-1}$ there exists $V \in \mathcal{M}$ such that

$$\sum_{i=1}^{k-1} \alpha_i V(S_i) \ne 1.$$

The following three conditions follow from **A1** and **A2**, yet we find it convenient to state them separately.

B1: For every odd $S \subseteq \{1, 2, \dots k\}$ there exist $v_1, v_2 \in \mathcal{M}$ such that for $i \in \{1, 2\}$ we have

$$\prod_{j \in S} v_i(j) = (-1)^i;$$

B2: for any set of real numbers $\{\beta_v\}_{v \in \mathcal{M}}$ such that $\sum_{v\in\mathcal{M}} \beta_v v$ is a zero vector, we have $\sum_{v \in \mathcal{M}} \beta_v = 0$;

B3: every two elements of $\mathcal{M}$ differ in at least two coordinates.

Conditions **B1** and **B3** follow immediately from **A2** and **A1** respectively, while conditions **B2** and **A1** are equivalent by a standard linear algebra argument. We say that a set $\mathcal{V}$ of $(1, -1)$-vectors of length k is *admissible* if it satisfies conditions **B1**, **B2** and **B3**, and we say that $\mathcal{V}$ is *strongly admissible* if it satisfies conditions **A1** and **A2**. Every realizable set is strongly admissible, and every strongly admissible set is admissible.

We say that sets $\mathcal{V}$ and $\mathcal{W}$ of $(1, -1)$-vectors of length k are *equivalent* if $\mathcal{W}$ can be obtained from $\mathcal{V}$ as follows: for some permutation π of the set $\{1, 2, \dots, k\}$ and some $S \subseteq \{1, 2, \dots, k\}$ apply π to the coordinates of all vectors in $\mathcal{V}$ and change the signs of all coordinates with indices in S for all vectors in $\mathcal{V}$. The above is clearly an equivalence relation. Trivially, if the sets $\mathcal{V}$ and $\mathcal{W}$ are equivalent then $\mathcal{V}$ is admissible (strongly admissible, realizable) if and only if $\mathcal{W}$ is.

Lemma 2.2. *No set of* $(1,-1)$*-vectors of length two is admissible.*

Proof. Suppose $\mathcal{V}$ is an admissible set of $(1,-1)$-vectors of length two. Clearly $\mathcal{V}$ is equivalent to a set containing $(1,1)$ and therefore without loss of generality we assume $(1,1) \in \mathcal{V}$. By **B2** we know that $(-1,-1) \notin \mathcal{V}$ and therefore by **B1** applied to $S = \{1\}$ we have $(-1,1) \in \mathcal{V}$ in contradiction with **B3**. $\square$

Lemma 2.3. *No set of* $(1,-1)$*-vectors of length three is admissible.*

Proof. Again without loss of generality we assume $(1,1,1) \in \mathcal{V}$. It implies by **B2** that $(-1,-1,-1) \notin \mathcal{V}$ and by **B1** applied to $S = \{1,2,3\}$ and equivalence we may assume $(1,1,-1) \in \mathcal{V}$ in contradiction with **B3**. $\square$

The next theorem follows immediately from Lemmas 2.2 and 2.3 and the observations above.

Theorem 2.4. *Every* 3*-Pfaffian graph is Pfaffian.*

Next we examine (strongly) admissible sets of vectors of length four and five. Denote the set $\{(-1,1,1,1),\ (1,-1,1,1),\ (1,1,-1,1),(1,1,1,-1)\}$ of all $(-1,1)$-sequences of length 4 with exactly one negative entry by $\mathcal{S}$.

Lemma 2.5. *Every admissible set* $\mathcal{V}$ *of* $(1,-1)$*-vectors of length four is equivalent to* $\mathcal{S}$*.*

Proof. For a vector V of length four we denote $\sum_{i=1}^{4} v_i$ by $\sigma(V)$. Without loss of generality we assume $(1,1,1,1) \in \mathcal{V}$. By **B2** and **B3** we have $\sigma(V) \in \{-2,0,4\}$ for every $V \in \mathcal{V}$. Let n denote the number of elements $V \in \mathcal{V}$ with $\sigma(V) = -2$. We have $n \leq 3$ by **B2**. We claim that $n = 0$.

