Z. angew. Math. Phys. 50 (1999) 617–633
0044-2275/99/040617–21 \$ 1.50+0.20/0

Zeitschrift für angewandte Mathematik und Physik ZAMP

Asymptotic behavior of solutions for a degenerate hyperbolic system of viscous conservation laws with boundary effect

Ming Mei*

Abstract. This paper studies the asymptotic behavior of solutions to the initial boundary value problem on the half space R_+ a 2×2 degenerate system of viscous conservation laws. We show that the solution of the IBVP exists globally in time and tends toward a shifted critical viscous shock wave as time goes to infinity. We also prove that the system does not exhibit "phase transition" and remains in the degenerate hyperbolic state, when the initial perturbations are small. The proof is given by a weighted energy method.

Mathematics Subject Classification (1991). 35L67, 35B35, 76L05, 76N10, 76E99.

Keywords. Degenerate hyperbolic system, critical viscous shock wave, stability, boundary layer.

1. Introduction

We consider a one-dimensional 2×2 system of viscous conservation laws on the half space $R_+ = [0, +\infty)$ in the form

$$\begin{cases} v_t - u_x = 0, \\ u_t - \sigma(v)_x = \mu u_{xx}, \end{cases} \quad (x,t) \in R_+ \times R_+ \tag{1.1}$$

with the initial and the Dirichlet boundary conditions

$$\begin{cases} (v,u)|_{t=0} = (v_0, u_0)(x), \quad x \in R_+, \\ u|_{x=0} = u_-, \quad (v,u)|_{x=+\infty} = (v_+, u_+). \end{cases} \tag{1.2}$$

Such a system physically describes one-dimensional motion of viscoelastic materials and viscous isentropic gas. In the model of viscoelasticity (resp. viscous gas), v

*The author was partly supported by the JSPS Research Fellowship for Young Scientists and the Grant-in-Aid for JSPS No. PD-96169 from the Ministry of Culture, Education and Science of Japan.

is the strain (resp. the specific volume), u the velocity, $\mu(>0)$ the viscous constant, $\sigma(v)$ the stress function (resp. the pressure) , $u_\pm$ and v_+ is given constants. We assume $u_+ > u_-$. The initial data $(v_0, u_0)(x)$ is assumed to tend toward (v_+, u_+) as $x \to +\infty$, and satisfies the compatibility condition $u_0(0) = u_-$.

The usual assumption on $\sigma(v)$ is $\sigma'(v) > 0$ for all v under consideration. In this case, the system (1.1) with $\mu = 0$ is a strictly hyperbolic system. The stability theory of viscous shock waves to the Cauchy problem of system (1.1) with $\sigma'(v) > 0$ has been studied by many people, see [3,5-8,12-17] and references therein. If $\sigma'(v)$ changes sign, we call this the mixed case. If $\sigma'(v) < 0$ in a region, the characteristic roots of (1.1) $\lambda_1(v) = -\sqrt{\sigma'(v)}$ and $\lambda_2(v) = +\sqrt{\sigma'(v)}$ are imaginary, so the system (1.1) with $\mu = 0$ is elliptic in this region. If $\sigma'(v) \geq 0$ in a region, the characteristic roots $\lambda_1(v)$ and $\lambda_2(v)$ are real. At critical points v_* with $\sigma'(v_*) = 0$ we have $\lambda_1(v_*) = \lambda_2(v_*)$. Here the system (1.1) with $\mu = 0$ is degenerate hyperbolic. The prototypes of this situation are the cases $\sigma(v) = v^k$ with even k for $v \in (-\infty, \infty)$ in the viscoelastic model, and $-\sigma(v) = \frac{R\theta}{v-b} - \frac{a}{v^2}$ with positive constants R, θ, a and b satisfying $R\theta b/a < (2/3)^3$ for $v > b$ in the model of van der Waals gas, cf. [1,5,6,12,14,15,16]. In the mixed case, the corresponding stability theory is quite incomplete, see [1,2,4]. In [1], Chern-Mei studied the Cauchy problem of system (1.1) under the assumption $\sigma'(v) > 0$ for all $v > 0$ but $\sigma'(v) = 0$ only at $v = 0$, which implies that the system (1.1) is degenerate hyperbolic for $v \geq 0$, and that the degenerate hyperbolicity only occurs at the critical point $v = 0$. For a critical viscous shock wave of such a degenerate hyperbolic system of viscous conservation laws, that is, a traveling wave solution with one end state being the critical point $v = 0$, we proved its stability by the weighted energy method in [1]. It is both of mathematical and physical interest to investigate the asymptotic behavior of solution with a boundary effect. It seems there are no results on the asymptotics toward the critical viscous shock waves for the degenerate hyperbolic viscous system case with the boundary effect.

To study such a problem, we have to overcome two difficulties. One is to determine the effect of boundary layer on the wave shift. Another difficulty arises from the degenerate hyperbolicity. Now let us recall the progress on the stability theory for the initial-boundary value problem. For Burgers equation $u_t + uu_x = u_{xx}$, on the half space $R_- = (-\infty, 0]$ with the Dirichlet boundary condition, the first framework concerning the asymptotics toward the viscous shock wave was given in Liu-Yu [10] (see also Yu [18]) by the pointwise method, while the problem for the generalized Burgers equation $u_t + f(u)_x = u_{xx}$ has been deeply analyzed by Liu-Nishihara [9] based on the elementary energy method. To locate the wave's shift, they have to choose the shift to be a function dependent on t and to have some asymptotical properties, in order to overcome the difficulty caused by the viscous term at the boundary. However, their approaches cannot be straightforwardly applied to the system case. Very recently, for the physically viscous p-system

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v)_x = \mu(u_x/v)_x, \\ p(v) = av^{-\gamma},\ a > 0,\ \gamma \geq 1, \end{cases}$$

by a heuristic argument of how the shift is determined, Matsumura-Mei [11] succeeded in handling such a problem for the first time. We formally locate the shift as an explicit constant based on two basic observations: One is that there are no standing waves for the p-system; even if there are some waves with negative speed, they are expected to be reflected at the boundary and finally be captured by the front shock wave. This makes the variations of asymptotic behavior of the solution simpler than that for the Cauchy problem case. Another observation is that the viscous p-system is not uniformly parabolic, the first equation is inviscid and specifies the shift a constant. The same shift is available for the second equation, since we can expect that the value of $v(x, t)$ at the boundary is automatically controlled by the effects of boundary and viscosity. These considerations are also applicable to equations (1.1) since it is a viscous p-system too. Thus, we determine the shift for our critical viscous shock wave to be constant. To overcome the difficulty caused by the degenerate hyperbolicity, we will borrow Chern-Mei's idea in [1] and introduce a suitable weight function.

In this paper, we consider system (1.1) with the degenerate hyperbolicity in the state space $v \geq 0$, under the basic assumptions on $\sigma(v)$

$$\sigma'(v) > 0 \text{ for } v > 0, \text{ and } \sigma'(0) = 0. \tag{1.3}$$

We also suppose that at the critical point $v = 0$

$$\sigma'(0) = \cdots = \sigma^{(k-1)}(0) = 0,\ \sigma^k(0) \neq 0 \tag{1.4}$$

for some integer $k \geq 3$. Then $\sigma'(v) = O(1)v^{k-1}$ for v near 0. Our main purpose is to prove that, under the assumptions (1.3) and (1.4), the unique global solution $(v, u)(x, t)$ of the IBVP for (1.1) and (1.2) exists, and asymptotically aproximates a shifted critical viscous front shock wave $(V, U)(x - st + \alpha)$ on the half space R_+ as $t \to +\infty$. The boundary layer helps to push the critical viscous front shock away from the boundary. We also prove that the degenerate hyperbolic equations (1.1) do *not* exhibit "phase transition" under suitable small initial perturbations, namely, the solution v satisfies $v(x, t) \geq 0$ so that the system (1.1) retains the degenerate hyperbolicity. For the proofs, a weight function of exponential type will be introduced to treat the degenerate hyperbolicity of the system (1.1) at the critical point $v = 0$.

