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Applied Mathematics Letters 20 (2007) 712–718

Applied Mathematics Letters

www.elsevier.com/locate/aml

Analysis on the critical speed of traveling waves

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Received 22 March 2006; received in revised form 15 June 2006; accepted 22 August 2006

Abstract

The note is concerned with a time-delayed reaction–diffusion equation with nonlocality for the population dynamics of single species. For the critical speed of traveling waves, we give a detailed analysis on its location and asymptotic behavior with respect to the parameters of the diffusion rate and mature age, respectively.

Keywords: Time-delayed reaction–diffusion equation; Nonlocality; Traveling wavefronts; Critical wave speed

1. Introduction

The growth dynamics of single-species population with age structure and diffusion, for example, Nicholson's blowflies population dynamics, has been a hot research topic and widely studied; see [1–10] and the references therein. In this note, we consider the initial value problem for a nonlocal time-delayed reaction–diffusion equation as follows:

$$\frac{\partial v}{\partial t} - D_m \frac{\partial^2 v}{\partial x^2} + d_m v = \varepsilon \int_{-\infty}^{\infty} b(v(t-r, x-y)) f_\alpha(y) \mathrm{d}y, \quad t \in [0, \infty), x \in R, \tag{1.1}$$

where $v(t, x)$ denotes the total population of mature species after the mature age $r > 0$ at time t and location x, $D_m > 0$ is the diffusion rate of the mature species, $d_m > 0$ is the death rate of the mature species, $\varepsilon > 0$ is an impact factor of the death rate of the immature species, and $\alpha > 0$ denotes the total amount of diffusion for the immature species. The parameters α, r and D_m satisfy

$$\alpha \le r D_m$$

as shown in [2,5,8], namely, the immature diffusion is always less than that of the adult species. $f_\alpha(y) = \frac{1}{\sqrt{4\pi\alpha}} \mathrm{e}^{-\frac{y^2}{4\alpha}}$ is the heat kernel satisfying the normalized condition $\int_{-\infty}^{\infty} f_\alpha(y) \mathrm{d}y = 1$. $b(v)$ is the birth rate, which is related only

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0893-9659/$ - see front matter
doi:10.1016/j.aml.2006.08.006

to the mature species. In particular, we choose the birth rate $b(v)$ as Nicholson's blowflies rate (cf. [1,4,8,9])

$$b(v) = pve^{-av}, \tag{1.2}$$

where $p > 0$ and $a > 0$ are constants. Eq. (1.1) can be derived from Metz and Diekmann's dynamical population model [6]

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial a} - D(a)\frac{\partial^2 u}{\partial x^2} + d(a)u = 0$$

by setting

$$v(t,x) = \int_r^\infty u(t,a,x)\mathrm{d}a,$$

where a denotes the age of the species and $u(t,a,x)$ represents the density of the species with age a at location x at time t. For the detailed derivation of Eq. (1.1), we refer the reader to [2,5,8].

Note that Eq. (1.1) has two constant equilibria obtained by solving the equation

$$d_m v = \varepsilon p \int_{-\infty}^{\infty} v e^{-av} f_\alpha(y)\mathrm{d}y,$$

which are

$$v_- = 0 \quad \text{and} \quad v_+ = \frac{1}{a}\ln\frac{\varepsilon p}{d_m}. \tag{1.3}$$

If $\frac{\varepsilon p}{d_m} > 1$, then $v_- < v_+$.

In [8], So, Wu and Zou proved that for Eq. (1.1) there exists a wavefront $\phi(x+ct)$ with the speed $c > c_*$, where $c_* > 0$ is the critical speed. Then Liang and Wu [2] further extended the result on the existence of traveling waves to the generalized case of birth rate

$$b(v) = pve^{-av^q}, \quad q \geq 1.$$

Later, Mei and So [5] showed that, if the wave speed c is suitably large that

$$c > 2\sqrt{D_m(3\varepsilon p - 2d_m)}, \tag{1.4}$$

then the wavefront is time-asymptotically stable in a weighted Sobolev space. However, since the critical speed c_* was not specified in [8], we do not know how far the wave speed c in (1.4) is from c_*, and what the stability is for the wave with any speed c close to c_*. In this note, we are going to answer the first question, which is necessary for the second question, and the second question will be discussed later in [3].

By a detailed computation, as shown below, we give the exact bounds of c_*, and show its asymptotic behavior as the diffusion rate of mature species $D_m \to 0^+$, and $D_m \to +\infty$, and the mature age $r \to 0^+$ and $r \to +\infty$, respectively.

