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NONLINEAR STABILITY OF TRAVELING WAVEFRONTS IN AN AGE-STRUCTURED REACTION-DIFFUSION POPULATION MODEL

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Abstract. The paper is devoted to the study of a time-delayed reaction-diffusion equation of age-structured single species population. Linear stability for this model was first presented by Gourley [4], when the time delay is small. Here, we extend the previous result to the nonlinear stability by using the technical weighted-energy method, when the initial perturbation around the wavefront decays to zero exponentially as $x \to -\infty$, but the initial perturbation can be arbitrarily large on other locations. The exponential convergent rate (in time) of the solution is obtained. Numerical simulations are carried out to confirm the theoretical results, and the traveling wavefronts with a large delay term in the model are reported.

1. Introduction.

The population of a single species with age-structure is usually described as a time delayed reaction-diffusion equation

$$\frac{\partial v}{\partial t} = d_m \frac{\partial^2 v}{\partial x^2} + \varepsilon b(v(x, t-\tau)) - d(v), \qquad t \in [0, \infty),\ x \in R, \tag{1}$$

where $v(x,t)$ denotes the total population of mature species after the mature age $\tau > 0$ at time t and location x. Here, $d_m > 0$ is the diffusion rate of the mature species, $d(v) > 0$ is the death function, $b(v)$ is the birth function, and $\varepsilon > 0$ is an impact factor of the death rate. Equation (1) can be derived from the Metz and Diekmann's dynamical population model [13]

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial a} - d_i(a)\frac{\partial^2 u}{\partial x^2} + \gamma(a)u = 0$$

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by setting

$$v(t,x)=\int_{\tau}^{\infty}u(t,a,x)\mathrm{d}a,$$

where a denotes the age of the species and $u(t,a,x)$ represents the density of the species with age a at a location x and in time t, $d_i(a)$ and $\gamma(a)$ are the diffusion and death rates of the immatures. For the detailed derivation of equation (1), we refer the reader to [4, 5, 12, 20].

As shown in [4] (see also [1, 2, 3]), for simplicity, by choosing the birth and death functions of the matures as $b(v(x,t-\tau))=\alpha v(x,t-\tau)$ and $d(v)=\beta v^2$, and let the death rates of the immatures $\gamma(a)=\gamma$ be a constant which determines the impact factor of the death rate according to $\varepsilon=e^{-\gamma\tau}$, and denoting the diffusion rate as $d_m=d$, Eq.(1) can be rewritten as

$$\frac{\partial v}{\partial t}=d\frac{\partial^2 v}{\partial x^2}+\alpha e^{-\gamma\tau}v(x,t-\tau)-\beta v^2, \qquad t\in[0,\infty),\ x\in R. \tag{2}$$

It is noticed that two constant equilibria exist in equation (2); namely,

$$v_-=0 \quad \text{ and } \quad v_+=\frac{\alpha}{\beta}e^{-\gamma\tau}.$$

Let c denote the speed of the wave solution, which depends on the maturation delay τ. In [3], Al-Omari and Gourley proved that for all speeds c exceeding a certain minimum value, equation (2) possesses a monotone traveling wavefront solution which connects the constant equilibria v_- and v_+. Later, by applying the weighted energy method as shown in [11], Gourley showed in [4] that the traveling wavefronts with speed c greater than the critical speed c_0 are linearly stable, if the initial perturbations around the wavefronts are exponentially decay in space as $x\to-\infty$.

In the present paper, we study equation (2) with the following initial value condition:

$$v(x,s)=v_0(x,s)\to v_\pm, \quad s\in[-\tau,0] \quad \text{as } x\to\pm\infty. \tag{3}$$

The goal of the present work is to extend Gourley's linear stability of the wavefronts to the nonlinear stability. Although we require the initial perturbation around the wavefront to decay as x near $-\infty$, the initial perturbations could be arbitrarily large in other locations. This is the so-called large initial perturbation problem for the stability, and the result is of particularly interest in mathematical and physical sciences. The method adopted here is based on the comparison principle together with the weighted L^2-energy method, which was first applied by Lin and Mei [6] to prove the stability of traveling waves with large initial perturbations for the Nicholson's blowflies equation. For the wavefronts and their stabilities related to the other models, we refer to, for example, [8]-[12], [14], [16]-[25], and the references therein.

Throughout the paper, $C>0$ denotes a generic constant, while $C_i>0$ ($i=0,1,2,\cdots$) represents a specific constant. Let I be an interval, typically $I=\mathbf{R}$. $L^2(I)$ is the space of the square integrable functions on I, and $H^k(I)$ ($k\geq 0$) is the Sobolev space of the L^2-functions $f(x)$ defined on the interval I whose derivatives $\frac{d^i}{dx^i}f$, $i=1,\cdots,k$, also belong to $L^2(I)$. $L^2_w(I)$ represents the weighted L^2-space with the weight $w(x)>0$ and its norm is defined by

$$\|f\|_{L^2_w}=\Big(\int_I w(x)f(x)^2dx\Big)^{1/2}.$$

$H^k_w(I)$ is the weighted Sobolev space with the norm

$$\|f\|_{H^k_w} = \Big(\sum_{i=0}^{k} \int_I w(x) \Big| \frac{d^i}{dx^i} f(x) \Big|^2 dx \Big)^{1/2}.$$

Let $T > 0$ and let $\mathbf{B}$ be a Banach space, we denote by $C^0([0,T];\mathbf{B})$ the space of the $\mathbf{B}$-valued continuous functions on $[0,T]$, and $L^2([0,T];\mathbf{B})$ as the space of the $\mathbf{B}$-valued L^2-functions on $[0,T]$. The corresponding spaces of the $\mathbf{B}$-valued functions on $[0,\infty)$ are defined similarly.

The paper is organized as follows. In Section 2, we introduce the traveling wavefronts and a weight function, the main result of this work, namely, the nonlinear stability of the traveling wavefronts is then presented. In Section 3, after having established the comparison principle and some key energy estimates in the weighted Sobolev's spaces, we prove the nonlinear stability. In Section 4, we report numerical simulations for several case studies, and the computed solutions confirm our theoretical results stated in Section 2.

2. **Nonlinear stability.** A traveling wavefront of Equation (2) is a monotone solution $\phi(x+ct)$ connecting with two constant states $v_- = 0$ and $v_+ = (\alpha/\beta)e^{-\gamma\tau}$, and it satisfies

$$\begin{cases} d\phi''(\xi) - c\phi'(\xi) + \alpha e^{-\gamma\tau}\phi(\xi - c\tau) - \beta\phi^2(\xi) = 0, \\ \phi(-\infty) = 0 = v_-, \qquad \phi(\infty) = (\alpha/\beta)e^{-\gamma\tau} = v_+, \end{cases} \tag{4}$$

where $\xi = x + ct$, and $' = \frac{d}{d\xi}$.