Suppose not. If $n = 3$ without loss of generality we assume $(1,-1,-1,-1)$, $(-1,1,-1,-1)$, $(-1,-1,1,-1) \in \mathcal{V}$. By **B1** applied to $S = \{1,2,3\}$ we may assume $(-1,1,1,-1) \in \mathcal{V}$ in contradiction with **B3**. If $n = 2$ we assume $(1,-1,-1,-1)$, $(-1,1,-1,-1) \in \mathcal{V}$ and **B1** applied to $S = \{1,2,3\}$ and **B3** again lead to a contradiction. If $n = 1$ by equivalence, **B1** applied to $S = \{1,2,3\}$ and **B3** we may assume $(1,-1,-1,-1)$, $(-1,1,1,-1) \in \mathcal{V}$ and apply **B1** to $S = \{1,2,4\}$ for a contradiction.

Condition **B1** applied to all subsets of $\{1,2,3,4\}$ of size 3 implies $|\mathcal{V}| \geq 4$. By **B2** we know that for every $V_1, V_2 \in \mathcal{V}$ we have $V_1 + V_2 \neq 0$. Therefore up to equivalence $\mathcal{V} = \{(1,1,1,1),(-1,1,1,-1),(1,-1,1,-1),\ (1,1,-1,-1)\}$ or $\mathcal{V} = \{(1,1,1,1),(1,1,-1,-1),(1,-1,1,-1),\ (1,-1,-1,1)\}$. Applying **B1** to $S = \{1\}$ we have $\mathcal{V} = \{(1,1,1,1),(-1,1,1,-1),(1,-1,1,-1),\ (1,1,-1,-1)\}$ and is equivalent to $\mathcal{S}$. $\square$

Lemma 2.6. *No set of* $(1,-1)$*-vectors of length five is strongly admissible.*

Proof. The only argument we were able to find proceeds by exhaustive case analysis. We therefore defer the proof of the lemma to the Appendix. $\square$

The theorem below immediately follows from Lemma 2.6.

Theorem 2.7. *Every 5-Pfaffian graph is 4-Pfaffian.*

We now need to introduce some additional notation. Let V and W be $(1,-1)$-vectors of length m and n, respectively. We denote by $V \times W$ the vector of length mn defined by

$$(V \times W)((j-1)n+i) = V(i)W(j)$$

for all $1 \le i \le m, 1 \le j \le n$. For sets of $(1,-1)$-vectors $\mathcal{V}$ and $\mathcal{W}$ of length m and n correspondingly let $\mathcal{V} \otimes \mathcal{W} = \{V \times W \mid V \in \mathcal{V}, W \in \mathcal{W}\}$. Let $\otimes^n \mathcal{V}$ denote the n^{th} power of $\mathcal{V}$ under this product operation. We use the convention $\otimes^0 \mathcal{V} = \{(1)\}$.

Conjecture 2.8. *Let G be a labeled graph that is k-Pfaffian, but not $(k-1)$-Pfaffian, for some integer $k \ge 1$. Then $k = 4^g$ for some non-negative integer g and there exists a set of $(1,-1)$-vectors $\mathcal{M}$ of length k, realized by G, such that $\mathcal{M} \subseteq \otimes^g \mathcal{S}$.*

Note that the results of this section imply that Conjecture 2.8 holds for $k \le 5$. Tardos [7] pointed out that there exists a strongly admissible set of $(1,-1)$-vectors of length six, namely the set of all vectors with exactly two negative coordinates. Therefore to prove Conjecture 2.8 one needs to use stronger properties of realizable sets than strong admissibility.

3. Drawing k-Pfaffian Graphs on surfaces

The following theorem is the main result of this section.