2. Critical speed of traveling waves

A traveling wave of Eq. (1.1) with the birth rate (1.2) connected with two constant states $v_\pm$ is the special solution of Eq. (1.1) in the form of $\phi(x+ct)$ ($c > 0$ is the wave speed) which satisfies a nonlocal delayed ordinary differential equation as follows:

$$\begin{cases} c\phi'(\xi) - D_m\phi''(\xi) + d_m\phi(\xi) = \varepsilon p \displaystyle\int_{-\infty}^{\infty} \phi(\xi - cr - y)e^{-a\phi(\xi - cr - y)} f_\alpha(y)\mathrm{d}y, \\ \phi(\pm\infty) = v_\pm, \end{cases} \tag{2.1}$$

where $\xi = x + ct$ and $' = \frac{\mathrm{d}}{\mathrm{d}\xi}$. Using the upper–lower solution method, So et al. [8] proved the existence of monotone wavefronts $\phi(\xi)$ with $\phi'(\xi) > 0$; see also [7].

Lemma 2.1 (*So–Wu–Zou [8]*). *If $1 < \frac{\varepsilon p}{d_m} \le e$, then there exists a critical number $c_* \ge 0$ such that for every $c > c_*$, Eq. (2.1) has a traveling wavefront solution $\phi(\xi)$ connecting $v_\pm$, with $\phi'(\xi) > 0$ and $v_- < \phi(\xi) < v_+$ for all $\xi \in (-\infty, \infty)$. Here, the critical speed c_* is the unique solution of*

$$\Delta_{c_*}(\lambda_*) = 0, \qquad \frac{\partial}{\partial \lambda}\Delta_{c_*}(\lambda)\Big|_{\lambda=\lambda_*} = 0, \tag{2.2}$$

where $\Delta_c(\lambda)$ is defined as

$$\Delta_c(\lambda) = \varepsilon p e^{\alpha\lambda^2 - \lambda c r} - [c\lambda + d_m - D_m\lambda^2]. \tag{2.3}$$

Now we are going to specify the critical speed c_*. Our main theorem is

Theorem 2.2. *Let $1 < \frac{\varepsilon p}{d_m} \le e$. Then the critical wave speed c_* satisfies:*

1. *Upper and lower bounds of c_*:*
   If $\alpha = rD_m$, then

$$0 \le c_* \le \min\left\{2\sqrt{D_m(\varepsilon p - d_m)},\ 2\sqrt{\frac{D_m}{r}\ln\frac{\varepsilon p}{d_m}}\right\}. \tag{2.4}$$

   If $\alpha < rD_m$, then

$$0 \le c_* \le \min\left\{2\sqrt{D_m(\varepsilon p - d_m)},\ \sqrt{\frac{D_m^2}{rD_m - \alpha}\ln\frac{\varepsilon p}{d_m}}\right\}. \tag{2.5}$$

2. *Asymptotic behavior of c_* with respect to the diffusion coefficient D_m:*
   Let D_m be free, and the other parameters ε, p, d_m, α and r be fixed; then

$$c_* \to 0 \quad \text{as } D_m \to 0^+, \tag{2.6}$$

$$c_* = O(\sqrt{D_m}) \to +\infty \quad \text{as } D_m \to +\infty. \tag{2.7}$$

3. *Asymptotic behavior of c_* with respect to the mature age r:*
   Let r be free, and the other parameters ε, p, d_m, D_m and α be fixed; then

$$c_* \to 2\sqrt{D_m(\varepsilon p - d_m)} \quad \text{as } r \to 0^+, \tag{2.8}$$

$$c_* = O(r^{-\frac{1}{2}}) \to 0 \quad \text{as } r \to +\infty. \tag{2.9}$$

To confirm our theoretical results on the asymptotic behavior of the critical speed c_*, we show two numerical results in Fig. 2.1. In the first graph of Fig. 2.1, for fixed parameters d_m, ε, p, α and r, we show a curve of c_* with respect to D_m, which indicates the asymptotic behaviors c_* in (2.6) and (2.7). In the second graph of Fig. 2.1, for fixed parameters d_m, ε, p, α and D_m, we show a curve of c_* with respect to r, which indicates the asymptotic behaviors c_* in (2.8) and (2.9).