In [3], Al-Omari and Gourley proved the existence of the traveling wavefronts of equation (2) connecting $v_\pm$.

Proposition 1 (Existence of Traveling Wavefronts, Al-Omari and Gourley [3]**).** *There exists a minimum speed $c_0 = c_0(\tau)$ satisfying*

$$F_{c_0}(\lambda_{c_0}) = G_{c_0}(\lambda_{c_0}), \qquad F'_{c_0}(\lambda_{c_0}) = G'_{c_0}(\lambda_{c_0}), \tag{5}$$

where

$$F_c(\lambda) = 2\alpha e^{-\gamma\tau} e^{-\lambda c\tau/2}, \qquad G_c(\lambda) = c\lambda - \frac{1}{2} d\lambda^2, \tag{6}$$

(c_0, λ_{c_0}) is the tangent point of $F_c(\lambda)$ and $G_c(\lambda)$, and c_0 is the solution of the following equation

$$\alpha \ exp \Big(1 - \gamma\tau - \frac{c_0^2\tau}{2d} - \frac{1}{2d}\sqrt{4d^2 + c_0^4\tau^2}\Big) = \frac{1}{c_0^2\tau^2}\Big(-2d + \sqrt{4d^2 + c_0^4\tau^2}\Big), \tag{7}$$

which implies $c_0^2 < 4\alpha d e^{-\gamma\tau}$. Then for all $c > c_0$, the traveling wavefront $\phi(x+ct)$ of equation (2) connecting $v_\pm$ exists.

As shown in [4], it is easily seen from Figure 1 that the critical wave speed c_0 is determined when the two curves $F_c(\lambda)$ and $G_c(\lambda)$ have a unique touched point. When $c > c_0$, the two curves intersect at two points, says λ_{1c} and λ_{2c}. Let λ_c be a point between λ_{1c} and λ_{2c}; then

$$G_c(\lambda_c) > F_c(\lambda_c). \tag{8}$$

Since $\lambda = 2c/d$ is a nonzero root of the equation $G_c(\lambda) = 0$, we have $\lambda_c < 2c/d$. From the first graph of Figure 1, it is verified that $G_{c_0}(\lambda) < F_{c_0}(\lambda)$ for all $\lambda > 0$

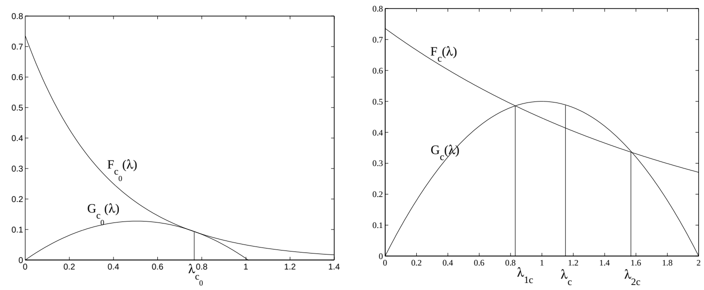

FIGURE 1. The graphs of $F_c(\lambda)$ and $G_c(\lambda)$ for $c = c_0$ and $c > c_0$, respectively

except at the touched point $\lambda = \lambda_{c_0}$, and the maximum of $G_{c_0}(\lambda)$ is given by $G_{c_0}(\frac{c_0}{d}) = \frac{c_0^2}{2d}$, in which

$$\frac{c_0^2}{2d} = G_{c_0}(\frac{c_0}{d}) < F_{c_0}(\frac{c_0}{d}) < F_{c_0}(0) = 2\alpha e^{-\gamma\tau}.$$

This implies $c_0 < \sqrt{4\alpha d e^{-\gamma\tau}}$ as mentioned in Proposition 1.

It has been reported in [11, 12] that when the wave is faster, namely, the wave speed is larger, then the wave is usually stable time-asymptotically. However, the more interesting and difficult case is to study the nonlinear stability of the slower waves, in particular, the wave with speed very close to the critical speed c_0. Hence, in the present paper, we are interested in the waves with speed c satisfying

$$c_0 < c < \sqrt{4\alpha d e^{-\gamma\tau}}. \tag{9}$$

As technically assumed in [4], we restrict the delay τ to be small, such that

$$4\alpha\tau e^{-\gamma\tau} < \cosh^{-1}(2); \tag{10}$$

then (9) is reduced to

$$c_0 < c < \sqrt{\frac{d\ \cosh^{-1}(2)}{\tau}}. \tag{11}$$

Thus, from $\lambda_c < 2c/d$, we have

$$\cosh\frac{\lambda_c c\tau}{2} < \cosh\frac{c^2\tau}{d} < 2. \tag{12}$$

Obviously, it holds

$$4\beta v_+ = 4\alpha e^{-\alpha\tau} > 2\alpha e^{-\alpha\tau}\cosh\frac{\lambda_c c\tau}{2}.$$

Since $\lim_{\xi\to\infty}\phi(\xi) = v_+$ and $\phi(\xi)$ is increasing, there exists a number ξ_0 such that

$$4\beta\phi(\xi_0) > 4\beta\phi(\xi_0 - c\tau) > 2\alpha e^{-\gamma\tau}\cosh\frac{\lambda_c c\tau}{2}. \tag{13}$$

Let $\tilde{v} = v - \phi$ be the linear perturbation around the wavefront ϕ; then $\tilde{v}$ satisfies

$$\begin{cases} \frac{\partial\tilde{v}}{\partial t} = d\frac{\partial^2\tilde{v}}{\partial x^2} + \alpha e^{-\gamma\tau}\tilde{v}(x, t-\tau) - 2\beta\phi\tilde{v}, & (x,t) \in R \times R_+, \\ \tilde{v}(x,s) = v_0(x,s) - \phi(x+cs), & (x,s) \in R \times [-\tau, 0]. \end{cases} \tag{14}$$

For the linear stability of the traveling wavefronts, defining the weight function as

$$w(\xi) = \begin{cases} e^{-\lambda_c \xi}, & \xi \le \xi_0, \\ e^{-\lambda_c \xi_0}, & \xi > \xi_0, \end{cases} \tag{15}$$

Gourley [4] showed the following result.

Proposition 2 (Linear Stability, Gourley [4]). *Let the delay τ satisfy* (10)*, and $\phi(x+cs)$ be a wavefront with speed c satisfying* (11). *If the initial perturbation satisfies $v_0(x,s)-\phi(x+cs) \in H^1_w(R)$ for $s \in [-\tau, 0]$, where $w = w(x+cs)$ $(for \quad s \in [-r, 0])$ is the weight function given in* (15)*, then the wavefront $\phi(x+ct)$ is linearly asymptotically stable in the sense that*

$$\sup_{x\in R} |\tilde{v}(x,t)| \le Ce^{-\mu t}, \quad t > 0,$$

where $\mu = \mu(\alpha, \beta, d, \tau, c, \lambda_c) > 0$ is a specific constant.

