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# Optimal Convergence Rates to Diffusion Waves for Solutions of the Hyperbolic Conservation Laws with Damping

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**Abstract.** This paper is devoted to study the asymptotic behaviors of the solutions to a model of hyperbolic balance laws with damping on the quarter plane $(x,t) \in \mathbb{R}_+ \times \mathbb{R}_+$. We show the optimal convergence rates of the solutions to their corresponding nonlinear diffusion waves, which are the solutions of the corresponding nonlinear parabolic equation given by the related Darcy's law. The optimal $L^p$-rates $(1+t)^{-(1-\frac{1}{2p})}$ for $2 \le p \le \infty$ obtained in the present paper improve those $(1+t)^{-(\frac{3}{4}-\frac{1}{2p})}$ in the previous works on the IBVP by K. Nishihara and T. Yang [*J. Differential Equations* **156** (1999), 439–458] and by P. Marcati and M. Mei [*Quart. Appl. Math.* **56** (2000), 763–784]. Both the energy method and the method of Fourier transform are efficiently used to complete the proof.

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**Keywords.** Asymptotic behavior, Darcy's law, decay rate, energy method, initial-boundary value problems, nonlinear diffusion wave, optimal convergence rate.

## 1. Introduction

Subsequent to the previous works by Nishihara–Yang [18] and Marcati–Mei [10], we continue to consider the following model of hyperbolic equations with damping, on the quarter plane $\mathbb{R}_+ \times \mathbb{R}_+$, ($\mathbb{R}_+ = (0, +\infty)$), given by

$$\begin{cases} v_t - u_x = 0 \\ u_t + p(v)_x = -\alpha u, \end{cases} \qquad (x,t) \in \mathbb{R}_+ \times \mathbb{R}_+, \tag{1.1}$$

which models a compressible flow with dissipative external force field in Lagrangian coordinates. The external force term $-\alpha u$ appears in the momentum equation. Here, $v > 0$ is the specific volume, $u$ is the velocity, the pressure $p(v)$ is a smooth function of $v$ such that $p(v) > 0$, $p'(v) < 0$, and $\alpha > 0$ is the damping constant. Such a system was firstly studied by van Duyn and Peletier [3].

For the corresponding Cauchy problem, Marcati–Milani [11] in the case of weak solutions, and Hsiao–Liu [4] and Nishihara [15] in the case of smooth solutions proved that the solutions $(v, u)(x, t)$ to the corresponding Cauchy problem of (1.1) tend time-asymptotically to the nonlinear self-similar diffusion wave solutions $(\bar{v}, \bar{u})(x, t)$ ($\bar{v}(x, t) = \phi(x/\sqrt{1+t})$) of the porous media equation

$$\begin{cases} \bar{v}_t = -\frac{1}{\alpha} p(\bar{v})_{xx} \\ p(\bar{v})_x = -\alpha \bar{u}, \end{cases} \qquad i.e., \quad \begin{cases} \bar{v}_t - \bar{u}_x = 0 \\ p(\bar{v})_x = -\alpha \bar{u}. \end{cases} \tag{1.2}$$

For the related initial-boundary value problems (IBVPs), Marcati–Mei [10] and Nishihara–Yang [18] studied the convergence to diffusion waves on the quarter plane $\mathbb{R}_+ \times \mathbb{R}_+$, but the decay rates they showed are not optimal. Moreover, Hsiao–Pan [7] studied the convergence to diffusion waves on the bounded domain $x \in [0, 1]$. We note also that, by using the method of pointwise estimates and the approximating Green function for a parabolic equation, Nishihara–Wang–Yang [17] succeeded in obtaining the optimal rates for the Cauchy problem case, which improves the previous works by Hsiao–Liu [4] and Nishihara [15]. But the method used in [17] is hard to extend to the IBVP case because of the difficulty of constructing a suitable approximating Green function. In the present paper the main purpose is to obtain the optimal decay rates for the convergence. The approach we adopt is the Fourier transform together with the energy method. We first reduce the fundamental solutions for the IBVPs to the corresponding linear equation and show the energy decay rates for the fundamental solutions by the Fourier transform method. Basing on it, we can see the optimal rates we expect for the nonlinear problems. Then we apply the basic energy estimates of the linear IBVP to the nonlinear cases and improve the previous convergence $L^p$-rates $(1+t)^{-(\frac{3}{4}-\frac{1}{2p})}$ for $2 \le p \le \infty$ in [10, 18] to the optimal ones $(1+t)^{-(1-\frac{1}{2p})}$, which is the main part of this paper.

