Graphs with full rank 3-color matrix and few 3-colorings

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Abstract

We exhibit a a counterexample to a conjecture of Thomassen stating that the number of distinct 3-colorings of every graph whose 3-color matrix has full column rank is superpolynomial in the number of vertices.

1 Introduction

Let G be a graph with vertices $v_1, v_2, \ldots, v_n$. A 3-*coloring* of G is a mapping that assigns to each vertex an element of the 3-element field $\mathbb{GF}_3$ such that adjacent vertices are given distinct elements. To each 3-coloring c of G we associate the vector $(c(v_1), c(v_2), \ldots, c(v_n))$. The 3-*color matrix* of G is the matrix whose rows are all these vectors. This matrix was introduced by Jensen and Thomassen [2]. They showed that this matrix has full column rank for every triangle-free planar graph, thereby strengthening a classical result by Grötzsch [1] that every triangle-free plane graph is 3-colorable. Thomassen [3] conjectured the following.

Conjecture 1. *There exists a positive number ε such that every graph on n vertices whose 3-color matrix has full column rank has at least 2^{n^ε} distinct 3-colorings.*

Thomassen [3] showed that every triangle-free planar graph has $2^{n^{1/12}/20\,000}$ distinct 3-colorings, where n is the number of vertices, verifying Conjecture 1

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for planar graphs. We disprove Conjecture 1 for general graphs in the next section, by exhibiting a family of triangle-free graphs with full column rank 3-color matrix and yet linearly many 3-colorings.

2 The counterexample

Throughout this section, we fix a positive integer k. The graph G_k has vertex set $V_k := \{v_0, v_1, \ldots, v_{3k-2}\}$, and for each $j \in \{0, 1, \ldots, 3k-2\}$, the vertex v_j is joined to the vertices $v_{j+k}, v_{j+k+1}, \ldots, v_{j+2k-1}$ where the indices are modulo $3k-1$.

The graph G_k is triangle-free, and hence the neighborhood of each vertex is an independent set. For each vertex $v_j \in V_k$, we define c_j to be the following 3-coloring, where the indices are modulo $3k-1$.

- The vertices $v_j, v_{j+1}, \ldots, v_{j+k-2}$ are colored 0;
- the vertices $v_{j+k-1}, v_{j+k}, \ldots, v_{j+2k-2}$ are colored 1; and
- the vertices $v_{j+2k-1}, v_{j+2k}, \ldots, v_{j+3k-2}$ are colored 2.

In the graph G_k, neighborhoods of the vertices are the only maximal independent sets. Consequently, up to a permutation of the colors, every 3-coloring of G_k is one of the colorings c_j. Therefore the graph G_k has linearly many 3-colorings, that is $6|V_k|$.

On the other hand, the 3-color matrix of G_k has full column rank. Indeed, fix a vertex $v_j \in V_k$. Let c be the 3-coloring of G_k obtained from c_{j+1} by recoloring v_j with 0. The difference of the two rows corresponding to c and c_{j+1} has a 1 on the entry corresponding to v_j, and 0 everywhere else.

We are unable to construct examples of families of graphs of girth five with full rank 3-color matrix and few 3-colorings. In fact, all the counterexamples to Conjecture 1 that we are aware of have large complete bipartite subgraphs. Thus it is natural to ask whether Conjecture 1 holds for graphs of girth five, or for graphs embeddable on a fixed surface.

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