3. Proof of the main theorem

As we mentioned before, the diffusion rate of the immature species is always less than that of the adult species, i.e., $\alpha \le rD_m$; now we are going to prove Theorem 2.2 for the following two cases: $\alpha = rD_m$ and $\alpha < rD_m$, respectively.
Case 1: $\alpha = rD_m$.
Let

$$F_c(\lambda) := \varepsilon p e^{\alpha\lambda^2 - cr\lambda}, \qquad G_c(\lambda) := c\lambda + d_m - D_m\lambda^2.$$

So, $\Delta_c(\lambda) = F_c(\lambda) - G_c(\lambda)$. For given c, the corresponding critical points of $F_c(\lambda)$ and $G_c(\lambda)$ are $\lambda_1 = \frac{cr}{2\alpha}$ and $\lambda_2 = \frac{c}{2D_m}$, respectively, i.e., $F_c'(\lambda_1) = 0$ and $G_c'(\lambda_2) = 0$, so then $F_c(\lambda)$ reaches the minimum $\underline{F_c} := \varepsilon p e^{-\frac{c^2 r^2}{4\alpha}}$, and

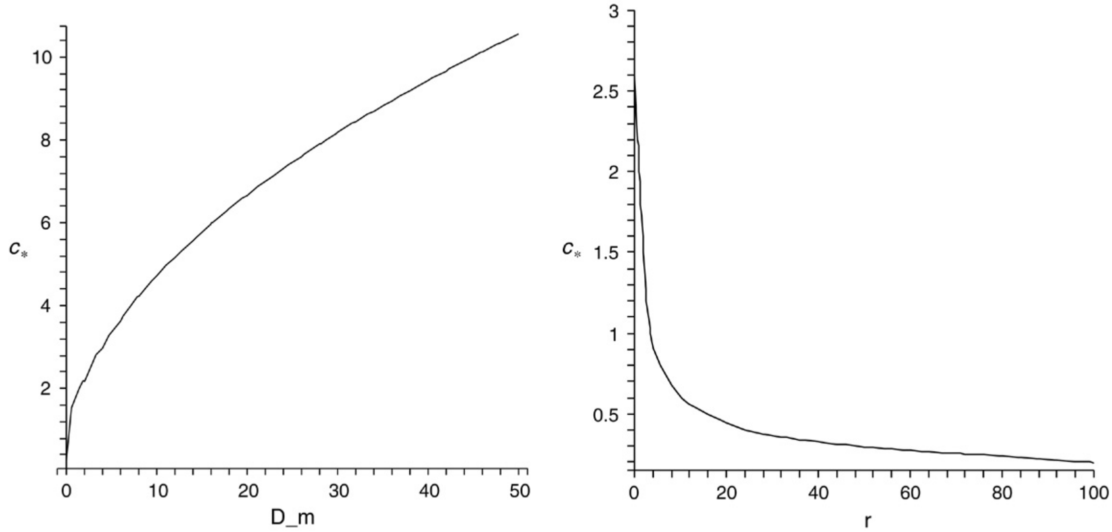

Fig. 2.1. Asymptotic behavior of the critical speed c_* with respect to D_m and r, respectively.

$G_c(\lambda)$ arrives at the maximum $\overline{G_c} := \frac{c^2}{4D_m} + d_m$. Since $\alpha = D_m r$, it can be verified that $\lambda_1 = \lambda_2$; namely, at the same point both $F_c(\lambda)$ and $G_c(\lambda)$ have extreme values. Now we denote this critical point as

$$\lambda_* := \frac{cr}{2\alpha} = \frac{c}{2D_m}. \tag{3.1}$$

Note that $F_c'(\lambda_*) = G_c'(\lambda_*) = 0$; it automatically holds that

$$\frac{\partial}{\partial\lambda}\Delta_c(\lambda)\bigg|_{\lambda=\lambda_*} = F_c'(\lambda_*) - G_c'(\lambda_*) = 0. \tag{3.2}$$

Therefore, once for some c it is satisfied that the minimum of $F_c(\lambda)$ is exactly the same as the maximum of $G_c(\lambda)$, i.e., $F_c(\lambda_*) = G_c(\lambda_*)$, then such a speed c is just the critical speed c_* satisfying (2.2). Therefore, from $\underline{F_c} = \overline{G_c}$ we get

$$\varepsilon p \mathrm{e}^{-\frac{c^2 r^2}{4\alpha}} = \frac{c^2}{4D_m} + d_m. \tag{3.3}$$

Again, by use of $\alpha = D_m r$, the above equation is reduced to

$$c_*^2 = 4D_m\left(\varepsilon p \mathrm{e}^{-\frac{c_*^2 r}{4D_m}} - d_m\right). \tag{3.4}$$