This paper is organized as follows. After stating the notations in the rest of this section, the properties of the critical viscous shock waves, the location of shift and the main theorem are stated in Section 2. Section 3 proves the main theorem by reformulating the original problem to a new IBVP, and proves the global exisitence and the asymptotic behavior of the solution for the reformed IBVP by the local existence result together with the *a priori* estimates. Section 4 is the proof of the *a priori* estimates by the weighted energy method.

Notations. Let L^2 and $H^l (l \geq 0)$ denote the L^2-space and Sobolev spaces on $R_+ = [0, \infty)$, respectively. Their norms are denoted by $\|\cdot\|$ and $\|\cdot\|_l$. Let L^2_w denote the weighted L^2-space with the weighted norm $\|f\|_w = (\int_0^\infty w(x)|f(x)|^2 dx)^{1/2}$, where $w > 0$ is the weighting function. Similarly, $H^l_w (l \geq 0)$ denotes the weighted Sobolev space with the weighted norm $\|f\|_{l,w} = (\sum_{j=0}^{l} \|\partial_x^j f\|^2_w)^{1/2}$. In what follows, C denote generic positive constants, and $f(x) \sim g(x)$ means $C^{-1} g \leq f \leq Cg$ for some positive constant C.

2. Preliminaries and main result

To analyze the asymptotic behavior of the solution of (1.1) and (1.2) approximating to a critical viscous shock, we are first going to recall the critical viscous shock waves and their properties. We then will locate the shift by a heuristic argument, cf. [11]. Finally, we will state our main result on the asymptotics toward a front critical shock wave on the half space R_+ as $t \to +\infty$ and the solution $v(x, t)$ is invariant on the region $v \geq 0$.

Critical viscous shock waves. A critical viscous shock wave of the system (1.1) is a travelling wave solution of (1.1) on the whole space $(-\infty, \infty)$ in the form

$$\begin{cases} (v, u)(x, t) = (V, U)(\xi), \\ (V, U)(\xi) \to (v_\pm, u_\pm), \end{cases} \quad \xi = x - st,$$

where s is the speed of wave, $(v_\pm, u_\pm)$ are the constant end states, and one of its end states, say v_+, is the critical state of the equations (1.1) at which $\sigma'(v_+) = 0$. In our case, see (1.3), that is $v_+ = 0$. If the speed $s > 0$, we call it a critical viscous *front* shock wave, while $s < 0$ a *back* wave. We plug $(u, v)(x, t) = (U, V)(x - st)$ into (1.1), then we arrive at

$$\begin{cases} -sV' - U' = 0, \\ -sU' - \sigma(V)' = \mu U'', \\ (V, U)(\pm\infty) = (v_\pm, u_\pm), \end{cases} \tag{2.1}$$

where $' = d/d\xi$, $\xi = x - st$. Integrating (2.1) and eliminating U, we obtain a single ordinary differential equation for $V(\xi)$:

$$\mu s V' = -s^2 V + \sigma(V) - \bar{c} \equiv: h(V), \tag{2.2}$$

where $\bar{c}$ is the integral constant given by

$$\bar{c} = -s^2 v_\pm + \sigma(v_\pm). \tag{2.3}$$

From (2.1), we can easily see $(v_\pm, u_\pm)$ and s satisfy the Rankine-Hugoniot condition

$$\begin{cases} -s(v_+ - v_-) - (u_+ - u_-) = 0 \\ -s(u_+ - u_-) - (\sigma(v_+) - \sigma(v_-)) = 0. \end{cases} \tag{2.4}$$

If we add the entropy condition

$$\frac{1}{s} h(v) \equiv \frac{1}{s}[-s^2(v - v_\pm) + \sigma(v) - \sigma(v_\pm)] \begin{cases} < 0, & \text{if } \ v_+ < v < v_- \\ > 0, & \text{if } \ v_- < v < v_+, \end{cases} \tag{2.5}$$

then the Rankine-Hugoniot condition (2.4) and the entropy condition (2.5) are sufficient and necessary for the existence of the travelling wave solutions of (2.1), unique up to the shift, see Kawashima-Matsumura [6], also Nishihara [15]. In this paper, we focus on

$$0 = v_+ < v_-, \tag{2.6}$$

and the convexity of $\sigma(v)$

$$\sigma''(v) > 0 \ \text{ for } \ v > 0, \tag{2.7}$$

which means that the entropy condition (2.5) is equivalent to the Lax's shock condition

$$\sigma'(v_+) < s^2 < \sigma'(v_-). \tag{2.8}$$

Also we have $s > 0$ from the Rankine-Hugoniot condition (2.4) and $u_+ > u_-$ and $v_+ < v_-$.

The standard arguments on the ordinary differential equations assert the existence of the critical viscous shock as follows.

Proposition 2.1. *Under the assumptions (2.4), (2.6)-(2.8), then there exists a unique front critical viscous shock wave* $(V, U)(\xi)$ $(\xi = x - st,\ s > 0)$ *up to a shift, satisfying*

$$v_- > V(\xi) > v_+ = 0, \quad u_- < U(\xi) < u_+, \quad s\mu V_\xi = h(V) < 0, \tag{2.9}$$

$$|V(\xi) - v_\pm| = O(1)e^{-c_\pm|\xi|}, \quad |U(\xi) - u_\pm| = O(1)e^{-c_\pm|\xi|}, \tag{2.10}$$

as $\xi \to \pm\infty$, *where* $c_\pm = |\sigma'(v_\pm) - s^2|/\mu s > 0$, *i.e,* $c_+ = s/\mu$ *due to* $\sigma'(v_+) = 0$.

Location of the shift. By the same basic observations in [11], we now make a heuristic argument to determine which of the shifted front critical viscous shock wave the solution of IBVP tends toward. Firstly, fixing a front critical viscous shock wave $(V,U)(x-st)$, we consider the situation where the initial data $(v_0,u_0)(x)$ are given in a neighborhood of $(V,U)(x-st-\beta)$ for some shift constant $\beta>0$, such that we can describe how $(V,U)(x-\beta)$ is far from the boundary by taking $\beta>0$ large, and the solution of the IBVP asymptotically converges to the shifted shock $(V,U)(x-st+\alpha-\beta)$ for some constant α determined below.

From the first equation of (1.1), we have

$$(v-V)_t=(u-U)_x, \quad (V,U)=(V,U)(x-st+\alpha-\beta). \tag{2.11}$$

Integrating (2.11) over R_+ with respect to x and by the boundary condition (1.2) yields

$$\frac{d}{dt}\int_0^\infty [v(x,t)-V(x-st+\alpha-\beta)]dx=(u-U)|_{x=0}^{\infty}=U(-st+\alpha-\beta)-u_-,$$

which implies, by the integration with respect to t,

$$\begin{aligned}&\int_0^\infty [v(x,t)-V(x-st+\alpha-\beta)]dx\\&=\int_0^\infty [v_0(x)-V(x+\alpha-\beta)]dx+\int_0^t [U(-st+\alpha-\beta)-u_-]dt.\end{aligned}$$

We want to determine some α such that

$$\int_0^\infty [v(x,t)-V(x-st+\alpha-\beta)]dx\to 0 \ \text{ as } \ t\to+\infty. \tag{2.12}$$

Let

$$I(\alpha):=\int_0^\infty [v_0(x)-V(x+\alpha-\beta)]dx+\int_0^\infty [U(-st+\alpha-\beta)-u_-]dt, \tag{2.13}$$

the shift α must be determined so that $I(\alpha)=0$, namely, (2.12) holds. We use (2.1) and note $v_+=0$ to have

$$\begin{aligned}I'(\alpha)&=-\int_0^\infty V'(x+\alpha-\beta)dx+\int_0^\infty U'(-st+\alpha-\beta)dt\\&=-v_++V(\alpha-\beta)-\frac{1}{s}[u_--U(\alpha-\beta)]\\&=-v_++v_-=v_-.\end{aligned}$$

Thus, we have by the integration on $I'(\alpha)$ with respect to α

$$I(\alpha)=I(0)+v_-\alpha=\int_0^\infty [v_0(x)-V(x-\beta)]dx+\int_0^\infty [U(-st-\beta)-u_-]dt+v_-\alpha.$$