In this paper, by using the comparison principle together with the weighted energy method, we prove the following nonlinear stability of traveling wavefronts even when the initial perturbations are not small.

Theorem 2.1 (Nonlinear Stability). *Let the delay τ satisfy* (10)*, for a given traveling wavefront $\phi(x+ct)$ with speed c satisfying* (11)*, and the initial datum satisfying*

$$v_0(x,s) - \phi(x+cs) \in C^0([-r,0]; H^1_w(R)), \tag{16}$$

where $w = w(x+cs)$ $(for \quad s \in [-r, 0])$ is the weight function given in (15)*, then the unique solution $v(x,t)$ of the Cauchy problem* (2) *and* (3) *exists globally*

$$v(t,x) - \phi(x+ct) \in C^0([0,\infty); H^1_w(R)) \cap L^2([0,\infty); H^2_w(R))$$

and it converges to the traveling wavefront $\phi(x+ct)$ time-asymptotically

$$\sup_{x\in R} |v(x,t) - \phi(x+ct)| \le Ce^{-\mu t}, \quad 0 \le t \le \infty. \tag{17}$$

Remark 1. For the wave stability, we usually require the initial perturbation around the wave to be sufficiently small. However, such a restriction is not necessary in the present stability; namely, the initial perturbation $\|v_0(\cdot,s)-\phi(\cdot+cs)\|_{H^1_w}$ $(s \in [-\tau, 0])$ can be large. This is the so-called large initial perturbation problem, and is quite significant and interesting in the studies of many problems in mathematical physics.

3. **Proof of nonlinear stability.** By using the energy method reported in [11], we can prove that equations (2) and (3) admit a unique global solution. The key step in the proof of the nonlinear stability is to establish the comparison principle and some energy estimates in the weighted space $L^2_w(R)$. Now, we first demonstrate the positivity of the solution $v(x,t)$ of the Cauchy problem (2) and (3), and then establish a comparison principle for the solution $v(x,t)$. Consequently, the convergence of the solution to the wavefront is obtained.

Lemma 3.1 (Positivity). *Let $v(x,t)$ be a bounded solution of the Cauchy problem* (2) *and* (3) *with a nonnegative initial data $v_0(x,s) \ge 0$ for $(x,s) \in R \times [-\tau, 0]$; then $v(x,t) \ge 0$ for $(x,t) \in R \times [0,\infty)$.*

Proof. For $t \in [0, \tau]$, i.e., $t - \tau \in [-\tau, 0]$, we have $v(x, t-\tau) = v_0(x, t-\tau) \geq 0$ for all $x \in R$. Hence, equation (2) becomes

$$\begin{cases} v_t - dv_{xx} + \beta v^2 = \alpha e^{-\gamma\tau} v(x, t-\tau) \geq 0, & (x,t) \in R \times [0, \tau] \\ v(x, 0) = v_0(x, 0) \geq 0, & x \in R, \end{cases} \tag{18}$$

which implies

$$v(x,t) \geq 0 \quad \text{for } (x,t) \in R \times [0, \tau]. \tag{19}$$

In fact, if we set $v = we^{\nu t}$ by choosing a large ν such that $a(x,t) := \nu + \beta v(x,t) \geq 0$ because of the boundedness of $v(x,t)$, equation (18) is reduced to

$$\begin{cases} w_t - dw_{xx} + a(x,t)w \geq 0, & (x,t) \in R \times [0, \tau] \\ w(x, 0) = v_0(x, 0) \geq 0, & x \in R. \end{cases}$$

Applying the maximum principle (c.f. [15]), the above equation ensures that $w(x,t) \geq 0$ for $(x,t) \in R \times [0, \tau]$. Therefore, $v(x,t) \geq 0$ for $(x,t) \in R \times [0, \tau]$.

For $t \in [\tau, 2\tau]$, i.e., $t - \tau \in [0, \tau]$, from (19), we have $v(x, t-\tau) \geq 0$ for $(x,t) \in R \times [\tau, 2\tau]$. Thus, equation (2) implies that

$$\begin{cases} v_t - dv_{xx} + \beta v^2 = \alpha e^{-\gamma\tau} v(x, t-\tau) \geq 0, & (x,t) \in R \times [\tau, 2\tau] \\ v(x, \tau) \geq 0, & x \in R. \end{cases} \tag{20}$$

Consequently,

$$v(x,t) \geq 0 \quad \text{for } (x,t) \in R \times [\tau, 2\tau]. \tag{21}$$

Combining (19) and (21), we prove

$$v(x,t) \geq 0 \quad \text{for } (x,t) \in R \times [0, 2\tau]. \tag{22}$$

By repeating this procedure, we have

$$v(x,t) \geq 0 \quad \text{for } (x,t) \in R \times [0, \infty).$$

The proof is complete. □

Lemma 3.2 (Comparison Principle)**.** *Let $\overline{v}(x,t)$ and $\underline{v}(x,t)$ be positive and bounded for $(x,t) \in R \times R_+$, and they satisfy*

$$\begin{cases} \overline{v}_t - d\overline{v}_{xx} + \beta \overline{v}^2 - \alpha e^{-\gamma\tau} \overline{v}(x, t-\tau) \\ \qquad \geq \underline{v}_t - d\underline{v}_{xx} + \beta \underline{v}^2 - \alpha e^{-\gamma\tau} \underline{v}(x, t-\tau), & (x,t) \in R \times R_+ \\ \overline{v}(x, s) \geq \underline{v}(x, s), & (x, s) \in R \times [-\tau, 0], \end{cases} \tag{23}$$

then

$$\overline{v}(x,t) \geq \underline{v}(x,t), \quad \textit{for } \ (x,t) \in R \times R_+. \tag{24}$$

Proof. Let

$$U(x,t) = \overline{v}(x,t) - \underline{v}(x,t), \quad \text{and} \quad U_0(x,s) = \overline{v}_0(x,s) - \underline{v}_0(x,s).$$

From (23), it can be verified that

$$\begin{cases} U_t - dU_{xx} + \beta a_1(x,t) U \geq \alpha e^{-\gamma\tau} U(x, t-\tau), & (x,t) \in R \times R_+ \\ U|_{t=s} = U_0(x,s), \quad (x,s) \in R \times [-\tau, 0] \end{cases} \tag{25}$$

where $a_1(x,t) := \overline{v}(x,t) + \underline{v}(x,t) > 0$ is bounded.