For the other model equations dealing with the stability theory of diffusion waves, we refer to [1, 2, 5, 6, 8, 9, 14, 16] and the references therein.

## 2. Main results

In this paper, we mainly consider the IBVP studied by Nishihara–Yang [18]

$$\begin{cases} v_t - u_x = 0 \\ u_t + p(v)_x = -\alpha u, \quad (x, t) \in \mathbb{R}_+ \times \mathbb{R}_+ \\ (v, u)(x, 0) = (v_0, u_0)(x) \to (v_+, u_+) \ \text{ as } x \to \infty \\ u(0, t) = 0. \end{cases} \tag{2.1}$$

The main goal is to improve the previous stability of diffusion waves and to show the optimal convergence rates. As for the other IBVP studied by Marcati–Mei [10], since it can be similarly treated, we will state the improved convergence rates in the last part of this paper but without proof.

In [18], Nishihara and Yang selected the linear diffusion waves $(\bar{v}, \bar{u})(x, t)$ as the asymptotic profile of (2.1):

$$\bar{v}_t - \kappa\bar{v}_{xx} = 0, \quad \bar{v}_x(0,t) = 0, \quad \bar{v}(+\infty,t) = v_+, \quad \text{and } \bar{u}(x,t) := \kappa\bar{v}_x(x,t), \tag{2.2}$$

which solve explicitly

$$\bar{v}(x,t) = v_+ + \frac{\delta_0}{\sqrt{4\kappa\pi(t+1)}}\exp\Big(-\frac{x^2}{4\kappa(t+1)}\Big), \tag{2.3}$$

where

$$\kappa = -p'(v_+)/\alpha > 0, \quad \delta_0 = 2\bigg(\int_0^\infty (v_0(x)-v_+)dx - \frac{u_+}{\alpha}\bigg). \tag{2.4}$$

Then they proved its stability with algebraic decay as follows

$$\|\partial_x^k(v-\bar{v})(t)\|_{L^2(\mathbb{R}_+)} = O(1)(1+t)^{-(k+1)/2}, \quad k=0,1,2, \tag{2.5}$$

$$\|\partial_x^k(u-\bar{u})(t)\|_{L^2(\mathbb{R}_+)} = O(1)(1+t)^{-(k+2)/2}, \quad k=0,1, \tag{2.6}$$

$$\|(v-\bar{v})(t)\|_{L^p(\mathbb{R}_+)} = O(1)(1+t)^{-(\frac{3}{4}-\frac{1}{2p})}, \quad 2 \le p \le \infty. \tag{2.7}$$

However, these rates are not optimal, because the linear diffusion wave $(\bar{v},\bar{u})(x,t)$ is not the right asymptotic profile of (2.1), and causes a slower delay in $L^1$ by the term $(p'(\bar{v})-p'(v_+))\bar{v}_x$. According to Darcy's law, we believe that the optimal profile to the solution $(v,u)(x,t)$ of (2.1) is the following nonlinear diffusion wave $(\bar{\bar{v}},\bar{\bar{u}})(x,t)$:

$$\alpha\bar{\bar{v}}_t + p(\bar{\bar{v}})_{xx} = 0, \quad (x,t) \in \mathbb{R}_+ \times \mathbb{R}_+ \tag{2.8}$$

with the boundary restrictions

$$\bar{\bar{v}}_x|_{x=0} = 0, \quad \bar{\bar{v}}|_{x=+\infty} = v_+. \tag{2.9}$$

Here

$$\bar{\bar{u}}(x,t) = -\frac{1}{\alpha}p(\bar{\bar{v}})_x. \tag{2.10}$$

We shall prove the convergence rates of $(v,u)(x,t)$ to $(\bar{\bar{v}},\bar{\bar{u}})(x,t)$ to be better than the above.