So, due to $I(\alpha) = 0$, the constant shift $\alpha = \alpha(\beta)$ can be determined explicitly by

$$\alpha := -\frac{1}{v_-}\Big\{ \int_0^\infty [v_0(x) - V(x-\beta)]dx + \int_0^\infty [U(-st-\beta) - u_-]dt \Big\}, \tag{2.14}$$

and it holds that

$$\begin{aligned}
&\int_0^\infty [v(x,t) - V(x-st+\alpha-\beta)]dx \\
&= \int_0^\infty [v_0(x) - V(x+\alpha-\beta)]dx + \int_0^t [U(-st+\alpha-\beta) - u_-]dt \\
&= I(\alpha) - \int_t^\infty [U(-s\tau+\alpha-\beta) - u_-]d\tau \\
&= -\int_t^\infty [U(-s\tau+\alpha-\beta) - u_-]d\tau \to 0, \quad \text{as } t \to \infty.
\end{aligned} \tag{2.15}$$

On the other hand, we analyze the second equation of (1.1) by the similar equation

$$(u-U)_t = (\sigma(v) - \sigma(V) + \mu u_x - \mu U')_x$$

which means that by the integration over $R_+ \times [0,t]$

$$\begin{aligned}
&\int_0^\infty [u(x,t) - U(x-st+\alpha-\beta)]dx \\
&= \int_0^\infty [u_0(x) - U(x+\alpha-\beta)]dx - \int_0^t [\sigma(v(0,\tau)) - \sigma(V(-s\tau+\alpha-\beta))]d\tau \\
&\quad - \mu \int_0^t [u_x(0,\tau) - U'(-s\tau+\alpha-\beta)]d\tau.
\end{aligned}$$

In order to have

$$\int_0^\infty [u_0(x) - U(x+\alpha-\beta)]dx \to 0 \ \text{ as } \ t \to +\infty \tag{2.16}$$

for the same shift α determined in (2.14), we need

$$\begin{aligned}
&\int_0^\infty [u_0(x) - U(x+\alpha-\beta)]dx - \int_0^\infty [\sigma(v(0,t)) - \sigma(V(-st+\alpha-\beta))]dt \\
&\qquad -\mu \int_0^\infty [u_x(0,t) - U'(-st+\alpha-\beta)]dt = 0.
\end{aligned} \tag{2.17}$$

Since we don't specify the boundary values of $v(0,t)$, $u_x(0,t)$, we expect $v(0,t)$ and $u_x(0,t) = v_t(0,t)$ to be automatically controlled by the effect of the boundary and viscosity so that (2.17) holds with the same shift α. Indeed, it is true; for the details, we refer to Theorem 2.4 and the remark below.

We define the weight functions as follows

$$w(x,t) = \begin{cases} 1, & x \le st - \alpha + \beta \\ e^{c_+(k-1)(x-st+\alpha-\beta)}, & x \ge st - \alpha + \beta, \end{cases} \tag{2.18}$$

where $k \ge 2$ is given in (1.4). We also mark $w := w(x,t)$ and $w_0 := w_0(x) = w(x,0)$, that is

$$w_0(x) = \begin{cases} 1, & x \le -\alpha + \beta \\ e^{c_+(k-1)(x+\alpha-\beta)}, & x \ge -\alpha + \beta, \end{cases} \tag{2.19}$$

for the simplicity. It can be easily seen that $w(x,t)(\ge 1) \in C^0(R_+ \times R_+)$ but $w \notin C^1$. We have the property of the critical viscous shock wave as follows.

Lemma 2.2. *It follows that*

$$\sigma'(\xi)^{-1} \sim V(\xi)^{-(k-1)} \sim w(x,t), \tag{2.20}$$

$$\Big|\frac{\sigma''(V)V_\xi}{\sigma'(V)}\Big| \le C, \tag{2.21}$$

where $\xi = x - st + \alpha - \beta$.

Proof. Since $\sigma'(v_+) = \cdots = \sigma^{(k-1)}(v_+) = 0$ and $\sigma^{(k)}(v_+) \ne 0$, see (1.4), which implies $|\sigma'(V)| = O(1)|V|^{k-1}$ for V near $v_+ = 0$, and $|h(V)| = O(1)|V - v_\pm| = O(1)e^{-c_\pm|x-st+\alpha-\beta|}$ as $x - st + \alpha - \beta \to \pm\infty$ (see (2.10)), we can see that (2.20) and (2.21) are true. □

Suppose that

$$v_0(x) - V(x-\beta) \in H^1_{w_0},\ u_0(x) - U(x-\beta) \in H^1_{w_0} \tag{2.22}$$

for some $\beta > 0$, and set

$$(\Phi_0, \Psi_0)(x) := -\int_x^\infty (v_0(y) - V(y-\beta), u_0(y) - U(y-\beta))dy. \tag{2.23}$$

We further assume

$$\Phi_0 \in L^2 \ \text{ and } \ \Psi_0 \in L^2_{w_0}. \tag{2.24}$$

Then, we have the asymptotic behavior for the constant shift α as follows.

Lemma 2.3. *Under (2.22) and (2.24), it holds that* $\Phi_0 \in H^2$, $\Phi_{0,x} \in H^1_{w_0}$, $\Psi_0 \in H^2_{w_0}$ *and*

$$\alpha \to 0 \tag{2.25}$$

as $\|(\Phi_0, \Psi_0)\|_2 \to 0$ *and* $\beta \to +\infty$.

Proof. Without any difficulty, we can see that (2.22) and (2.24) imply $\Phi_0 \in H^2$, $\Phi_{0,x} \in H^1_{w_0}$, $\Psi_0 \in H^2_{w_0}$. To prove (2.25), we first note

$$0 < U(-st-\beta) - u_- \le Ce^{-c_-|-st-\beta|} = Ce^{-c_-(st+\beta)}$$

due to (2.9), (2.10) and $\beta > 0$, which means $\left|\int_0^\infty [U(-st-\beta)-u_-]dt\right| \le Ce^{-c_-\beta}$. So, we use the definition of α to obtain

$$\begin{aligned}|\alpha| &\le \frac{1}{v_-}\Big(\Big|\int_0^\infty [v_0(x) - V(x-\beta)]dx\Big| + \Big|\int_0^\infty [U(-st-\beta)-u_-]dt\Big|\Big)\\ &\le C(|\Phi_0(0)| + e^{-c_-\beta} \le C(\|\Phi_0\|_2 + e^{-c_-\beta}) \to 0\end{aligned}$$

as $\|(\Phi_0, \Psi_0)\|_2 \to 0$ and $\beta \to +\infty$. □

Main theorem. Let

$$(\phi_0, \psi_0)(x) := -\int_x^\infty [(v_0, u_0)(y) - (V, U)(y+\alpha-\beta)]dy, \quad x \in R_+. \tag{2.26}$$

We now state our main result as follows.

Theorem 2.4. *Suppose that (1.3), (1.4), (2.4), (2.6)-(2.8) and* $\phi_0 \in H^2$, $\phi_{0,x} \in H^1_{w_0}$, $\psi_0 \in H^2_{w_0}$ *hold. Furthermore, we assume*

$$-\sigma''(V)h(V)/2s^2\sigma'(V) < 1 \quad \textit{for} \quad V \in [v_+, v_-]. \tag{2.27}$$

Then there exists a positive constant δ_1 *such that if* $\|\phi_0\|_2 + \|\phi_{0,x}\|_{1,w_0} + \|\psi_0\|_{2,w_0} + \beta^{-1} < \delta_1$*, then (1.1) and (1.2) has a uniquely global solution* $(v,u)(x,t)$ *satisfying*

$$\begin{aligned} v - V &\in C^0([0,\infty); H^1_w) \cap L^2([0,\infty); H^1_w),\\ u - U &\in C^0([0,\infty); H^1_w) \cap L^2([0,\infty); H^2_w)\end{aligned}$$

and the asymptotic behaviors

$$\sup_{x\in R_+} |(v,u)(x,t) - (V,U)(x-st+\alpha-\beta)| \to 0 \quad \textit{as} \quad t \to +\infty, \tag{2.28}$$

$$\sup_{x\in R_+} \Big|\int_x^\infty [(v,u)(y,t) - (V,U)(y-st+\alpha-\beta)]dy\Big| \to 0 \quad \textit{as} \quad t \to +\infty, \tag{2.29}$$

$$v(x,t) \ge v_+ = 0 \quad \textit{for all} \quad (x,t) \in R_+ \times R_+. \tag{2.30}$$

Remark. 1. (2.29) includes the fact $\int_0^\infty [(v,u)(x,t) - (V,U)(x-st+\alpha-\beta)]dx \to 0$ as $t \to +\infty$, which means (2.16), or say, (2.17) is true for the shift α defined in (2.14).