As shown in Lemma 3.1, for $t \in [0,\tau]$, (i.e., $t-\tau \in [-\tau,0]$), $U(x,t-\tau) = U_0(x,t-\tau) \geq 0$, and equation (24) is reduced to

$$\begin{cases} U_t - dU_{xx} + \beta a_1(x,t)U \geq 0, & (x,t) \in R \times [0,\tau] \\ U|_{t=s} = U_0(x,s) \geq 0, & (x,s) \in R \times [-\tau,0], \end{cases} \tag{26}$$

which leads to

$$U(x,t) \geq 0, \quad i.e., \quad \overline{v}(x,t) \geq \underline{v}(x,t), \qquad \text{for} \ \ (x,t) \in R \times [0,\tau].$$

Similarly, we can prove

$$U(x,t) \geq 0, \quad i.e., \ \overline{v}(x,t) \geq \underline{v}(x,t), \qquad \text{for} \ \ (x,t) \in R \times [\tau,2\tau].$$

Hence, by repeating the procedure, we obtain

$$U(x,t) \geq 0, \quad i.e., \quad \overline{v}(x,t) \geq \underline{v}(x,t), \qquad \text{for} \ \ (x,t) \in R \times R_+.$$

The proof is complete. □

We now prove the stability of the traveling wavefronts by the weighted energy method. Let

$$\begin{cases} v_0^+(x,\tau) = \max\{v_0(x,\tau), \phi(x+c\tau)\}, \\ v_0^-(x,\tau) = \min\{v_0(x,\tau), \phi(x+c\tau)\}, \end{cases} \quad \text{for } (x,\tau) \in R \times [-\tau,0], \tag{27}$$

so

$$v_0^-(x,\tau) \leq v_0(x,\tau) \leq v_0^+(x,\tau) \quad \text{for } (x,\tau) \in R \times [-\tau,0] \tag{28}$$

$$v_0^-(x,\tau) \leq \phi(x+c\tau) \leq v_0^+(x,\tau) \quad \text{for } (x,\tau) \in R \times [-\tau,0]. \tag{29}$$

Denote $v^+(x,t)$ and $v^-(x,t)$ as the corresponding solutions of equations (2) and (3) with respect to the above mentioned initial data $v_0^+(x,\tau)$ and $v_0^-(x,\tau)$; i.e.,

$$\begin{cases} v_t^\pm - dv_{xx}^\pm + \beta(v^\pm)^2 = \alpha e^{-\gamma\tau} v^\pm(x,t-\tau), & (x,t) \in R \times R_+ \\ v^\pm(x,s) = v_0^\pm(x,s), & x \in R, \ s \in [-\tau,0]. \end{cases} \tag{30}$$

By the Comparison Principle, we have

$$v^-(x,t) \leq v(x,t) \leq v^+(x,t) \quad \text{for } (x,t) \in R \times R_+, \tag{31}$$

$$v^-(x,t) \leq \phi(x+ct) \leq v^+(x,t) \quad \text{for } (x,t) \in R \times R_+. \tag{32}$$

The convergence of the wave solutions with the initial data $v_0^+(x,\tau)$, $v_0^-(x,\tau)$, and $v_0(x,\tau)$ are discussed as follows.

Case 1: The convergence of $v^+(x,t)$ to $\phi(x+ct)$

Lemma 3.3. *It holds*

$$\sup_{x\in R} |v^+(x,t) - \phi(x+ct)| \leq Ce^{-\mu t}, \qquad t \geq 0. \tag{33}$$

Proof. Let $\xi := x+ct$ and

$$z(\xi,t) := v^+(x,t) - \phi(x+ct), \quad z_0(\xi,\tau) = v_0^+(x,\tau) - \phi(x+c\tau); \tag{34}$$

then by (29) and (32), we have

$$z(\xi,t) \geq 0, \qquad z_0(\xi,\tau) \geq 0. \tag{35}$$

Since $v^+(x,t)$ and $\phi(x+ct)$ satisfy equation (2), it can be verified that $z(\xi,t)$ satisfies

$$\begin{cases} cz_\xi + z_t - dz_{\xi\xi} - \alpha e^{-\gamma\tau} z(\xi - c\tau, t-\tau) + 2\beta\phi(\xi)z + \beta z^2 & = 0,\ (\xi,t)\in R\times R_+, \\ z(\xi,\tau) = z_0(\xi,\tau), & (\xi,\tau)\in R\times[-\tau,0]. \end{cases} \tag{36}$$

Multiplying (36) by $e^{2\mu t}w(\xi)z(\xi,t)$, we obtain

$$\begin{aligned} &\Big(\frac{1}{2}e^{2\mu t}wz^2\Big)_t + e^{2\mu t}\Big(\frac{1}{2}cwz^2 - dwzz_\xi\Big)_\xi + de^{2\mu t}wz_\xi^2 + de^{2\mu t}w'zz_\xi \\ &-\mu e^{2\mu t}wz^2 - \frac{1}{2}ce^{2\mu t}\frac{w'}{w}wz^2 - \alpha e^{-\gamma\tau}e^{2\mu t}w(\xi)z(\xi,t)z(z-c\tau,t-\tau) \\ &+2\beta e^{2\mu t}w\phi z^2 + \beta e^{2\mu t}wz^3 = 0. \end{aligned} \tag{37}$$

Note that, for any $\eta_1 > 0$,

$$|de^{2\mu t}w'zz_\xi| \le d\eta_1 e^{2\mu t}wz_\xi^2 + \frac{d}{4\eta_1}e^{2\mu t}\Big(\frac{w'}{w}\Big)^2 wz^2,$$

and dropping the positive term $\beta e^{2\mu t}wz^3$ (i.e., the last term in (37)), because $z(\xi,t)\ge 0$, we have

$$\begin{aligned} &\Big(\frac{1}{2}e^{2\mu t}wz^2\Big)_t + e^{2\mu t}\Big(\frac{1}{2}cwz^2 - dwzz_\xi\Big)_\xi + d(1-\eta_1)e^{2\mu t}wz_\xi^2 \\ &-\mu e^{2\mu t}wz^2 - \frac{1}{2}ce^{2\mu t}\frac{w'}{w}wz^2 - \frac{d}{4\eta_1}e^{2\mu t}wz^2 \\ &-\alpha e^{-\gamma\tau}e^{2\mu t}w(\xi)z(\xi,t)z(z-c\tau,t-\tau) + 2\beta e^{2\mu t}w\phi z^2 \le 0. \end{aligned} \tag{38}$$