In order to construct such a nonlinear diffusion wave, let us consider a function $\phi(x,t+1)$ (here, using $t+1$ instead of $t$ is to avoid the singularity of solution decay at the point $t=0$) which satisfies

$$\alpha\bar{\delta}_0\phi_t + p(v_+ + \bar{\delta}_0\phi)_{xx} = 0, \quad (x,t) \in \mathbb{R}_+ \times \mathbb{R}_+ \tag{2.11}$$

namely,

$$\phi_t - \frac{-p'(v_+)}{\alpha}\phi_{xx} = -\frac{1}{\alpha\bar{\delta}_0}[p(v_+ + \bar{\delta}_0\phi) - p(v_+) - p'(v_+)\bar{\delta}_0\phi]_{xx}, \quad (x,t) \in \mathbb{R}_+ \times \mathbb{R}_+ \tag{2.12}$$

with the initial boundary values

$$\phi_x|_{x=0} = 0, \quad \phi|_{x=+\infty} = 0, \quad \text{and} \quad \phi|_{t=0} = \phi(x,1) =: \phi_0(x), \tag{2.13}$$

where $\phi_0(x)$ is a given smooth function such that

$$\phi_0(x) \in L^1(\mathbb{R}_+) \quad \text{and} \quad \int_0^\infty \phi_0(x)dx \neq 0, \tag{2.14}$$

and $\bar{\delta}_0$ is a constant such that

$$\int_0^\infty [v_0(x) - v_+]dx - \bar{\delta}_0 \int_0^\infty \phi_0(x)dx - \frac{u_+}{\alpha} = 0, \tag{2.15}$$

namely,

$$\bar{\delta}_0 := \left( \int_0^\infty [v_0(x) - v_+]dx - \frac{u_+}{\alpha} \right) \Big/ \int_0^\infty \phi_0(x)dx. \tag{2.16}$$

By making an iteration $\phi_{n+1}(x, t+1)$ for $n = 0, 1, \cdots$

$$\begin{aligned}\partial_t \phi_{n+1} - \frac{-p'(v_+)}{\alpha} \partial_x^2 \phi_{n+1} &= -\frac{1}{\alpha \bar{\delta}_0} \partial_x^2 [p(v_+ + \bar{\delta}_0 \phi_n) - p(v_+) - \bar{\delta}_0 p'(v_+)\phi_n] \\ &=: -\frac{1}{\alpha \bar{\delta}_0} \partial_x^2 H(\phi_n)\end{aligned}$$

and by using the integral form

$$\begin{aligned}&\phi_{n+1}(x, t+1) \\ &= \int_0^\infty G(x, y; t+1, 0)\phi_0(y)dy - \frac{1}{\alpha \bar{\delta}_0} \int_0^t \int_0^\infty G(x, y; t+1, \tau)\partial_y^2 H(\phi_n)dyd\tau \\ &= \int_0^\infty G(x, y; t+1, 0)\phi_0(y)dy - \frac{1}{\alpha \bar{\delta}_0} \int_0^t \int_0^\infty G_{yy}(x, y; t+1, \tau)H(\phi_n)dyd\tau,\end{aligned}$$

where $G(x, y; t, \tau)$ is the Green function of the Nuemann IBVP to the heat equation

$$G(x, y; t, \tau) = \frac{1}{\sqrt{4\pi\kappa(t-\tau)}} \Big[ e^{-\frac{(x-y)^2}{4\kappa(t-\tau)}} + e^{-\frac{(x+y)^2}{4\kappa(t-\tau)}} \Big], \quad \kappa := -\frac{p'(v_+)}{\alpha},$$

then we can prove that $\{\phi_n\}$ is the Cauchy series and converges to a limit, say $\phi$, which is the unique global solution of the IBVP (2.11) and (2.13). Furthermore, by using the Green function method and energy estimates, we can prove the following decay rates

$$\|\partial_t^j \partial_x^k \phi(t)\|_{L^2} = O(1)\bar{\delta}_0(1+t)^{-(4j+2k+1)/4}, \tag{2.17}$$

$$\|\phi_{xt}(t)\|_{L^1} = O(1)\bar{\delta}_0(1+t)^{-3/2}. \tag{2.18}$$

Now we construct our diffusion wave as

$$\bar{\bar{v}}(x, t) := v_+ + \bar{\delta}_0 \phi(x, t+1). \tag{2.19}$$

From (2.17), (2.18) and (2.19), we get

$$\|\partial_t^j \partial_x^k (\bar{\bar{v}} - v_+)(t)\|_{L^2(\mathbb{R}_+)} = O(1)\bar{\delta}_0(1+t)^{-(4j+2k+1)/4}, \tag{2.20}$$

$$\|\bar{\bar{v}}_{xt}(t)\|_{L^1(\mathbb{R}_+)} = O(1)\bar{\delta}_0(1+t)^{-3/2}. \tag{2.21}$$