From (3.4), we immediately have the upper bound for c_*

$$c_*^2 \leq 4D_m(\varepsilon p - d_m), \quad i.e., c_* \leq 2\sqrt{D_m(\varepsilon p - d_m)}. \tag{3.5}$$

Furthermore, (3.4) and $c_*^2 \geq 0$ give also

$$\varepsilon p \mathrm{e}^{-\frac{c_*^2 r}{4D_m}} - d_m \geq 0,$$

which implies

$$c_* \leq 2\sqrt{\frac{D_m}{r}\ln\frac{\varepsilon p}{d_m}}. \tag{3.6}$$

Thus, (3.5) and (3.6) leads to (2.4), i.e.,

$$0 \le c_* \le \min\left\{\sqrt{D_m(\varepsilon p - d_m)},\ 2\sqrt{\frac{D_m}{r}\ln\frac{\varepsilon p}{d_m}}\right\}.$$

Obviously,

$$c_*^2 = 4D_m\left(\varepsilon p \mathrm{e}^{-\frac{c_*^2 r}{4D_m}} - d_m\right) \le 4D_m\,(\varepsilon p - d_m) \to 0, \quad \text{as } D_m \to 0^+,$$

which implies (2.6): $c_* \to 0$, as $D_m \to 0^+$.

For the proof of (2.7), we set

$$z = \frac{c_*^2 r}{4D_m}, \tag{3.7}$$

and Eq. (3.4) is equivalent to

$$z = r\varepsilon p \mathrm{e}^{-z} - r d_m. \tag{3.8}$$

It is easily seen that Eq. (3.8) has a unique solution $z_* > 0$, where z_*, satisfying $0 < z_* < \ln\frac{\varepsilon p}{d_m}$, is an absolute constant and is independent of D_m. In fact, the decreasing curve $w_1(z) := r\varepsilon p \mathrm{e}^{-z} - r d_m$ intersects the straight line $w_2(z) := z$ at a unique point z_* between 0 and $\ln\frac{\varepsilon p}{d_m}$. Thus, from (3.7), we have $\frac{c_*^2 r}{4D_m} = z_*$, which implies

$$c_* = \sqrt{\frac{4D_m z_*}{r}} \to +\infty, \quad \text{as } D_m \to +\infty. \tag{3.9}$$

This proves (2.7).

When $r \to 0^+$, since c_* is bounded (see (3.4))

$$0 < c_* \le 2\sqrt{D_m(\varepsilon p - d_m)},$$

and then $\lim_{r\to 0^+} c_*$ is also bounded, which implies that $\lim_{r\to 0^+} \mathrm{e}^{-\frac{c_*^2 r}{4D_m}} = \mathrm{e}^0 = 1$. Now, letting $r \to 0^+$, from Eq. (3.4), we have

$$\lim_{r\to 0^+} c_*^2 = 4D_m\left(\varepsilon p \lim_{r\to 0^+} \mathrm{e}^{-\frac{c_*^2 r}{4D_m}} - d_m\right) = 4D_m(\varepsilon p - d_m),$$

namely,

$$\lim_{r\to 0^+} c_* = 2\sqrt{D_m(\varepsilon p - d_m)}.$$

This proves (2.8).

When $r \to +\infty$, from (3.4), i.e.,

$$\mathrm{e}^{\frac{c_*^2 r}{4D_m}} = \frac{4D_m\varepsilon p}{c_*^2 + 4D_m d_m}, \tag{3.10}$$

we must have

$$\lim_{r\to+\infty} c_* = 0. \tag{3.11}$$

In fact, if $\lim_{r\to+\infty} c_* > 0$, then

$$\lim_{r\to\infty} \mathrm{e}^{\frac{c_*^2 r}{4D_m}} = \infty, \quad \text{and} \quad \lim_{r\to\infty} \frac{4D_m\varepsilon p}{c_*^2 + 4D_m d_m} < \infty,$$

which leads to a contradiction when we take the limits in (3.10). Therefore, $\lim_{r\to+\infty} c_* = 0$.

Now we are going to estimate its decay rate. Since (3.10) can be reduced to

$$c_* = 2\sqrt{\frac{D_m}{r}\ln\frac{4D_m\varepsilon p}{c_*^2+4D_m d_m}},$$

we obtain

$$c_* \sim 2\sqrt{\frac{D_m}{r}\ln\frac{\varepsilon p}{d_m}} \to 0, \quad \text{as } r\to\infty.$$

This proves (2.9).
Case 2: $\alpha < rD_m$.