Integrating (38) with respect to (ξ,t) over $R\times[0,t]$, we obtain

$$\begin{aligned} &e^{2\mu t}\|z(t)\|^2_{L^2_w} + 2d(1-\eta_1)\int_0^t e^{2\mu s}\|z_\xi(s)\|^2_{L^2_w}ds \\ &+\int_0^t\int_R e^{2\mu s}\Big[-c\frac{w'(\xi)}{w(\xi)} - \frac{d}{2\eta_1}\Big(\frac{w'(\xi)}{w(\xi)}\Big)^2 + 4\beta\phi(\xi) - 2\mu\Big]w(\xi)z^2(\xi,s)\ d\xi ds \\ &\le \|z(0)\|^2_{L^2_w} + 2\alpha e^{-\gamma\tau}\int_0^t\int_R e^{2\mu s}w(\xi)z(\xi,s)z(\xi-c\tau,s-\tau)\ d\xi ds. \end{aligned} \tag{39}$$

For the last term in (39), by using the Cauchy-Schwartz inequality $2xy \le \eta_2 x^2 + (1/\eta_2)y^2$ with $\eta_2 > 0$, we can estimate that

$$\begin{aligned} &2\alpha e^{-\gamma\tau}\int_0^t\int_R e^{2\mu s}w(\xi)z(\xi,s)z(\xi-c\tau,s-\tau)\ d\xi ds \\ &\le \alpha e^{-\gamma\tau}\int_0^t\int_R e^{2\mu s}w(\xi)\Big(\eta_2 z^2(\xi,s) + \frac{1}{\eta_2}z^2(\xi-c\tau,s-\tau)\Big)\ d\xi ds \\ &= \eta_2\alpha e^{-\gamma\tau}\int_0^t\int_R e^{2\mu s}w(\xi)z^2(\xi,s)\ d\xi ds \\ &\quad + \frac{1}{\eta_2}\alpha e^{-\gamma\tau}\int_0^t\int_R e^{2\mu s}w(\xi)z^2(\xi-c\tau,s-\tau)\ d\xi ds. \end{aligned} \tag{40}$$

By applying the change of variables, $\xi - c\tau \to \xi$, $s - \tau \to s$, to the last term of (40), we have

$$\begin{aligned}
&\frac{1}{\eta_2}\alpha e^{-\gamma\tau}\int_0^t\int_R e^{2\mu s}w(\xi)z^2(\xi - c\tau, s-\tau)\ d\xi ds\\
&= \frac{1}{\eta_2}\alpha e^{-\gamma\tau}\int_{-\tau}^{t-\tau}\int_R e^{2\mu(s+\tau)}w(\xi+c\tau)z^2(\xi,s)\ d\xi ds\\
&\le \frac{1}{\eta_2}\alpha e^{-\gamma\tau+2\mu\tau}\int_0^t\int_R e^{2\mu s}w(\xi+c\tau)z^2(\xi,s)\ d\xi ds\\
&\quad +\frac{1}{\eta_2}\alpha e^{-\gamma\tau+2\mu\tau}\int_{-\tau}^0\int_R e^{2\mu s}w(\xi+c\tau)z_0^2(\xi,s)\ d\xi ds. \qquad (41)
\end{aligned}$$

Substituting (41) into (40), it yields

$$\begin{aligned}
&2\alpha e^{-\gamma\tau}\int_0^t\int_R e^{2\mu s}w(\xi)z(\xi,s)z(\xi - c\tau, s-\tau)\ d\xi ds\\
&\le \eta_2\alpha e^{-\gamma\tau}\int_0^t\int_R e^{2\mu s}w(\xi)z^2(\xi,s)\ d\xi ds\\
&\quad +\frac{1}{\eta_2}\alpha e^{-\gamma\tau+2\mu\tau}\int_0^t\int_R e^{2\mu s}w(\xi+c\tau)z^2(\xi,s)\ d\xi ds\\
&\quad +\frac{1}{\eta_2}\alpha e^{-\gamma\tau+2\mu\tau}\int_{-\tau}^0\int_R e^{2\mu s}w(\xi+c\tau)z_0^2(\xi,s)\ d\xi ds. \qquad (42)
\end{aligned}$$

Applying the above inequality (42) into (39), we have

$$\begin{aligned}
&e^{2\mu t}\|z(t)\|^2_{L^2_w} + 2d(1-\eta_1)\int_0^t e^{2\mu s}\|z_\xi(s)\|^2_{L^2_w}ds\\
&+\int_0^t\int_R e^{2\mu s}B(\mu,\xi)w(\xi)z^2(\xi,s)\ d\xi ds\\
&\le \|z(0)\|^2_{L^2_w} + \frac{1}{\eta_2}\alpha e^{-\gamma\tau+2\mu\tau}\int_{-\tau}^0\int_R e^{2\mu s}w(\xi+c\tau)z_0^2(\xi,s)\ d\xi ds, \qquad (43)
\end{aligned}$$

where

$$\begin{aligned}
B(\mu,\xi) &= -c\frac{w'(\xi)}{w(\xi)} - \frac{d}{2\eta_1}\Big(\frac{w'(\xi)}{w(\xi)}\Big)^2 + 4\beta\phi(\xi) - 2\mu - \eta_2\alpha e^{-\gamma\tau}\\
&\quad -\frac{1}{\eta_2}\alpha e^{-\gamma\tau+2\mu\tau}\ \frac{w(\xi+c\tau)}{w(\xi)}\\
&= -c\frac{w'(\xi)}{w(\xi)} - \frac{d}{2\eta_1}\Big(\frac{w'(\xi)}{w(\xi)}\Big)^2 + 4\beta\phi(\xi) - \eta_2\alpha e^{-\gamma\tau} - \frac{1}{\eta_2}\alpha e^{-\gamma\tau}\ \frac{w(\xi+c\tau)}{w(\xi)}\\
&\quad -2\mu - \frac{1}{\eta_2}\alpha e^{-\gamma\tau}\frac{w(\xi+c\tau)}{w(\xi)}\Big(e^{2\mu\tau}-1\Big)\\
&= B(\xi) - 2\mu - \frac{1}{\eta_2}\alpha e^{-\gamma\tau}\frac{w(\xi+c\tau)}{w(\xi)}\Big(e^{2\mu\tau}-1\Big). \qquad (44)
\end{aligned}$$