In particular, if

$$\int_0^\infty [v_0(x) - v_+]dx - \frac{u_+}{\alpha} = 0, \tag{2.22}$$

(2.16) implies $\bar{\delta}_0 = 0$, that is, the nonlinear diffusion wave $\bar{\bar{v}}(x,t)$ degenerates to the state constant $v_+$ ($\bar{\bar{v}}(x,t) \equiv v_+$). The relationship (2.8)-(2.10) is also equivalent to

$$\begin{cases} \bar{\bar{v}}_t - \bar{\bar{u}}_x = 0 \\ p(\bar{\bar{v}})_x = -\alpha\bar{\bar{u}} \\ \bar{\bar{u}}(0,t) = 0, \quad (\bar{\bar{v}}, \bar{\bar{u}})(\infty, t) = (v_+, 0). \end{cases} \tag{2.23}$$

Here follows our main result.

**Theorem 2.1.** *Suppose that* $v_0 - v_+ \in L^1(\mathbb{R}_+)$, $(V_0, z_0)(x) := \Big( - \int_x^\infty [v_0(y) - \bar{\bar{v}}(y,0)]dy, u_0(x) - \bar{\bar{u}}(x,0)\Big) \in (H^3(\mathbb{R}_+) \cap L^1(\mathbb{R}_+)) \times (H^2(\mathbb{R}_+) \cap L^1(\mathbb{R}_+))$ *and that* $\|v_0 - v_+\|_{L^1(\mathbb{R}_+)} + \|V_0\|_{H^3(\mathbb{R}_+)} + \|z_0\|_{H^2(\mathbb{R}_+)} + \|V_0\|_{L^1(\mathbb{R}_+)} + \|z_0\|_{L^1(\mathbb{R}_+)} + |u_+| \ll 1$ *hold. Then there exists a unique time-global solution* $(v,u)(x,t)$ *of the IBVP* (2.1) *such that*

$$v - \bar{\bar{v}} \in C^k(0,\infty; H^{2-k}(\mathbb{R}_+)), \quad k = 0,1,2, \qquad u - \bar{\bar{u}} \in C^k(0,\infty; H^{1-k}(\mathbb{R}_+)), \quad k = 0,1$$

*and*

$$\|\partial_x^k (v - \bar{\bar{v}})(\cdot,t)\|_{L^2(\mathbb{R}_+)} = O(1)(1+t)^{-(2k+3)/4}, \quad k = 0,1, \tag{2.24}$$

$$\|(v - \bar{\bar{v}})(\cdot,t)\|_{L^p(\mathbb{R}_+)} = O(1)(1+t)^{-(2p-1)/(2p)}, \quad 2 \le p \le +\infty, \tag{2.25}$$

$$\|(u - \bar{\bar{u}})(\cdot,t)\|_{L^2(\mathbb{R}_+)} = O(1)(1+t)^{-5/4}. \tag{2.26}$$

**Remarks.** 1. It is easy to see that our new convergence rates in (2.24)–(2.26) are better than Nishihara–Yang's as stated in (2.5)–(2.7).

2. The smallness conditions are sufficient in this paper as well as in most of the previous works. In spite of the recent work by Zhao [20], might it be possible to remove them from the present paper? This is still unknown.

## 3. Proof of the main theorem

As in [4, 15, 10, 18], let us introduce a pair of auxiliary functions $(\hat{v}, \hat{u})(x,t)$:

$$(\hat{v}, \hat{u})(x,t) = \left( \frac{u_+ m_0(x)}{-\alpha} e^{-\alpha t}, u_+ \int_0^x m_0(y) dy e^{-\alpha t} \right), \tag{3.1}$$

where $m_0 \geq 0$ is a smooth function with compact support in $\mathbb{R}_+$ such that $\int_0^\infty m_0(y)dy = 1$. One can verify that $(\hat{v}, \hat{u})(x,t)$ satisfies

$$\begin{cases} \hat{v}_t - \hat{u}_x = 0 \\ \hat{u}_t = -\alpha \hat{u} \\ \hat{u}(0,t) = 0, \quad (\hat{v}, \hat{u})(\infty, t) = (0, u_+ e^{-\alpha t}). \end{cases} \tag{3.2}$$

By adding the first equations in (2.1), (2.23) and (3.2), and by integrating it over $(-\infty, \infty) \times [0,t]$, as well as by using (2.15), one gets

$$\int_0^\infty (v - \bar{\bar{v}} - \hat{v})(x,t)dx = \int_0^\infty [v_0(x) - v_+]dx - \bar{\bar{\delta}}_0 \int_0^\infty \phi_0(x)dx - \frac{u_+}{\alpha} = 0. \tag{3.3}$$