2. (2.30) means that the system (1.1) does not exhibit "phase transition", and remains in the degenerate hyperbolic state.

3. There is no restriction to weak shocks, i.e., we don't assume $|v_+ - v_-| \ll 1$.

3. Proof of main theorem

In order to prove Theorem 2.4, as in Matsumura-Mei [11], we introduce a pair of new variables as follows

$$(\phi,\psi)(x,t) := -\int_x^\infty [(v,u)(x,t)-(V,U)(y-st+\alpha-\beta)]dy, \tag{3.1}$$

for $(x,t)\in R_+\times R_+$, namely,

$$(v,u)(x,t)=(V,U)(x-st+\alpha-\beta)+(\phi_x,\psi_x)(x,t). \tag{3.2}$$

Substituting (3.2) into (1.1), and integrating the system on $[x,\infty)$ with respect to x, we conclude the system for $(\phi,\psi)(x,t)$ with the Neumann boundary in the form

$$\begin{cases} \phi_t-\psi_x=0, & \\ \psi_t-\sigma'(V)\phi_x-\mu\psi_{xx}=F, & (x,t)\in R_+\times R_+, \\ (\phi,\psi)|_{t=0}=(\phi_0,\psi_0)(x), & x\in R_+, \\ \psi_x|_{x=0}=u_- - U(-st+\alpha-\beta), & t\in R_+, \end{cases} \tag{3.3}$$

where F is defined as $F=\sigma(V+\phi_x)-\sigma(V)-\sigma'(V)\phi_x$ and satisfying

$$|F|=O(|\phi_x|^2). \tag{3.4}$$

If one can prove the IBVP (3.3) to have a unique solution $(\phi,\psi)(x,t)$ time-globally, then one can know that the IBVP (1.1) and (1.2) has a unique solution $(v,u)(x,t)$ time-globally. For any constant $T>0$, let

$$\begin{aligned} X(0,T)=\{&\phi\in C^0(0,T;H^2),\phi_x\in C^0(0,T;H^1_w)\cap L^2(0,T;H^1_w), \\ &\psi\in C^0(0,T;H^2_w),\psi_x\in L^2([0,T];H^2_w)\} \end{aligned}$$

and

$$\begin{aligned} N(t)&=\sup_{0\le\tau\le t}(\|\phi(\tau)\|_2+\|\phi_x(\tau)\|_{1,w}+\|\psi(\tau)\|_{2,w}), \\ N_0&=\|\phi_0\|_2+\|\phi_{0,x}\|_{1,w_0}+\|\psi_0\|_{2,w_0}, \end{aligned}$$

where the weight functions $w(x,t)$ and $w_0(x)$ are defined in (2.18) and (2.19), we state the following theorem corresponding to Theorem 2.4.

Theorem 3.1. *In addition to the assumptions in Theorem 2.4, then there exists a positive constant δ_2 such that if $N_0+\beta^{-1}\le\delta_2$, then (3.3) has a uniquely global solution $(\phi,\psi)\in X(0,\infty)$ satisfying*

$$\|\phi(t)\|_2^2+\|\phi_x(t)\|_{1,w}^2+\|\psi(t)\|_{2,w}^2+\int_0^t(\|\phi_x(\tau)\|_{1,w}^2+\|\psi_x(\tau)\|_{2,w}^2)d\tau\le C(N_0^2+e^{-c_-\beta}), \tag{3.5}$$

$$\int_0^t \Big|\frac{d}{dt}\|\phi_x(\tau)\|^2\Big| + \Big|\frac{d}{dt}\|\phi_x(\tau)\|^2\Big| d\tau \le C(N_0^2 + e^{-c_- \beta}) \tag{3.6}$$

for all $t \ge 0$. Moreover, the asymptotic behaviors of the solution $(\phi, \psi)(t, x)$ hold

$$\sup_{x \in R_+} |(\phi_x, \psi_x)(x,t)| \to 0 \quad as \quad t \to \infty, \tag{3.7}$$

$$\sup_{x \in R_+} |(\phi, \psi)(x,t)| \to 0 \quad as \quad t \to \infty, \tag{3.8}$$

$$|\phi_x(x,t)| \le C\delta_2 e^{-c_+(k-1)|\xi|/2} \quad as \quad \xi = x - st + \alpha - \beta \to +\infty, \tag{3.9}$$

where C is independent of δ_2.

Proof of Theorem 2.4. Once Theorem 3.1 is true, then Theorem 2.4 is easily proved from Theorem 3.1. In fact, from (3.1) (or (3.2)) and (3.7), (3.8), we conclude that the IBVP (1.1) and (1.2) has a unique global solution $(v,u)(x,t)$ satisfying (2.28) and (2.29).

To see (2.30), since we have by (3.2)

$$v(x,t) - v_+ = V(\xi) - v_+ + \phi_x(x,t),$$

and $V(\xi) - v_+ \ge C_1 e^{-c_+|\xi|} \to 0$ as $\xi = x - st + \alpha - \beta \to +\infty$ for some constant $C_1 > 0$ by (2.9) and (2.10), also notice (3.9), we then have

$$v(x,t) - v_+ \ge V(\xi) - v_+ - |\phi_x(x,t)| \ge (C_1 - C\delta_2 e^{-c_+(k-3)|\xi|/2})e^{-c_+|\xi|} \ge 0$$

for $k \ge 3$ and $\delta_2 \ll 1$. This completes the proof of Theorem 2.4. □

Therefore, to prove Theorem 3.1 will be our purpose here. We now state the local existence result and the *a priori* estimates for the IBVP (3.3) as follows.

Proposition 3.2. (Local Existence). *For any $\delta_0 > 0$, there exists a positive constant T_0 depending on δ_0 such that, if $N_0 + \beta^{-1} \le \delta_0$, then the problem (3.3) has a unique solution $(\phi, \psi) \in X(0, T_0)$ satisfying $N(t) \le 2\delta_0$ for $0 \le t \le T_0$.*

Proposition 3.3. (A Priori Estimates.) *Let $(\phi, \psi) \in X(o, T_0)$ be a solution of (3.3) for a positive T. Then there exists a positive constant δ_3 such that if $N(T) + \beta^{-1} < \delta_3$, then $(\phi, \psi)(x,t)$ satisfies the a priori estimates (3.5) and (3.6) for $0 \le t \le T$.*

The proof of Proposition 3.2 is standard, we omit its details. To prove Proposition 3.3 is our main effort. By Propositions 3.2 and 3.3, we can prove Theorem 3.1.

Proof of Theorem 3.1. Thanks to Propositions 3.2 and 3.3, by the standard continuation argument, we can obtain a uniquely global solution $(\phi, \psi)(x,t)$ satisfying (3.5) and (3.6) for all $t \in [0, \infty)$.

To prove (3.7) and (3.8), we consider the function $G(t) := \|\phi_x(t)\|^2 + \|\psi_x(t)\|^2$. By virtue of the uniform estimates (3.5) and (3.6) and by $\|\phi_x(t)\|_{1,w} + \|\psi_x(t)\|_{1,w} \geq \|\phi_x(t)\|_1 + \|\psi_x(t)\|_1$ due to $w(x,t) \geq 1$, we see that both $G(t)$ and $|G'(t)|$ are integrable over $t \geq 0$. So, it means that $G(t) \to 0$, i.e., $\|\phi_x(t)\| + \|\psi_x(t)\| \to 0$ as $t \to \infty$. Furthermore, $\|\phi(t)\|_2 + \|\psi(t)\|_2$ is uniformly bounded in $t \geq 0$ due to (3.5) By the Sobolev inequality, we then obtain

$$\sup_{x\in R_+} |(\phi_x, \psi_x)(x,t)|^2 \leq 2\{\|\phi_x(t)\|\|\phi_{xx}(t)\| + \|\psi_x(t)\|\|\psi_{xx}(t)\|\} \to 0,$$

$$\sup_{x\in R_+} |(\phi, \psi)(x,t)|^2 \leq 2\{\|\phi(t)\phi_x(t)\| + \|\psi(t)\|\|\psi_x(t)\|\} \to 0$$

as $t \to \infty$.