Here

$$B(\xi) := -c\frac{w'(\xi)}{w(\xi)} - \frac{d}{2\eta_1}\Big(\frac{w'(\xi)}{w(\xi)}\Big)^2 + 4\beta\phi(\xi) - \eta_2\alpha e^{-\gamma\tau} - \frac{1}{\eta_2}\alpha e^{-\gamma\tau}\ \frac{w(\xi+c\tau)}{w(\xi)} \qquad (45)$$

is the same as introduced by Gourley [4]. By selecting the weight function $w(\xi)$ as stated in (15) and taking $\eta_1 = 1$, $\eta_2 = e^{-\lambda_c c\tau/2}$, Gourley proved that (see [4]: (2.7),(2.8) on p.261, and (2.15), (2.16) on p.263) there exists a positive constant

$$\mu_c := \min\{\mu_c^{(1)}, \mu_c^{(2)}, \mu_c^{(3)}\},$$

where

$$\begin{aligned}
\mu_c^{(1)} : &= c\lambda_c - \frac{d}{2}\lambda_c^2 + 4\beta\phi(\xi_0 - c\tau) - 2\alpha e^{-\gamma\tau}\cosh\frac{\lambda_c c\tau}{2} > 0,\\
\mu_c^{(2)} : &= 4\beta\phi(\xi_0) - 2\alpha e^{-\gamma\tau}\cosh\frac{\lambda_c c\tau}{2} > 0,\\
\mu_c^{(3)} : &= c\lambda_c - \frac{d}{2}\lambda_c^2 - 2\alpha e^{-\gamma\tau}e^{-\lambda_c c\tau/2} > 0,
\end{aligned}$$

such that

$$B(\xi) \ge \mu_c. \tag{46}$$

Note that

$$\frac{w(\xi + c\tau)}{w(\xi)} < 1, \quad \text{for } \xi \in (-\infty, \infty) \tag{47}$$

then (44)-(47) imply

$$B(\mu, \xi) \ge \mu_c - 2\mu - \frac{1}{\eta_2}\alpha e^{-\gamma\tau}(e^{2\mu\tau} - 1).$$

Let $\mu > 0$ be sufficiently small, such that

$$B(\mu, \xi) \ge \mu_c - 2\mu - \frac{1}{\eta_2}\alpha e^{-\gamma\tau}(e^{2\mu\tau} - 1) =: C_0 > 0; \tag{48}$$

then for $\eta_1 = 1$, (43) becomes

$$e^{2\mu t}\|z(t)\|_{L_w^2}^2 + C_0\int_0^t e^{2\mu s}\|z(s)\|_{L_w^2}^2\, ds \le C_1\Big(\|z(0)\|_{L_w^2}^2 + \int_{-\tau}^0 \|z_0(s)\|_{L_w^2}^2\, ds\Big) \tag{49}$$

for some constant $C_1 > 0$.

Similarly, differentiating (36) with respect to ξ and multiplying by $e^{2\mu t}w(\xi)z_\xi(\xi, t)$, then by integrating the resultant equation with respect to (ξ, t) over $R \times [0, t]$, and applying (49), we obtain

$$e^{2\mu t}\|z_\xi(t)\|_{L_w^2}^2 + C_0\int_0^t e^{2\mu s}\|z_\xi(s)\|_{L_w^2}^2\, ds \le C_2\Big(\|z(0)\|_{H_w^1}^2 + \int_{-\tau}^0 \|z_0(s)\|_{H_w^1}^2\, ds\Big) \tag{50}$$

for some constant $C_2 > 0$.

Combining (49) and (50), we prove that

$$e^{2\mu t}\|z(t)\|_{H_w^1}^2 \le C_3\Big(\|z(0)\|_{H_w^1}^2 + \int_{-\tau}^0 \|z_0(s)\|_{H_w^1}^2\, ds\Big) \tag{51}$$

for some constant $C_3 > 0$.

Since $H_w^1(R) \hookrightarrow H^1(R) \hookrightarrow C^0(R)$ for the weight $w(\xi)$ given in (15) (for the proof of the above Sobolev space embedding, we refer to Mei-Nishihara [9]), we obtain

$$\sup_{\xi\in R}|z(\xi, t)| \le C\|z(t)\|_{H^1} \le C\|z(t)\|_{H_w^1} \le Ce^{-\mu t}$$

for all $t \ge 0$. This implies (33), and proof is complete. □

Case 2: The convergence of $v^-(x, t)$ to $\phi(x + ct)$

Lemma 3.4. *It holds*

$$\sup_{x \in R} |v^-(x,t) - \phi(x+ct)| \le Ce^{-\mu t}, \qquad t \ge 0. \tag{52}$$

Proof. Let $z(\xi,t) = \phi(x+ct) - v^-(x,t)$, $\xi = x+ct$, and $z_0(\xi,s) = \phi(x+cs) - v_0^-(x,s)$; then $z(\xi,t)$ satisfies the Cauchy problem (36). As shown in Lemma 3.3, we can similarly prove Lemma 3.4. The detail is omitted. □

Case 3: The convergence of $v(x,t)$ to $\phi(x+ct)$

Now we will prove Theorem 2.1, namely, the following convergence result.

Lemma 3.5. *It holds*

$$\sup_{x \in R} |v(x,t) - \phi(x+ct)| \le Ce^{-\mu t}, \qquad t \ge 0. \tag{53}$$

Proof. Since $v_0^-(x,s) \le v_0(x,s) \le v_0^+(x,s)$, from Lemma 3.2, the corresponding solutions satisfy

$$v^-(x,t) \le v(x,t) \le v^+(x,t), \quad (x,t) \in R \times R_+.$$

Thanks to Lemmas 3.3 and 3.4, we have the following convergence results:

$$\sup_{x \in R} |v^-(x,t) - \phi(x+ct)| \le Ce^{-\mu t}, \qquad \sup_{x \in R} |v^+(x,t) - \phi(x+ct)| \le Ce^{-\mu t}.$$

Then, by using the Squeeze Theorem, we finally prove

$$\sup_{x \in R} |v(x,t) - \phi(x+ct)| \le Ce^{-\mu t}.$$

The proof is complete. □

4. **Numerical simulations.** To investigate the stability of the traveling waves, we perform numerical simulations to confirm the theoretical results presented in Section 2.

The mathematical formulation given in equation (2) and (3) is a nonlinear time-delayed partial differential equation, and the nonlinearity is due to the term βv^2. The computational results reported in this section are based on the following finite-difference approximation with a forward scheme for the time derivative and a central scheme for the spatial derivative:

$$\frac{v_j^n - v_j^{n-1}}{\Delta t} = d\frac{v_{j-1}^{n-1} - 2v_j^{n-1} + v_{j+1}^{n-1}}{(\Delta x)^2} + \alpha e^{-\gamma\tau} v_j^{n-\tau} - \beta v_j^{n-1} v_j^{n-1}, \tag{54}$$

where Δt and Δx denote the step size in time and space, respectively. It is noted that the nonlinear term is calculated at the $(n-1)$th time step; hence, it is an explicit scheme. The advantage of this simple scheme is that the resulting finite-difference equations are linear, and the solutions can easily be computed. The numerical scheme given in (54) is of first-order accurate in time and second-order accurate in space. We have also implemented an implicit numerical scheme in which the nonlinear term is computed at the nth time step, but the accuracy is the same as the above explicit scheme. For implicit formulation, an iterative method is required to solve the resulting nonlinear difference equations in each time step. However, we observe that the computed solutions are almost the same as those obtained from the explicit scheme.