Thus, it is reasonable to introduce the following perturbations as our new variables

$$V(x,t) := -\int_x^\infty (v - \bar{\bar{v}} - \hat{v})(y,t)dy \tag{3.4}$$

$$z(x,t) := u(x,t) - \bar{\bar{u}}(x,t) - \hat{u}(x,t). \tag{3.5}$$

Finally, one obtains a new IBVP on $(V,z)(x,t)$ as follows

$$\begin{cases} V_t - z = 0 \\ z_t + (p'(\bar{\bar{v}})V_x)_x + \alpha z = -F, \quad (x,t) \in \mathbb{R}_+ \times \mathbb{R}_+ \\ (V,z)|_{t=0} = (\bar{V}_0, \bar{z}_0)(x) \\ V|_{x=0} = 0, \end{cases} \tag{3.6}$$

where

$$F := \frac{1}{\alpha} p(\bar{\bar{v}})_{xt} + (p(V_x + \bar{\bar{v}} + \hat{v}) - p(\bar{\bar{v}}) - p'(\bar{\bar{v}})V_x)_x, \tag{3.7}$$

$$\bar{V}_0(x) := V_0(x) + \int_x^\infty \hat{v}(y,0)dy = -\int_x^\infty [v_0(y) - \bar{\bar{v}}(y,0) - \hat{v}(y,0)]dy, \tag{3.8}$$

$$\bar{z}_0(x) := z_0(x) - \hat{u}(x,0) = u_0(x) - \bar{\bar{u}}(x,0) - \hat{u}(x,0). \tag{3.9}$$

Without any difficulty, we can prove the following stability with slower decay rates by using a similar argument of [18, 10]. Since the proof is tedious but similar as in the previous works (see Theorem in [18] and Theorem 2.1 in [10]), we omit its details.

**Theorem 3.1.** *Under the assumptions in Theorem 2.1 there exists a unique time-global solution $(V,z)(x,t)$ of the IBVP (3.6) such that*

$$V \in C^k(0,\infty; H^{3-k}(\mathbb{R}_+)), \quad k = 0,1,2,3, \qquad z \in C^k(0,\infty; H^{2-k}(\mathbb{R}_+)), \quad k = 0,1,2$$

*and*

$$\|\partial_x^k V(t)\|_{L^2(\mathbb{R}_+)} = O(1)(1+t)^{-k/2}, \quad k = 0,1,2,3, \tag{3.10}$$

$$\|\partial_x^k z(t)\|_{L^2(\mathbb{R}_+)} = O(1)(1+t)^{-(k+2)/2}, \quad k = 0,1,2, \tag{3.11}$$

*and*

$$(1+t)^2\|z_t(t)\|_{L^2(\mathbb{R}_+)}+(1+t)^{-5/2}(\|z_{xt}(t)\|_{L^2(\mathbb{R}_+)}+\|z_{tt}(t)\|_{L^2(\mathbb{R}_+)})=O(1). \quad (3.12)$$

Furthermore, we improve the decay rates in Theorem 3.1 to be optimal as follows.

**Theorem 3.2.** *Under the assumptions in Theorem 2.1, the solution* $(V,z)(x,t)$ *decays time-asymptotically as*

$$\|\partial_x^k V(t)\|_{L^2(\mathbb{R}_+)}=O(1)(1+t)^{-(2k+1)/4}, \qquad k=0,1,2, \quad (3.13)$$

$$\|z(t)\|_{L^2(\mathbb{R}_+)}=O(1)(1+t)^{-5/4}. \quad (3.14)$$

The proof of Theorem 3.2 will be completed in the next section. Based on the above theorem, we are going to prove Theorem 2.1.