Theorem 3.1. *For a labeled graph G and a non-negative integer g the following are equivalent*

(1) *There exists a drawing of G on an orientable surface of genus g such that $cr(M)$ is even for every perfect matching M of G.*
(2) *There exist orientations $D_0, D_1, \ldots, D_{4^g-1}$ of G such that for every perfect matching M of G*

$$(D_0(M), D_1(M), \ldots, D_{4^g-1}(M)) \in \otimes^g \mathcal{S}.$$

It is convenient to prove Theorem 3.1 in terms of special kinds of planar drawings. Consider a plane with a fixed collection of g disjoint closed squares $Q_1, Q_2, \ldots, Q_g$. We say that $Q_1, Q_2, \ldots, Q_g$ are *singularities* and that a drawing of G in the plane is a g-drawing if the images of all the vertices of G lie outside $Q_1 \cup Q_2 \cup \ldots \cup Q_g$ and the images of the edges of G intersect each Q_i in a finite number of straight line segments which are parallel to the sides of Q_i. Figure 1 shows an example of a g-drawing.

For each singularity Q_i fix one of its sides. For $e \in E(G)$ let e' be its image in Γ. Denote by $s_\Gamma(i,e)$ the number of segments in $e' \cap Q_i$ parallel to the fixed side of Q_i and by $s'_\Gamma(i,e)$ the number of segments in $e' \cap Q_i$ perpendicular to this side. For $e, f \in E(G)$ let $cr'_\Gamma(e,f)$ denote the number of times the edges e and f cross outside of singularities, i. e.

$$cr'_\Gamma(e,f) = cr_\Gamma(e,f) - \sum_{i=1}^{g} (s_\Gamma(i,e)s'_\Gamma(i,f) + s'_\Gamma(i,e)s_\Gamma(i,f)).$$

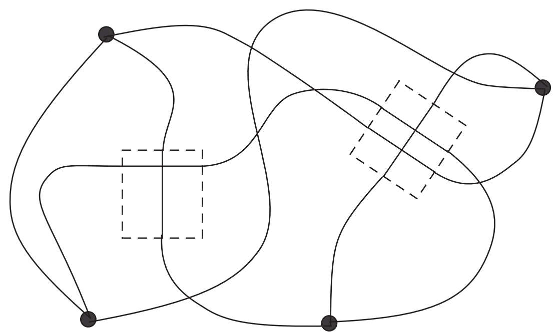

FIGURE 1. A 2-drawing of K_4

In the notation introduced above we omit index Γ when the drawing is understood from context. The definitions of g-drawings and of $cr'(e,f)$ are motivated by the following observation. For every drawing Γ of a graph G on an orientable surface of genus g there exists a g-drawing Γ' of a graph G in the plane such that $cr_\Gamma(e,f) = cr'_{\Gamma'}(e,f)$ for all $e,f \in E(G)$ and vice versa.

We say that $S \subseteq E(G)$ is a *marking* of a g-drawing Γ of G if $cr'_\Gamma(M)$ and $|M \cap S|$ have the same parity for every perfect matching M of G, where $cr'_\Gamma(M) = \sum_{\{e,f\}\subseteq M} cr'_\Gamma(e,f)$. Let L be a line in the plane and let H be one of the open half-planes determined by L such that all the singularities lie in H. We say that a g-drawing Γ of a labeled graph G is *standard* if the images of the vertices of G lie on L in order, and the images of the edges of G lie in $H \cup L$.

Lemma 3.2. *For a labeled graph G and a non-negative integer g the following are equivalent*

1. *There exists a standard g-drawing Γ of G and a marking S of Γ.*
2. *There exist orientations $D_0, D_1, \ldots, D_{4^g-1}$ of G such that for every perfect matching M of G*