To prove (3.9), firstly, we can easily check $(w\phi_x)(x,t) \in H^1$ since $\phi(x,t) \in X(0,\infty)$. By the Sobolev inequality, we have $|w(x,t)\phi_x(x,t)| \leq C\|(w\phi_x)(t)\|_1 \leq C\|\phi_x(t)\|_{1,w}$. This fact together with (3.5) yield (3.9). The proof of Theorem 3.1 is complete. □

4. Proof of a priori estimates

Let $(\phi, \psi) \in X(0,T)$ be a solution of (3.3) for a positive constant T. We firstly give the boundary estimates.

Theorem 4.1. *The boundary estimates hold*

$$\left|\int_0^t \phi\psi|_{x=0} d\tau\right| \leq CN(t)e^{-c_-\beta}, \tag{4.1}$$

$$\left|\int_0^t [\sigma'(V)^{-1}\psi\psi_x]|_{x=0} d\tau\right| \leq CN(t)e^{-c_-\beta}, \tag{4.2}$$

$$\left|\int_0^t \phi_x\psi_x|_{x=0} d\tau\right| \leq CN(t)e^{-c_-\beta}, \tag{4.3}$$

$$\left|\int_0^t [\sigma'(V)^{-1}\psi_x\psi_t]|_{x=0} d\tau\right| \leq CN(t)e^{-c_-\beta}, \tag{4.4}$$

$$\left|\int_0^t [\sigma'(V)^{-1}\psi_x\psi_{xx}]|_{x=0} d\tau\right| \leq CN(t)e^{-c_-\beta}, \tag{4.5}$$

$$\left|\int_0^t [\sigma'(V)^{-1}\psi_{xt}\psi_{xx}]|_{x=0} d\tau\right| \leq CN(t)e^{-c_-\beta}, \tag{4.6}$$

for $t \in [0,T]$, where $c_- = |\sigma'(c_-) + s^2|/\mu s > 0$ stated in (2.10).

Proof. From the first equation of (3.3), we have $\phi_t|_{x=0} = \psi_x|_{x=0} = u_- - U(-st + \alpha - \beta)$, integrate it on t and note (2.26) or (2.15) to get

$$\begin{aligned}\phi(0,t) &= \phi_0(0) - \int_0^t [U(-s\tau + \alpha - \beta) - u_-]d\tau \\ &= -\int_0^\infty [v_0(x) - V(x + \alpha - \beta)dx - \int_0^t [U(-s\tau + \alpha - \beta) - u_-]d\tau \\ &= \int_0^\infty [U(-s\tau + \alpha - \beta) - u_-]d\tau]d\tau \equiv: A(t). \end{aligned} \tag{4.7}$$

Hence, by the facts $|-st + \alpha - \beta| = st + \beta - \alpha$ because of $\beta \gg |\alpha|$ (see (2.25)), and $|U(-st + \alpha - \beta) - u_-| \le Ce^{-c_-(\beta-\alpha)}e^{-c_-st} \le O(1)e^{-c_-\beta}e^{-st}$, we have

$$|\phi(0,t)| = |A(t)| \le O(1)\int_t^\infty e^{-c_-\beta}e^{-c_-st}d\tau \le O(1)e^{-c_-\beta}e^{-c_-st}. \tag{4.8}$$

Similarly, we have $\psi_{xt}|_{x=0} = A''(t)$, $A(t) \in W^{3,1}(0,\infty)$ and that

$$\left|\frac{d^l}{dt^l}A(t)\right| \le Ce^{-c_-\beta}e^{-c_-st} \ (l = 0,1,2,3), \ \|A\|_{W^{3,1}} \le Ce^{-c_-\beta}. \tag{4.9}$$

We note also the Sobolev inequalities as follows

$$|\phi(0,t)| \le \sup_{x\in R_+} |\psi(x,t)| \le CN(t), \ |\psi_x(0,t)| \le \sup_{x\in R_+} |\phi_x(x,t)| \le CN(t). \tag{4.10}$$

To prove (4.1), (4.2) and (4.3), since $-st + \alpha - \beta < 0$ by $\beta > |\alpha|$ and $V(-st + \alpha - \beta) > V(0)$ by the monotonicity of $V(\xi)$, we note $|\sigma'(V(-st + \alpha - \beta))|^{-1} \le |\sigma'(V(\alpha - \beta))|^{-1} \le C$ because of the monotonicity of $\sigma'(V)$, and use (4.7), (4.9) and (4.10) to have (4.1) and (4.2) as follows:

$$\left|\int_0^t \phi\psi|_{x=0}d\tau\right| \le \int_0^t |A(\tau)||\psi(0,\tau)|d\tau \le CN(t)e^{-c_-\beta}$$

and

$$\left|\int_0^t [\sigma'(V)^{-1}\psi\psi_x]|_{x=0}d\tau\right| \le C\int_0^t |A'(\tau)||\psi(0,\tau)|d\tau \le CN(t)e^{-c_-\beta}.$$

A similar fashion yields (4.3).

To prove (4.4), making use of the equality $\phi_{tx} = \psi_{xx}$, and integrating it by parts, also by (4.9) and (4.10), we have

$$\begin{aligned}&\left|\int_0^t \psi_x\psi_t|_{x=0}d\tau\right| = \left|\int_0^t A'(\tau)\psi_t(0,\tau)d\tau\right| \\ &\left|\int_0^t [\{A'(\tau)\psi(0,\tau)\}_t - A''(\tau)\psi(0,\tau)]d\tau\right| \le CN(t)e^{-c_-\beta}.\end{aligned}$$

By the same way, we can prove (4.6) and (4.7) without any difficulty. □

We are going to make the following basic energy estimates.

Lemma 4.2. *It follows that*

$$\begin{aligned}&\|\phi(t)\|^2+\|\psi(t)\|_w^2+\int_0^t\|\psi_x(\tau)\|_w^2d\tau\\&\le C\left\{\|\phi_0\|^2+\|\psi_0\|_{w_0}^2+N(t)e^{-c_-\beta}+N(t)\int_0^t\|\phi_x(\tau)\|_w^2d\tau\right\}.\end{aligned}\tag{4.11}$$

Proof. We multiply the first equation of (3.3) by ϕ and the second one by $\psi/\sigma'(V)$ respectively, and add them to yield

$$\begin{aligned}\left\{\frac{\phi^2}{2}+\frac{\psi^2}{2\sigma'(V)}\right\}_t-\left\{\phi\psi+\frac{\mu}{\sigma'(V)}\psi\psi_x\right\}_x-\frac{s\sigma''(V)V_x}{2\sigma'(V)^2}\psi^2&\\-\frac{\mu\sigma''(V)V_x}{\sigma'(V)^2}\psi\psi_x+\frac{\mu}{\sigma'(V)}\psi_x^2=\frac{F\psi}{\sigma'(V)}.&\end{aligned}\tag{4.12}$$