In computation, the sizes of the time step and space step are chosen as $\Delta t = 0.01$ and $\Delta x = 0.2$, so that the condition to ensure a stable numerical computation

TABLE 1. Case studies: Parameters and Initial data

Case	α	γ	τ	$v_+ = \frac{\alpha}{\beta}e^{-\gamma\tau}$	$4\alpha\tau e^{-\gamma\tau}$	Initial data $v_0(x,s)$ for $s \in (-\tau, 0)$
1	0.5	1	1	$\frac{1}{2e}$	0.7358	$\begin{cases} 0, & x \le -10, \\ \frac{1}{2e}, & x \ge -10. \end{cases}$
2	1	2	1	$\frac{1}{e^2}$	0.5413	$\begin{cases} 0, & x \le -20, \\ 40, & -20 \le x \le 20, \\ \frac{1}{e^2}, & x \ge 20. \end{cases}$
3	0.5	1	1	$\frac{1}{2e}$	0.7358	$\begin{cases} 0, & x \le -10, \\ -\frac{200e-1}{400e}x^2 + \frac{1}{40e}x + 50, & -10 \le x \le 10, \\ \frac{1}{2e}, & x \ge 10. \end{cases}$
4	0.5	0.1	10	$\frac{1}{2e}$	7.3576	$\begin{cases} 0, & x \le -10, \\ \frac{1}{2e}, & x \ge -10. \end{cases}$
5	0.5	0.1	10	$\frac{1}{2e}$	7.3576	$\begin{cases} 0, & x \le -10, \\ -\frac{200e-1}{400e}x^2 + \frac{1}{40e}x + 50, & -10 \le x \le 10, \\ \frac{1}{2e}, & x \ge 10. \end{cases}$
6	1	0.1	20	$\frac{1}{e^2}$	1.4653	$\begin{cases} 0, & x \le -20, \\ 40, & -20 \le x \le 20, \\ \frac{1}{e^2}, & x \ge 20. \end{cases}$

$\Delta t/(\Delta x)^2 < 1/2$ is satisfied. Although the original model assumes the spatial domain in $(-\infty, \infty)$, a finite computational domain $(-L, L)$ is imposed. Here, we let $L = 400$ so that the computational domain is sufficiently large and no artificial numerical reflection is introduced in the computed solution.

In this section, we report the numerical simulations for six test cases. We first choose $d = 1$ and $\beta = 1$; other parameters and the initial data for each case study are listed in Table 1. The corresponding numerical solutions are displayed in Figures 4.2 to 4.7.

In the first group of the case studies, we focus on Cases 1, 2, and 3, in which a small delay $\tau = 1$ is considered. It is noted that the condition $4\alpha\tau e^{-\gamma\tau} < \cosh^{-1}(2)$ is satisfied for each case, where $\cosh^{-1}(2) = 1.3710$. For Case 1, we observe that starting with a discontinuous initial data, a rapid smoothing effect results in a short time. When t=10, Figure 4.2 illustrates that a stable traveling wave is established and it is moving in the negative x-direction as time increases. In Case 2, the initial datum is discontinuous with a big jump from 0 to 40, which causes the initial perturbation around the wavefront larger than $40 - \frac{1}{e^2}$, because the monotone traveling wave is bounded between 0 and $\frac{1}{e^2}$. However, as shown in Figure 3, after time $t = 50$ the solution becomes a stable wavefront propagating from right to left. For Case 3, we test the continuous initial datum $v_0(x, s)$, and it has a maximum $v_0(\frac{5}{200e-1}, s) = 50 + \frac{5}{16(200e-1)}$ at $x = \frac{5}{200e-1}$. Consequently, the initial perturbation around the wavefront is larger than $50 + \frac{5}{16(200e-1)} - \frac{1}{2e}$. From Figure 4, we observe that, the solution behaves as a stable wavefront moving from right to left after time $t = 50$. As shown in Figures 3 and 4, Cases 2 and 3 demonstrate numerically that the wavefronts are asymptotically stable even if the initial perturbations are really large. This confirms our theoretical stability result in Section 2. With the parameters chosen for Cases 1 through 3, we expect that the speeds of the traveling waves for Cases 1 and 3 are identical; and the wavefront will propagate faster for Case 2 than for Cases 1 and 3. The numerical simulations presented in Figures 4.2 through 4.4 indeed confirm our theoretical prediction. We observe that $v(x, 50)$ and $v(x, 100)$ are essentially the same for Case 1 and 3.

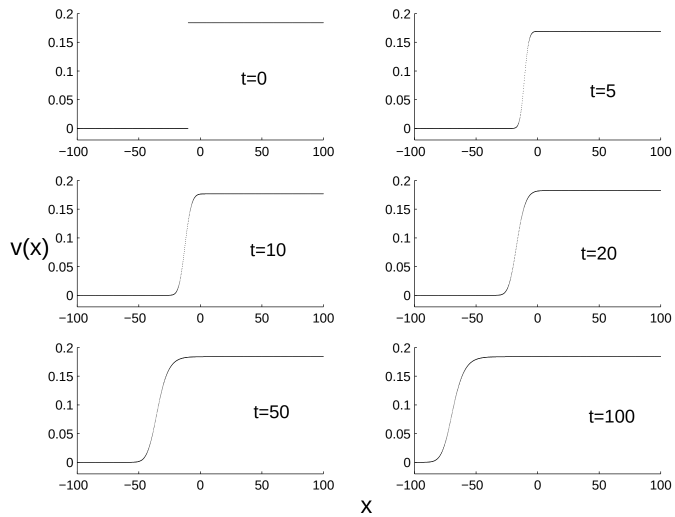

FIGURE 2. The graphs of $v(x,t)$ of Case 1 at $t = 0$, 5, 10, 20, 50, 100, respectively

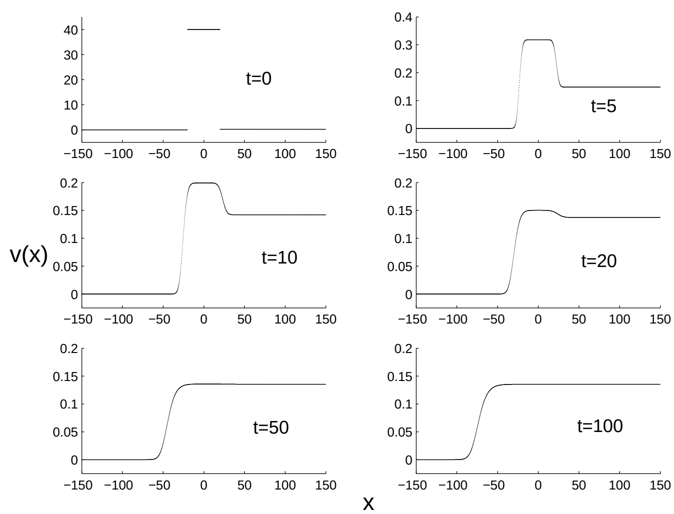

FIGURE 3. The graphs of $v(x,t)$ of Case 2 at $t = 0$, 5, 10, 20, 50, 100, respectively