$$(D_0(M), D_1(M), \ldots, D_{4^g-1}(M)) \in \otimes^g \mathcal{S}.$$

Proof. **(1)⇒(2).** Let $S' = \{e \in E(G) \mid \sum_{i=1}^{g} s(i,e)s'(i,e) \text{ is odd}\}$. For $i \in \{1,2,\ldots,g\}$ let $E(i,0) = \emptyset$, $E(i,1) = \{e \in E(G) \mid s(i,e) \text{ is odd}\}$, $E(i,2) = \{e \in E(G) \mid s'(i,e) \text{ is odd}\}$ and let $E(i,3) = E(i,1) \triangle E(i,2)$. For an integer j let j_i denote the i-th digit from the right in a base two representation of j and let $j_* = \sum_{i=1}^{2g} j_{2i-1}j_{2i}$. For an orientation D let $\chi(D) = \{uv \in E(D) | u > v\}$. Note that χ is a bijection between orientations of G and subsets of $E(G)$. For $j \in \{0,1,\ldots,4^g-1\}$ let

$$D'_j = \chi^{-1}(S \triangle S' \triangle E(1, j_1 + 2j_2) \triangle E(2, j_3 + 2j_4) \triangle \ldots \triangle E(g, j_{2g-1} + 2j_{2g})).$$

Let $D_j(M) = D'_j(M)$ if j_* is even and let $D_j(M)$ be obtained from $D'_j(M)$ by switching orientation of all the edges incident to vertex 1 if j_* is odd. We claim that $D_0, D_1, \ldots, D_{4^g-1}$ satisfy (2). It is not difficult to verify and is proved in

Lemma 4.2 in [5] that for an orientation D of G and a perfect matching M of G

$$(*)\qquad D(M) = (-1)^{cr(M)+|M\cap S(D)|}.$$

Let $s(i,M) = \sum_{e\in M} s(i,e)$ and $s'(i,M) = \sum_{e\in M} s'(i,e)$. For $j \in \{0,1,\dots,4^g-1\}$ the identities below hold modulo 2

$$cr(M) + |M\cap S(D'_j)| = cr(M) + |M\cap S| + |M\cap S'| + \sum_{i=1}^{g} |M\cap E(i, j_{2i-1}+2j_{2i})| =$$

$$= (cr'(M) - cr(M)) + |M\cap S'| + \sum_{i:j_{2i-1}=1} s(i,M) + \sum_{i:j_{2i}=1} s'(i,M) =$$

$$= \sum_{i=1}^{g}\left(\sum_{\{e,f\}\subseteq M}(s(i,e)s'(i,f)+s'(i,e)s(i,f)) + \sum_{e\in M} s(i,e)s'(i,e)\right) + \sum_{i:j_{2i-1}=1} s(i,M) +$$

$$+ \sum_{i:j_{2i}=1} s'(i,M) = \sum_{i=1}^{g}(s(i,M))(s'(i,M)) + \sum_{i:j_{2i-1}=1} s(i,M) + \sum_{i:j_{2i}=1} s'(i,M) =$$

$$= \sum_{i=1}^{g}(s(i,M)+j_{2i})(s'(i,M)+j_{2i-1}) + j_*.$$

Therefore

$$D'_j(M) = (-1)^{j_*}\prod_{i=1}^{g}(-1)^{(s(i,M)+j_{2i})(s'(i,M)+j_{2i-1})}.$$

and

$$D_j(M) = \prod_{i=1}^{g}(-1)^{(s(i,M)+j_{2i})(s'(i,M)+j_{2i-1})}.$$

Let $v_0 = (1,1,-1,1)$, $v_1 = (1,-1,1,1)$, $v_2 = (-1,1,1,1)$, and $v_3 = (1,1,1,-1)$. Note that for all $k \in \{0,1,2,3\}$ and $j \in \{1,2,3,4\}$ we have $v_k(j) = (-1)^{(j_1+k_1)(j_2+k_2)}$. Let $m(i) = r'_i + 2r_i$, where r_i and r'_i are the remainders modulo 2 of $s(i,M)$ and $s'(i,M)$ respectively. We claim that

$$(D_0(M), D_1(M), \dots, D_{4^g-1}(M)) = v_{m(1)} \times v_{m(2)} \times \dots \times v_{m(g)}.$$