Since

$$\begin{aligned}&-\frac{s\sigma''(V)V_x}{2\sigma'(V)^2}\psi^2-\frac{\mu\sigma''(V)V_x}{\sigma'(V)^2}\psi\psi_x+\frac{\mu}{\sigma'(V)}\psi_x^2\\&=-\frac{\sigma''(V)V_x}{2s\sigma'(V)^2}(s\psi-\mu\psi_x)^2+\frac{\mu}{\sigma'(V)}\left(1+\frac{\sigma''(V)h(V)}{2s^2\sigma'(V)}\right)\psi_x^2\\&\ge C\mu\psi_x^2/\sigma'(V),\end{aligned}\tag{4.13}$$

where we used the facts that $\mu sV_x=h(V)\le 0$, $\sigma''(V)\ge 0$ and $1+\frac{\sigma''(V)h(V)}{2s^2\sigma'(V)}>0$ for $V\in[v_+,v_-]$ which means $1+\frac{\sigma''(V)h(V)}{2s^2\sigma'(V)}\sim C$ for some positive constant C, see (2.9), (2.7) and (2.27), we substitute (4.13) into (4.12) to have

$$\left\{\frac{\phi^2}{2}+\frac{\psi^2}{2\sigma'(V)}\right\}_t-\left\{\phi\psi+\frac{\mu}{\sigma'(V)}\psi\psi_x\right\}_x+\frac{C\mu}{\sigma'(V)}\psi_x^2\le\frac{F\psi}{\sigma'(V)}.\tag{4.14}$$

Integrating (4.14) over $R_+\times[0,t]$ and using the boundary estimates (4.1) and (4.2), also noting $|F|=O(1)\phi_x^2$ (see (3.4)) and $\sigma'(V)^{-1}\sim w(x,t)$ (see (2.20)), we obtain

$$\begin{aligned}&\|\phi(t)\|^2+\|\psi(t)\|_w^2+\int_0^t\|\psi_x(\tau)\|_w^2d\tau\\&\le C\left(\|\phi_0\|^2+\|\psi_0\|_{w_0}^2+\int_0^t\int_0^\infty\left|\frac{F\phi}{\sigma'(V)}\right|dxd\tau+\left|\int_0^t[\phi\psi-\mu\sigma'(V)^{-1}\psi\psi_x]|_{x=0}d\tau\right|\right)\\&\le C\left(\|\phi_0\|^2+\|\psi_0\|_{w_0}^2+N(t)\int_0^t\|\phi_x(\tau)\|_w^2d\tau+N(t)e^{-\beta}\right).\end{aligned}\tag{4.15}$$

Thus, we have proved this lemma. □

We make the higher estimates for the solution of (3.3). Differentiating the equations of (3.3) with respect to x and multiplying the first equation by ϕ_x and the second one by $\sigma'(V)^{-1}\psi_x$, then adding them, we obtain

$$\left\{\frac{\phi_x^2}{2}+\frac{\psi_x^2}{2\sigma'(V)}\right\}_t-\left\{\phi_x\psi_x+\frac{\mu}{\sigma'(V)}\psi_x\psi_{xx}\right\}_x+\frac{\mu}{\sigma'(V)}\psi_{xx}^2-\frac{s\sigma''(V)V_x}{2\sigma'(V)^2}\psi_x^2$$
$$-\frac{\mu\sigma''(V)V_x}{\sigma'(V)^2}\psi_x\psi_{xx}=\frac{F_x\psi_x}{\sigma'(V)}+\frac{\sigma''(V)V_x}{\sigma'(V)}\phi_x\psi_x. \tag{4.16}$$

Since Lemma 2.2 means $|\sigma''(V)V_x/\sigma'(V)|\le C$ and $|\sigma''(V)V_x|\le C$, we get by the Cauchy inequality that

$$\left|\frac{\mu\sigma''(V)V_x}{\sigma'(V)^2}\psi_x\psi_{xx}\right|\le\frac{\mu\psi_{xx}^2}{2\sigma'(V)}+\frac{C\mu\psi_x^2}{2\sigma'(V)}, \tag{4.17}$$
$$\left|\frac{\mu\sigma''(V)V_x}{\sigma'(V)}\phi_x\psi_x\right|\le\frac{\varepsilon\phi_x^2}{\sigma'(V)}+\frac{C\psi_x^2}{4\varepsilon\sigma'(V)}, \tag{4.18}$$

where $0<\varepsilon<1$ is a constant to be chosen later (see (4.36)). Substituting (4.17) and (4.18) into (4.16), integrating the resulant inequality over $R_+\times[0,t]$, and applying Lemma 2.2 yield

$$\begin{aligned}&\|\phi_x(t)\|^2+\|\psi_x(t)\|_w^2+\int_0^t\|\psi_{xx}(\tau)\|_w^2d\tau\\&\le C\left\{\|\phi_{0,x}\|^2+\|\psi_{0,x}\|_{w_0}^2+\left|\int_0^t[\phi_x\psi_x-\mu\sigma'(V)^{-1}\psi_x\psi_{xx}]|_{x=0}d\tau\right|\right.\\&\quad+\varepsilon\int_0^t\|\phi_\xi\|_w^2d\tau+(1+\varepsilon^{-1})\int_0^t\|\psi_x(\tau)\|_w^2d\tau\\&\quad\left.+\left|\int_0^t\int_0^\infty\frac{F_x\psi_x}{\sigma'(V)}dxd\tau\right|\right\}.\end{aligned} \tag{4.19}$$

On the other hand, we have by (4.11)

$$\begin{aligned}&(1+\varepsilon^{-1})\int_0^t\|\psi_x(\tau)\|_w^2d\tau\\&\le C(1+\varepsilon^{-1})\left(\|\phi_0\|+\|\psi_0\|_{w_0}+N(t)e^{-c_-\beta}+N(t)\int_0^t\|\phi_x(\tau)\|_w^2d\tau\right),\end{aligned} \tag{4.20}$$

and have by the integration by parts and using (3.4) and (4.3),

$$\begin{aligned}
&\left|\int_0^t\int_0^\infty \frac{F_x\psi_x}{\sigma'(V)}dxd\tau\right| \le \left|\int_0^t\int_0^{+\infty} F\left\{\frac{\psi_x}{\sigma'(V)}\right\}_x dxd\tau\right| + \left|\int_0^t\left(\frac{F\psi_x}{\sigma'(V)}\right)\Big|_{x=0}d\tau\right| \\
&\le C\int_0^t\int_0^{+\infty}|\phi_x|^2\left(\left|\frac{\sigma''(V)V_x}{\sigma'(V)}\frac{\psi_x}{\sigma'(V)}\right| + \left|\frac{\psi_{xx}}{\sigma'(V)}\right|\right)dxd\tau + C\int_0^t\left|\frac{\phi_x^2\psi_x}{\sigma'(V)}\Big|_{x=0}\right|d\tau \\
&\le CN(T)\int_0^t(\|\phi_x(\tau)\|_w^2 + \|\psi_{xx}(\tau)\|_w^2)d\tau + CN(t)e^{-c_-\beta}. \qquad (4.21)
\end{aligned}$$

Plugging (4.20), (4.21) and the boundary estimates (4.3), (4.4) into (4.19), we have proved the following lemma.

Lemma 4.3. *It follows that*

$$\begin{aligned}
\|\phi_x(t)\|^2 + \|\psi_x(t)\|_x^2 + \int_0^t\|\psi_{xx}(\tau)\|_w^2d\tau \le C\Big\{&(1+\varepsilon^{-1})[\|\phi_0\|_1^2 + \|\psi_0\|_{1,w_0}^2 \\
&+N(t)e^{-c_-\beta}] + [\varepsilon + (1+\varepsilon^{-1})N(t)\int_0^t\|\phi_x(\tau)\|_w^2d\tau\Big\} \qquad (4.22)
\end{aligned}$$

for $N(t) \ll 1$, where C is a positive constant independent of ε.

To establish the *a priori* estimates (3.5) and (3.6), the following estimate for ϕ_x is key in this paper.

Lemma 4.4. *It holds that*

$$\begin{aligned}
\|\phi_x(t)\|_w^2 + \int_0^t\|\phi_x(\tau)\|_w^2d\tau \le C\Big\{&(1+\varepsilon^{-1})[\|\phi_0\|_1^2 + \|\phi_{0,x}\|_{w_0}^2 + \|\psi_0\|_{1,w_0}^2 \\
&+N(t)e^{-c_-\beta}] + [\varepsilon + (1+\varepsilon^{-1})N(t)\int_0^t\|\phi_x(\tau)\|_w^2d\tau\Big\} \qquad (4.23)
\end{aligned}$$

for $N(t) \ll 1$, where C is a positive constant independent of ε.