Now, we examine the test cases with large delay terms. Notice that except the parameters γ is reduced by 10 and τ is increased ten-fold, other parameters and initial data for Cases 4 and 5 are the same as those imposed for Cases 1 and 3. It is of interest to note that even when the value of $4\alpha\tau e^{-\gamma\tau}$ is increased tenfold and the condition (10) is not satisfied, stable monotone traveling wave solutions are obtained for Cases 4 and 5. The solutions shown in Figures 4.5 and 4.6 have essentially the same profiles as those displayed in Figures 4.2 and 4.4 for Cases 1 and 3, but they travel with a slower speed. Finally, Case 6 is constructed, so that the solutions can be compared with those corresponding to Case 2. Here, we

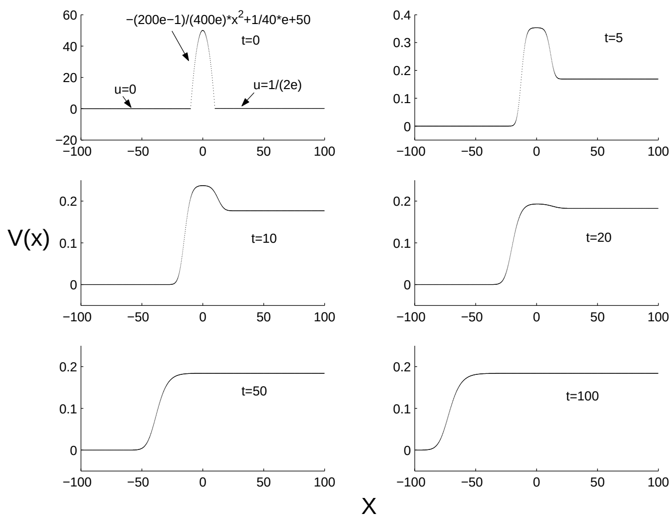

FIGURE 4. The graphs of $v(x,t)$ of Case 3 at $t = 0$, 5, 10, 20, 50, 100, respectively

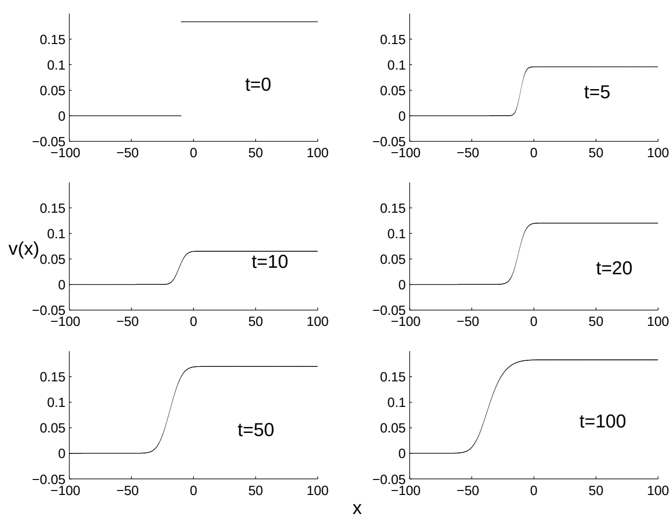

FIGURE 5. The graphs of $v(x,t)$ of Case 4 at $t = 0$, 5, 10, 20, 50, 100, respectively

reduce the value of γ in Case 2 by 20, but we enlarge the delay τ twentyfold. For this case, the condition (10) is not satisfied. From the computed solutions given in Figure 4.7, we observe that a stable monotone traveling wave solution connecting two equilibria is obtained when the time is sufficiently large.

The reported numerical simulations confirm the theoretical results presented in this work. In particular, we observe that stable traveling wavefronts are obtained even when the initial perturbation is not small. Moreover, the stable solution is not affected by a large delay term in which the condition (2.7) is not valid.

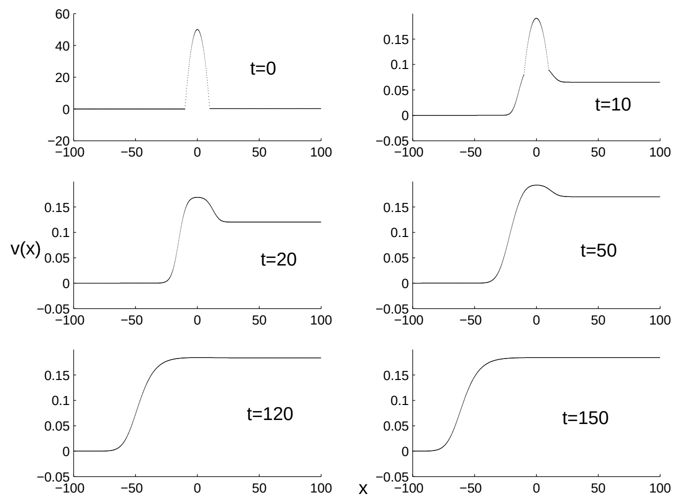

FIGURE 6. The graphs of $v(x,t)$ of Case 5 at $t = 0$, 10, 20, 50, 120, 150, respectively

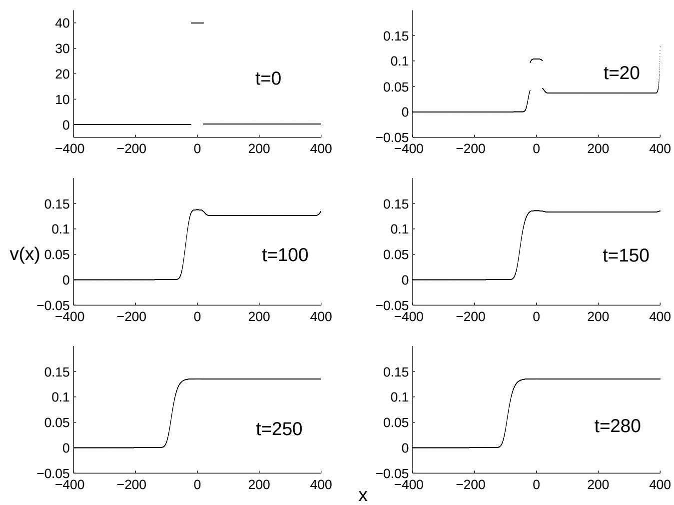

FIGURE 7. The Graphs of $v(x,t)$ of Case 6 at $t = 0$, 20, 100, 150, 250, 280, respectively

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