Indeed

$$v_{m(1)} \times v_{m(2)} \times \dots \times v_{m(g)}(j) = \prod_{i=1}^{g} v_{m(i)}(j_{2i-1}+2j_{2i}) =$$

$$= \prod_{i=1}^{g}(-1)^{(r'_i+j_{2i-1})(r_i+j_{2i})} = \prod_{i=1}^{g}(-1)^{(s'(i,M)+j_{2i-1})(s(i,M)+j_{2i})} = D_j(M).$$

(2)⇒(1). Denote by A_j the set of edges of G in which D_j differs from D_0. Let Γ be a standard drawing of G such that for every $e \in E(G)$ such that $s(i,e)$ is odd if and only if $e \in A_{2^{2i-2}}$ and $s'(i,e)$ is odd if and only if $e \in A_{2^{2i-1}}$. Such a drawing is not difficult to construct. We use the notation introduced in the proof of (1)⇒(2) implication. Let $S = S(D_0)\triangle S'$. We claim that S is a marking of Γ, i.e. that $cr'(M) + |M\cap S|$ is even for every perfect matching M of G. By $(*)$ this claim is equivalent to the statement $D_0(M) = (-1)^{cr'(M)-cr(M)+|M\cap S'|}$ for every perfect matching M of G. Repeating part of the argument above we have modulo 2

$$cr'(M) - cr(M) + |M \cap S'| = \sum_{i=1}^{g}(s(i,M))(s'(i,M)) = \sum_{i=1}^{g} |M \cap A_{2^{2i-1}}||M \cap A_{2^{2i}}|.$$

Note that $|M \cap A_j|$ is even if and only if $D_0(M)D_j(M) = 1$. Let

$$(D_0(M), D_1(M), \ldots, D_{4^g-1}(M)) = w_1 \times w_2 \times \ldots \times w_g$$

for some $w_1, w_2, \ldots, w_g \in \mathcal{S}$. Then

$$D_0(M)D_{2^{2i-1}}(M) = w_i(1)w_i(2) \text{ and } D_0(M)D_{2^{2i}}(M) = w_i(1)w_i(3).$$

It follows that $|M \cap A_{2^{2i-1}}||M \cap A_{2^{2i}}|$ is odd if and only if $w_i(1) = -1$ as in every element of $\mathcal{S}$ at most one coordinate is negative. Therefore

$$(-1)^{cr'(M)-cr(M)+|M\cap S'|} = \prod_{i=1}^{g} w_i(1) = D_0(M),$$

verifying the claim. □

We say that g-drawings Γ_1 and Γ_2 are *similar* if every vertex of G has the same image in Γ_1 and Γ_2 and for every edge of G the symmetric difference of its images in Γ_1 and Γ_2 is a union of a family of closed simple curves in the plane none of which intersects a singularity or has a singularity in its interior. In [5] the following lemma was proved for $g = 0$ (if $g = 0$ every two g-drawings of G are similar). The proof for $g > 0$ is analogous.

Lemma 3.3. *Let Γ_1 and Γ_2 be similar g-drawings of a labeled graph G. If there exists a marking of Γ_1 then there exists a marking of Γ_2.*

If there exists a marking of a g-drawing Γ of a labeled graph G then there exists a g-drawing Γ' of G similar to Γ such that $\emptyset$ is a marking of Γ'.

Let L be a line in the plane such that all the singularities lie in one of the open half planes determined by L. Clearly every g-drawing Γ of G can be transformed by some homeomorphism of the plane that is identical on the singularities to a g-drawing Γ', such that the images of the vertices of G in Γ' lie on L in order. Such Γ' is similar to some standard drawing. This observation and Lemma 3.2 imply Theorem 3.1.

We are now ready to prove Theorem 1.4

Proof of Theorem 1.4. By Theorem 3.1 there exist orientations $D_0, D_1, \ldots, D_{4^g-1}$ of G such that for every perfect matching M of G

$$(D_0(M), D_1(M), \ldots, D_{4^g-1}(M)) \in \otimes^g \mathcal{S}.$$

It is easy to verify that the sum of the coordinates of every element of $\otimes^g \mathcal{S}$ is 2^g. Therefore for every perfect matching M of G

$$\frac{1}{2^g} \sum_{i=0}^{4^g-1} D(i) = 1,$$

and G is 4^g-Pfaffian. □

Theorems 2.3, 3.1, 1.4 and Lemma 2.5 imply Theorem 1.3. By Theorems 3.1 and 1.4 Conjecture 2.8 implies the following conjecture.