Proof. We note that $\phi_{xt} - \psi_{xx} = 0$ from the first eqution of (3.3). Multiplying this equation by $w(x,t)\phi_{xx}$ for $x \in [0, st-\alpha+\beta]$ and $x \in [st-\alpha+\beta, \infty)$, respectively, yields

$$\{w\phi_{xx}\phi_x\}_t - w\phi_{xxt}\phi_x - w_t\phi_x\phi_{xx} - w\phi_{xx}\psi_{xx} = 0.$$

Then use $\phi_{xxt} = \phi_{xxx}$ and $w_t = -sw_x$ for all $x \in R_+$, especially, $w_t = -sw_x = 0$ for $x \in [0, st-\alpha+\beta]$, to obtain

$$\frac{1}{2}\{(w\phi_x^2)_x - w_x\phi_x^2\}_t - \{w\phi_x\psi_{xx} - \frac{s}{2}w_x\phi_x^2\}_x + w_x\phi_x\psi_{xx} - \frac{s}{2}w_{xx}\phi_x^2 = 0, \quad (4.24)$$

which hold for both $x \in [0, st - \alpha + \beta]$ and $x \in [0, st - \alpha + \beta, \infty)$, respectively. We firstly integrate (4.24) with respect to x from 0 to $st - \alpha + \beta$ and use $w(x,t) = 1$, $w_x = 0$ for $x \in [0, st - \alpha + \beta]$ to obtain

$$\frac{1}{2}\frac{d}{dt}(\phi_x^2|_{x=st-\alpha+\beta}) - (\phi_x\phi_{xx})|_{x=st-\alpha+\beta} = \frac{1}{2}\frac{d}{dt}(\phi_x^2|_{x=0}) - (\phi_x\phi_{xx})|_{x=0}. \quad (4.25)$$

Secondly, integrating (4.25) with respect to x from $st-\alpha+\beta$ to ∞ and making use of $w(x,t) = e^{c_+(k-1)(x-st+\alpha-\beta)}$, $w_x = c_+(k-1)w$, also noting $w(x,t)|_{x=st-\alpha+\beta} = 1$, after here we remark $x(t) := st - \alpha + \beta$ for simplicity, give us

$$\begin{aligned} -\frac{1}{2}\frac{d}{dt}(\phi_x^2|_{x=x(t)}) - c_+(k-1)\frac{d}{dt}\int_{x(t)}^{\infty} w\phi_x^2 dx + [\phi_x\psi_{xx} - \frac{s}{2}c_+(k-1)\phi_x^2]|_{x=x(t)} \\ +c_+(k-1)\int_{x(t)}^{\infty} w\phi_x\psi_{xx}dx - \frac{s}{2}c_+^2(k-1)^2\int_{x(t)}^{\infty} w\phi_x^2 dx = 0. \quad (4.26)\end{aligned}$$

Substituting (4.25) into (4.26) and using the continuity of $w(x,t)$ yield

$$\begin{aligned} c_+(k-1)\frac{d}{dt}\int_{x(t)}^{\infty} w\phi_x^2 dx + \frac{s}{2}c_+(k-1)w\phi_x^2|_{x=x(t)} + \frac{s}{2}c_+^2(k-1)^2\int_{x(t)}^{\infty} w\phi_x^2 dx \\ = c_+(k-1)\int_{x(t)}^{\infty} w\phi_x\psi_{xx}dx - \frac{1}{2}\frac{d}{dt}(\phi_x^2|_{x=0}) + (\phi_x\psi_{xx})|_{x=0}. \quad (4.27)\end{aligned}$$

Applying the Cauchy inequality

$$\begin{aligned} \left| c_+(k-1)\int_{x(t)}^{\infty} w\phi_x\psi_{xx}dx \right| &\le \frac{s}{4}c_+^2(k-1)^2\int_{x(t)}^{\infty} w\phi_x^2 dx + s^{-1}\int_{x(t)}^{\infty} w\psi_{xx}^2 dx \\ &\le \frac{s}{4}c_+^2(k-1)^2\int_{x(t)}^{\infty} w\phi_x^2 dx + s^{-1}\|\psi_{xx}(t)\|_w^2 \end{aligned}$$

and by the first equation (3.3)

$$\frac{1}{2}\frac{d}{dt}(\phi_x^2|_{x=0}) = (\phi_x\phi_{xt})|_{x=0} = (\phi_x\psi_{xx})|_{x=0}$$

to (4.27), and integrating the resultant inequality with respect to τ from 0 to t, dropping the positive term $\frac{s}{2}c_+(k-1)\phi_x^2|_{x=x(t)}$, we obtain

$$\int_{x(t)}^{\infty} w\phi_x^2 dx + \int_0^t \int_{x(\tau)}^{\infty} w\phi_x^2 dx d\tau \le C\Big\{\|\phi_{0,x}\|_{w_0}^2 + \int_0^t \|\psi_{xx}(\tau)\|_w^2 d\tau\Big\}. \quad (4.28)$$

Thanks to (4.22) and (4,28), we have

$$\int_{x(t)}^{\infty} w\phi_x^2 dx + \int_0^t \int_{x(\tau)}^{\infty} w\phi_x^2 dx d\tau \leq C\Big\{(1+\varepsilon^{-1})[\|\phi_0\|_1^2 + \|\phi_{0,x}\|_{w_0}^2 + \|\psi\|_{1,w_0}^2 + N(t)e^{-c_-\beta}] + [\varepsilon + (1+\varepsilon^{-1})N(t)] \int_0^t \|\phi_x(\tau)\|_w^2 d\tau\Big\}. \tag{4.29}$$

On the other hand, from equations (3.3) we have

$$\mu\phi_{xt} + \sigma'(V)\phi_x - \psi_t = -F. \tag{4.30}$$

We multiply (4.30) by ϕ_x to obtain

$$\{\frac{\mu}{2}\phi_x^2\}_t + \sigma'(V)\phi_x^2 - \phi_x\psi_t = -F\phi_x. \tag{4.31}$$

From $\phi_{xt} = \psi_{xx}$, we have

$$-\phi_x\psi_t = -\{\phi_x\psi\}_t + \phi_{xt}\psi = -\{\phi_x\psi\}_t + \{\psi_x\psi\}_x - \psi_x^2. \tag{4.32}$$

Substituting (4.32) into (4.31) yields

$$\{\frac{\mu}{2}\phi_x^2 - \psi\phi_x\}_t + \sigma'(V)\phi_x^2 + \{\psi\psi_x\}_x = \psi_x^2 - F\phi_x. \tag{4.33}$$

Integrating (4.33) over $R_+ \times [0,t]$, using the Cauchy inequality:

$$\left|\int_0^{+\infty} \psi\phi_x dx\right| \leq \frac{\mu}{4}\|\phi_x(t)\|^2 + \mu^{-1}\|\psi(t)\|^2 \leq \frac{\mu}{4}\|\phi_x(t)\|^2 + \mu^{-1}\|\psi(t)\|_w^2,$$

also noting $\sigma'(V(x - st + \alpha - \beta)) \geq \sigma'(V(0)) \geq C > 0$, for $x \leq st - \alpha + \beta$ due to $\sigma''(V) > 0$ and $V_x < 0$, by the boundary estimate (4.2) in Lemma 4.1, we then get

$$\begin{aligned}
&\frac{\mu}{4}\|\phi_x(t)\|^2 + \sigma'(V(0)) \int_0^t \int_0^{x(\tau)} \phi_x^2 dx d\tau + \int_0^t \int_{x(\tau)}^{\infty} \sigma'(V)\phi_x^2 dx d\tau \\
&\leq C\Big\{\|\phi_{0,x}\|^2 + \|\psi_0\|^2 + N(t)e^{-c_-\beta} + \|\psi(t)\|_w^2 \\
&\quad + \int_0^t \|\psi_x(t)\|^2 d\tau + N(t)\int_0^t \|\phi_x(\tau)\|^2 d\tau\Big\} \\
&\leq C\Big\{\|\phi_{0,x}\|^2 + \|\psi_0\|^2 + N(t)e^{-c_-\beta} + \|\psi(t)\|_w^2 \\
&\quad + \int_0^t \|\psi_x(\tau)\|_w^2 d\tau + N(t)\int_0^t \|\phi_x(\tau)\|_w^2 d\tau\Big\}.
\end{aligned} \tag{4.34}$$