Let $(\underline{c}_*, \underline{\lambda}_*)$ and $(\overline{c}_*, \overline{\lambda}_*)$ be the pairs of the critical speed and the critical λ for the cases $\alpha = rD_m$ and $\alpha < rD_m$, respectively. For $\alpha < rD_m$, it can be easily verified that $\lambda_1 = \frac{\overline{c}_* r}{2\alpha} > \frac{\overline{c}_*}{2D_m} = \lambda_2$, where λ_1 and λ_2 are the critical points of $F_c(\lambda)$ and $G_c(\lambda)$, and that $\lambda_1 > \overline{\lambda}_* > \lambda_2$, as well as that $F_{\overline{c}_*}(\lambda_1) < F_{\overline{c}_*}(\overline{\lambda}_*) = G_{\overline{c}_*}(\overline{\lambda}_*) < G_{\overline{c}_*}(\lambda_2) < \varepsilon p$, i.e.,

$$\varepsilon p e^{-\frac{\overline{c}_*^2 r^2}{4\alpha}} < d_m + \frac{\overline{c}_*^2}{4D_m} < \varepsilon p. \tag{3.12}$$

From the second inequality of (3.12), we immediately prove the boundedness of $\overline{c}_*$:

$$0 < \overline{c}_* < 2\sqrt{D_m(\varepsilon p - d_m)}. \tag{3.13}$$

On the other hand, for given λ, the graph of $F_c(\lambda)$ is always above the graph of $G_c(\lambda)$, except for the unique tangent point at $\lambda = \overline{\lambda}_*$. So when $\lambda = \frac{\overline{c}_*}{D_m}$, we have $F_{\overline{c}_*}(\frac{\overline{c}_*}{D_m}) > G_{\overline{c}_*}(\frac{\overline{c}_*}{D_m})$. Notice that $F_{\overline{c}_*}(\frac{\overline{c}_*}{D_m}) = \varepsilon p e^{\frac{\overline{c}_*^2}{D_m^2}(\alpha - D_m r)}$ and $G_{\overline{c}_*}(\frac{\overline{c}_*}{D_m}) = d_m$; by a straightforward calculation, we then obtain

$$0 < \overline{c}_* < \sqrt{\frac{D_m^2}{D_m r - \alpha}\ln\frac{\varepsilon p}{d_m}}. \tag{3.14}$$

Thus, (3.13) and (3.14) imply (2.5), i.e.,

$$0 \le c_* \le \min\left\{2\sqrt{D_m(\varepsilon p - d_m)},\ \sqrt{\frac{D_m^2}{rD_m - \alpha}\ln\frac{\varepsilon p}{d_m}}\right\}.$$

Comparing the first inequality of (3.12) for $\overline{c}_*$ (i.e., $e^{-\frac{\overline{c}_*^2 r^2}{4\alpha}} < d_m + \frac{\overline{c}_*^2}{4D_m}$) with the equality for $\underline{c}_*$ (see (3.3), i.e., $e^{-\frac{\underline{c}_*^2 r^2}{4\alpha}} = d_m + \frac{\underline{c}_*^2}{4D_m}$), it is verified that

$$0 < \underline{c}_* < \overline{c}_* < 2\sqrt{D_m(\varepsilon p - d_m)}. \tag{3.15}$$

Letting $D_m \to 0^+$ in (3.13), we obtain $\overline{c}_* \to 0$, which proves (2.6). On the other hand, letting $D_m \to \infty$ in (3.15), and noting $\underline{c}_* = O(\sqrt{D_m}) \to \infty$ (see (3.9)), we then have

$$\overline{c}_* = O(\sqrt{D_m}) \to \infty, \quad \text{as } D_m \to +\infty.$$

This proves (2.7).

Similarly, taking $r \to 0^+$ in (3.15) and noting $\lim_{r\to 0^+}\underline{c}_* = 2\sqrt{D_m(\varepsilon p - d_m)}$, we obtain

$$\lim_{r\to 0^+}\overline{c}_* = 2\sqrt{D_m(\varepsilon p - d_m)},$$

which proves (2.8).

Finally, we are going to prove (2.9). Taking $r \to +\infty$ in (3.14), we then obtain

$$\overline{c}_* = O(r^{-\frac{1}{2}}) \to 0, \quad \text{as } r \to +\infty.$$

The proof is complete.

Acknowledgements

The authors would like to express their thanks to the referees for valuable suggestions.

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