Conjecture 3.4. *For a graph G and a non-negative integer g the following are equivalent*

(1) *There exists a drawing of G on an orientable surface of genus g such that $cr(M)$ is even for every perfect matching M of G.*
(2) *G is 4^g-Pfaffian.*
(3) *G is $(4^{g+1}-1)$-Pfaffian.*

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Appendix A. Proof of Lemma 2.6

Let $\mathcal{V}$ be a strongly admissible set of vectors of length five.

For a vector V of length five let $S(V) = \{i \mid V(i) = 1\}$ and let $\sigma(V) = |S(V)|$. We assume without loss of generality that $(1,1,1,1,1) \in \mathcal{V}$. By **B2** and **B3** we have $\sigma(V) \in \{1,2,3,5\}$ for every $V \in \mathcal{V}$. Let n_k denote the number of elements $V \in \mathcal{V}$ with $\sigma(V) = k$. We consider cases depending on n_1. Note that by **B2**, we have $n_1 \leq 4$.

Case 1: $\mathbf{n_1 = 4}$. We assume without loss of generality that $(1,-1,-1,-1,-1)$, $(-1,1,-1,-1,-1)$, $(-1,-1,1,-1,-1)$, $(-1,-1,-1,1,-1) \in \mathcal{V}$. By **A1**, we have $V(1)+V(2)+V(3)+V(4)-3V(5) = 1$ for every $V \in \mathcal{V}$. Therefore $|\mathcal{V}| = 5$ and **B1** applied to $S = \{1,2,3,4,5\}$ yields a contradiction.

Case 2: $\mathbf{n_1 = 3}$. We assume that $(1,-1,-1,-1,-1)$, $(-1,1,-1,-1,-1)$, $(-1,-1,1,-1,-1) \in \mathcal{V}$. By **B1** applied to $S = \{1,2,3,4,5\}$ we have $n_2 > 0$ and therefore $(-1,-1,-1,1,1) \in \mathcal{V}$ by **B3**. We have $2(1,1,1,1,1)+(1,-1,-1,-1,-1)+(-1,-1,1,-1,-1)+(-1,-1,1,-1,-1)+(-1,-1,-1,1,1) = (0,0,0,0,0)$ in contradiction with **B2**.

Case 3: $\mathbf{n_1 = 2}$. We assume that $(1,-1,-1,-1,-1)$, $(-1,1,-1,-1,-1) \in \mathcal{V}$. By **B1** applied to $S = \{1,2,3,4,5\}$ and **B3** without loss of generality we have $(-1,-1,1,1,-1) \in \mathcal{V}$. By **B1** applied to $S = \{1,2,3\}$ there exists $W \in \mathcal{V}$ such that $S(W) \cap \{1,2,3\}$ is even. Suppose first $|S(W) \cap \{1,2,3\}| = 0$ then $W = \{-1,-1,-1,1,1\}$. It follows from **A1** that $V(1)+V(2)-2V(3)+3V(4)-2V(5) = 1$ for every $V \in \mathcal{V}$. In particular $|S(V) \cap \{1,2,4\}|$ is odd for every $V \in \mathcal{V}$ in contradiction with **B1**.

Therefore $|S(W) \cap \{1,2,3\}| = 2$. It follows from **B3** that $\sigma(W) = 3$. We consider all possible choices for W up to the symmetry between the first and second coordinates.

- **W=(1,1,-1,1,-1) or W=(1,-1,1,-1,1):** From **A1** we have $V(1)+V(2)+2V(3)-V(4)-2V(5) = 1$ for every $V \in \mathcal{V}$. Again it follows that $|S(V) \cap \{1,2,4\}|$ is odd for every $V \in \mathcal{V}$.
- **W=(1,1,-1,-1,1):** (-1,-1,1,1,-1) and W contadict **B2**.
- **W=(1,-1,1,1,-1):** (-1,-1,1,1,-1) and W contadict **B3**.