Using (4.11) into (4.34) and dropping the positive term $\int_0^t \int_{x(\tau)}^\infty \sigma'(V)\phi_x^2 dx d\tau$, we have

$$\begin{aligned}&\|\phi_x(t)\|^2 + \int_0^t \int_0^{x(\tau)} \phi_x^2 dx d\tau \\ &\le C\Big\{ \|\phi_0\|_1^2 + \|\psi_0\|_{w_0}^2 + N(t)e^{-c_-\beta} + N(t)\int_0^t \|\phi_x(\tau)\|_w^2 d\tau \Big\}.\end{aligned}$$

Therefore, adding the above inequality and (4.29) implies (4.23). □

Lemma 4.5. *It holds that*

$$\begin{aligned}\|\phi(t)\|_1^2 + \|\phi_x(t)\|_w^2 + \|\psi(t)\|_{1,w}^2 + \int_0^t \{\|\phi_x(\tau)\|_w^2 + \|\psi_x(\tau)\|_{1,w}^2 d\}\tau \\ \le C(\|\phi_0\|_1^2 + \|\phi_{0,x}\|_{w_0}^2 + \|\psi_0\|_{1,w_0}^2 + e^{-c_-\beta})\end{aligned} \tag{4.35}$$

for $N(T) \ll 1$.

Proof. Lemmas 4.2–4.4 imply that

$$\begin{aligned}&\|\phi(t)\|_1^2 + \|\phi_x(t)\|_w^2 + \|\psi(t)\|_{1,w}^2 + \int_0^t \{\|\phi_x(\tau)\|_w^2 + \|\psi(\tau)\|_{1,w}^2\} d\tau \\ &\le C_1 \Big\{ (1+\varepsilon^{-1})[\|\phi_0\|_1^2 + \|\phi_{0,x}\|_{w_0}^2 + \|\psi_0\|_{1,w_0}^2 + N(t)e^{-c_-\beta}] \\ &\quad + [\varepsilon + (1+\varepsilon^{-1})N(t)] \int_0^t \|\phi_x(\tau)\|_w^2 d\tau \Big\}\end{aligned}$$

for some positive constant C_1. Now, we choose ε $(0 < \varepsilon < 1)$ such that

$$C_1 \varepsilon \le \frac{1}{4}, \tag{4.36}$$

while choose $N(t)$ such that $C_1(1+\varepsilon^{-1})N(t) \le \frac{1}{4}$, then we obtain $C_1[\varepsilon + (1 + \varepsilon^{-1})N(t)] \le \frac{1}{2}$. Therefore, (4.35) is proved. □

The energy estimate for (ϕ_{xx}, ψ_{xx}) can be showed by repeating the same procedure in Lemmas 4.2–4.4. The estimates appearing in Lemma 4.1 and Lemma 4.5 are available for this proof. We list the result as follows but omit the details.

Lemma 4.6. *It holds that*

$$\|\phi_{xx}(t)\|_w^2 + \|\psi_{xx}(t)\|_w^2 + \int_0^t \{\|\phi_{xx}(\tau)\|_w^2 + \|\psi_{xxx}(\tau)\|_w^2\} d\tau \le C(N_0^2 + e^{-c_-\beta}) \tag{4.37}$$

for $t \in [0, T]$ *and* $N(t) \ll 1$.

Proof of Proposition 3.3. Combining Lemmas 4.5–4.6, we immediately prove (3.5) for $0 \le t \le T$. To prove (3.6) for $0 \le t \le T$, we differentiate the first equation of (3.3) with respect to x, and multiply it by ϕ_x, then integrate the resultant equality on x to have

$$\frac{d}{dt}\|\phi_x(t)\|^2 = 2\int_0^\infty \psi_{xx}\phi_x dx$$

for $0 \le t \le T$. Applying the estimate (3.5) to it, we have

$$\int_0^t \left|\frac{d}{dt}\|\phi_x(t)\|^2\right| d\tau \le \int_0^t (\|\psi_{xx}(\tau)\|^2 + \|\phi_x(t)\|^2) d\tau \le C(N_0^2 + e^{-c_- \beta}).$$

for $0 \le t \le T$. Similarly, the second equation of (3.3) and the *a priori* estimate (3.5) give

$$\int_0^t \left|\frac{d}{dt}\|\psi_x(t)\|^2\right| d\tau \le C(N_0^2 + e^{-c_- \beta})$$

for $0 \le t \le T$.

Thus, we complete the proof of Proposition 3.3. □

References

[1] I.-L. Chern and M. Mei, Asymptotic stability of critical viscous shock waves for a degenerate hyperbolic viscous conservation law, *Comm. Part. Diff. Eqs.* **23** (1998), 869–886.

[2] H. Frid and I.-S. Liu, Oscillation waves in Riemann problems inside elliptice regions conservation laws of mixed type, *J. Appl. Math. Phys. (ZAMP)* **46** (1995), 913–931.

[3] J. Goodman, Nonlinear asymptotic stability of viscous shock profiles for conservation laws, *Arch. Rational Mech. Anal.* **95** (1986), 325–344.

[4] D. Hoff and M. Khodja, Stability of coexisting phases for compressible van der Waals fluids, *SIAM J. Appl. Math.* **53** (1993), 1–14.

[5] S. Kawashima and A. Matsumura, Asymptotic stability of travelling wave solutions of system for one-dimensional gas motion, *Comm. Math. Phys.* **101** (1985), 97–127.

[6] S. Kawashima and A. Matsumura, Stability of shock profiles in viscoelsaticity with non-convex constitutive relations, *Comm. Pure Appl. Math.* **47** (1994), 1547–1569.

[7] T.-P. Liu, Nonlinear stability of shock waves for viscous conservation laws, *Mem. Amer. Math. Soc.* **56**(328) (1985).

[8] T.-P. Liu, Shock waves for compressible Navier–Stokes equations are stable, *Comm. Pure Appl. Math.* **39** (1986), 565–594.

[9] T.-P. Liu and K. Nishihara, Asymptotic behavior for scalar viscous conservation laws with boundary effect, *J. Diff. Eqs.* **133** (1997), 296–320.

[10] T.-P. Liu and S.-H. Yu, Propagation of stationary viscous Burgers shock under the effect of boundary, *Arch. Rational Mech. Anal.* **139** (1997), 57–82.

[11] A. Matsumura and M. Mei, Convergence to travelling fronts of solutions of the p-system with viscosity in the presence of a boundary, *Arch. Rational Mech. Anal.*, (1999), in press.

[12] A. Matsumura and M. Mei, Nonlinear stability of viscous shock profile for a non-convex system of viscoelasticity, *Osaka J. Math.* **34** (1997), 589–603.

[13] A. Matsumura and K. Nishihara, On the stability of travelling wave solutions of a one-dimensional model system for compressible viscous gas, *Japan J. Appl. Math.* **2** (1985), 17–25.

[14] M. Mei, Stability of traveling wave solution for non-convex equations of barotropic viscous gas, *Osaka J. Math.* **34** (1997), 308–318.

[15] M. Mei and K. Nishihara, Nonlinear stability of travelling waves for one dimensional viscous materials with non-convex nonlinearity, *Tokyo J. Math.* **20** (1997), 241–264.

[16] K. Nishihara, Stability of travelling waves with degenerate shock condition for one-dimensional viscoelastic model, *J. Diff. Eq.* **120** (1985), 304–318.

[17] A. Szepessy and Z.-P. Xin, Nonlinear stability of viscous shock waves, *Arch. Rational Mech. Anal.* **122** (1993), 53–104.

[18] S.-H. Yu, Burgers equations on the quarter plane, preprint.

Ming Mei
Department of Computational Science
Faculty of Science
Kanazawa University
Kakuma-machi, Kanazawa 920-1192
Japan
(e-mail: mei@kappa.s.kanazawa-u.ac.jp)

(Received: September 9, 1997; revised: April 6, 1998)