Case 4: $\mathbf{n_1 = 1}$. We assume that $(1,-1,-1,-1,-1) \in \mathcal{V}$. Note that by cases 1-3 we may assume that for every $V \in \mathcal{V}$ there exists at most one $W \in \mathcal{V}$ such that $|S(V) \triangle S(W)| = 4$. By **B1** applied to $S = \{1,2,3,4,5\}$ and **B3** we have $(-1,1,1,-1,-1) \in \mathcal{V}$ up to equivalence. We will proceed by considering subcases depending on n_3, but we would like to make a couple of observations first.

Note that if $W \in \mathcal{V}$, $\sigma(W) = 3$ then $1 \in S(W)$ by the observation above applied to $(1,-1,-1,-1,-1)$. Also $|S(W) \cap \{2,3\}| = 1$ by **B2** and **B3** applied to W and $(-1,1,1,-1,-1)$. Moreover note that if $\mathcal{S}$ is a strongly admissible set and V is a $(1,-1)$-vector that lies in the affine space spanned by $\mathcal{S}$ then $\mathcal{S} \cup V$ is strongly admissible.

4.1: $\mathbf{n_3 \geq 3}$. We assume without loss of generality that $(1,1,-1,1,-1)$, $(1,1,-1,-1,1)$, $(1,-1,1,1,-1)$, $(1,-1,1,-1,1) \in \mathcal{V}$. Indeed, no other vector W with $\sigma(W) = 3$ can lie in $\mathcal{V}$ by an observation above, and these four vectors are affinely dependent: $(1,1,-1,1,-1) + (1,1,-1,-1,1) - (1,-1,1,1,-1) - (1,-1,1,-1,1) = (0,0,0,0,0)$. From **A1** we have $2V(1) + V(2) + V(3) - V(4) - V(5) = 2$ for every $V \in \mathcal{V}$. It follows that $|\mathcal{V}| = 7$. We have

$$\frac{1}{2}(V(1) + V(\{1,2,4\})) + V(\{1,2,5\} - V(\{1,2,3\})) = 1$$

for every $V \in \mathcal{V}$ in contradiction with **A2**. Note that the set $\mathcal{V}$ is admissible.

4.2: $\mathbf{1 \leq n_3 \leq 2}$. We assume without loss of generality that $(1,-1,1,-1,1) \in \mathcal{V}$. If $(1,1,-1,-1,1) \in \mathcal{V}$ or $(1,-1,1,1,-1) \in \mathcal{V}$ then we again can conclude that $2V(1) + V(2) + V(3) - V(4) - V(5) = 2$ for every $V \in \mathcal{V}$ for a contradiction. By **B1** applied to $S = \{1,2,4\}$ there must exist $W \in \mathcal{V}$ with $\sigma(W) = 2$ such that $S(W) \cap \{1,2,4\}$ is even. If $S(W) \cap \{1,2,4\} = \emptyset$ then W and $(1,-1,1,-1,1)$ contradict **B2** and if $S(W) \subseteq \{1,2,4\}$ then W and $(1,-1,1,-1,1)$ contradict **B3**.

4.3: $\mathbf{n_3 = 0}$. Let $\mathcal{V}' = \{V \in \mathcal{V} \mid \sigma(V) = 2\}$. Note that $1 \notin S(W)$ for every $W \in \mathcal{V}'$ by **B3** applied to W and $(1,-1,-1,-1,-1)$. Also $S(W_1) \cap S(W_2) \neq \emptyset$ for every $W_1, W_2 \in V'$ by **B2** applied to $W_1, W_2, (1,-1,-1,-1,-1)$ and $(1,1,1,1,1)$. It follows that up to renumbering of the coordinates $\mathcal{V}$ is a subset of one of the following sets

$$\mathcal{V}_1 = \{(1,1,1,1,1), (1,-1,-1,-1,-1), (-1,1,1,-1,-1),$$
$$(-1,1,-1,1,-1), (-1,-1,1,1,-1)\}$$

or

$$\mathcal{V}_2 = \{(1,1,1,1,1), (1,-1,-1,-1,-1), (-1,1,1,-1,-1),$$
$$(-1,1,-1,1,-1), (-1,1,-1,-1,1)\}.$$

Moreover, $\mathcal{V}_1$ and $\mathcal{V}_2$ are equivalent. To verify that, consider changing signs of the last four coordinates of all vectors in $\mathcal{V}_1$. Therefore we assume $\mathcal{V} \subseteq \mathcal{V}_1$. Then

$$\frac{1}{2}(V(\{1,2,3\})) + V(\{1,2,4\} - V(\{1,2,5\}) - V(1)) = 1$$

for every $V \in \mathcal{V}$ in contradiction with **A2**.

Case 5: $\mathbf{n_1 = 0}$. Note that by the preceding cases we may assume that $|S(V) \triangle S(W)| \leq 3$ for all $V, W \in \mathcal{V}$. By **B1** applied to $S = \{1, \ldots, 5\}$ we assume without loss of generality that $(1, 1, -1, -1, -1) \in \mathcal{V}$. Let $\mathcal{V}' = \{V \in \mathcal{V} \mid \sigma(V) = 2\}$ be defined as before. By the observation above either there exists $x \in \{1, 2, 3, 4, 5\}$ such that $x \in W$ for every $W \in \mathcal{V}'$, or $n_2 = 3$ and there exists $S \subseteq \{1, 2, 3, 4, 5\}$ such that $|S| = 3$ and $S(W) \subset S$ for every $W \in V'$. Suppose first that the second outcome holds. Without loss of generality $\mathcal{V}' = \{(1, 1, -1, -1, -1), (1, -1, 1, -1, -1), (-1, 1, 1, -1, -1)\}$. By **A2** there must exist $U \in V$ such that $U(4) \neq U(5)$. It follows however that $\sigma(U) = 3$, $|S(U) \cap \{1, 2, 3\}| = 2$ and therefore there exists $W \in \mathcal{V}'$ such that $|S(W) \triangle S(U)| = 1$ in contradiction with **B3**.

We assume now without loss of generality that $1 \in W$ for every $W \in \mathcal{V}'$. By **B1** applied to $S = \{1\}$ there exists $U \in V$ such that $U(1) = -1$ and $\sigma(U) = 3$. Without loss of generality $U = \{-1, 1, 1, 1, -1\}$. By observations above $|S(W) \triangle S(U)| = 1$ for every $W \in V'$. By **B1** applied to $S = \{2, 3, 4\}$ there exists $T \in V$ such that $\sigma(T) = 3$ and $|S(T) \cap \{2, 3, 4\}| = 2$. Without loss of generality $T = (-1, 1, 1, -1, 1)$, as $|S(T) \triangle S(U)| \leq 3$. It also follows that $n_3 = 2$. Indeed if $Z \in \mathcal{V}$, $Z \neq U, T$ and $\sigma(Z) = 3$ then $|S(Z) \cap \{2, 3, 4\}| = |S(T) \cap \{2, 3, 5\}| = 2$ and therefore $S(Z) = \{2, 4, 5\}$ or $S(Z) = \{1, 2, 3\}$ in contradiction with **B2** or **B3** respectively. By **B1** applied to $S = \{2\}$ we have $(1, -1, 1, -1, -1) \in \mathcal{V}$. In fact it follows that

$$\begin{aligned} \mathcal{V} = \{&(1, 1, 1, 1, 1), (1, 1, -1, -1, -1), (1, -1, 1, -1, -1), \\ &\qquad (-1, 1, 1, 1, -1), (-1, 1, 1, -1, 1)\}. \end{aligned}$$

But then $S(V) \cap \{1, 4, 5\}$ is odd for every $V \in \mathcal{V}$ in contradiction with **B1**.

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