*Proof of Theorem 2.1.* Thanks to Theorem 3.2, and by noticing that $V_x=v-\bar{\bar{v}}-\hat{v}$, $z=u-\bar{\bar{u}}-\bar{u}$, and $(\partial_x^k\hat{v},\partial_x^k\hat{u})(x,t)$ decays like $e^{-\alpha t}$, we have

$$\begin{aligned}
&\|\partial_x^k(v-\bar{\bar{v}})(t)\|_{L^2}=\|\partial_x^k(V_x+\hat{v})(t)\|_{L^2}\\
&\le\|\partial_x^k V_x(t)\|_{L^2}+\|\partial_x^k\hat{v}(t)\|_{L^2}\\
&\le C(1+t)^{-(2(k+1)+1)/4}+Ce^{-\alpha t}\\
&\le C(1+t)^{-(2k+3)/4}
\end{aligned} \quad (3.15)$$

and

$$\begin{aligned}
&\|(u-\bar{\bar{u}})(t)\|_{L^2}=\|(z+\hat{u})(t)\|_{L^2}\\
&\le\|z(t)\|_{L^2}+\|\hat{u}(t)\|_{L^2}\\
&\le C(1+t)^{-5/4}+Ce^{-\alpha t}\\
&\le C(1+t)^{-5/4}.
\end{aligned} \quad (3.16)$$

This proved (2.24) and (2.26).

For the proof of (2.25), by using (2.24) and Sobolev's inequalities $\|f\|_{L^\infty}\le\sqrt{2}\|f\|_{L^2}^{1/2}\|f_x\|_{L^2}^{1/2}$ and $\|f\|_{L^p}\le\|f\|_{L^\infty}^{(p-2)/p}\|f\|_{L^2}^{2/p}$ for $2\le p\le+\infty$, we get

$$\begin{aligned}
&\|(v-\bar{\bar{v}})(t)\|_{L^p}\\
&\le\|(v-\bar{\bar{v}})(t)\|_{L^\infty}^{(p-2)/p}\|(v-\bar{\bar{v}})(t)\|_{L^2}^{2/p}\\
&\le\Big(\sqrt{2}\|(v-\bar{\bar{v}})(t)\|_{L^2}^{1/2}\|\partial_x(v-\bar{\bar{v}})(t)\|_{L^2}^{1/2}\Big)^{(p-2)/p}\|(v-\bar{\bar{v}})(t)\|_{L^2}^{2/p}
\end{aligned}$$

$$
\begin{aligned}
&= 2^{\frac{p-2}{2p}} \|(v-\bar{\bar{v}})(t)\|_{L^2}^{\frac{p-2}{2p}+\frac{2}{p}} \|\partial_x(v-\bar{\bar{v}})(t)\|_{L^2}^{\frac{p-2}{2p}} \\
&\le C(1+t)^{-\frac{3}{4}\times(\frac{p-2}{2p}+\frac{2}{p})}(1+t)^{-\frac{5}{4}\times\frac{p-2}{2p}} \\
&\le C(1+t)^{-(1-\frac{1}{2p})}. \qquad (3.17)
\end{aligned}
$$

This proved (2.25). The proof is complete. □

## 4. Proof of Theorem 3.2

Now we are going to prove Theorem 3.2. For the system (3.6), by substituting the first equation $z = V_t$ into the second one, we obtain

$$
\begin{cases} V_{tt} + \alpha V_t - \beta V_{xx} = G, & (x,t) \in \mathbb{R}_+ \times \mathbb{R}_+ \\ (V, V_t)(x,0) = (\bar{V}_0, \bar{z}_0)(x)), & x \in \mathbb{R}_+ \\ V(0,t) = 0, \end{cases} \tag{4.1}
$$

where $\beta = -p'(v_+)$ and

$$
G = -F - ((p'(\bar{\bar{v}}) - p'(v_+))V_x)_x. \tag{4.2}
$$

In what follows, we deduce the fundamental solutions and their basic energy estimates for the linear damped wave equation with the null-Dirichlet boundary condition. Based on this, we know what are the optimal rates we may obtain for the linear and nonlinear damped wave equations. In such a sense, it is essential.

Let us consider the linear damped wave equation

$$
\phi_{tt} + \alpha\phi_t - \beta\phi_{xx} = 0, \quad (x,t) \in \mathbb{R}_+ \times \mathbb{R}_+, \tag{4.3}
$$

with the initial data

$$
(\phi, \phi_t)|_{t=0} = (\phi_0, \phi_1)(x), \quad x \in \mathbb{R}_+ \tag{4.4}
$$

and the Dirichlet boundary condition

$$
\phi|_{x=0} = 0. \tag{4.5}
$$

In order to solve the IBVP (4.3)–(4.5), we make an odd extension

$$
\psi(x,t) := \begin{cases} \phi(x,t), & x \ge 0 \\ -\phi(-x,t), & x < 0, \end{cases} \qquad \psi_i(x) := \begin{cases} \phi_i(x), & x \ge 0 \\ -\phi_i(-x), & x < 0, \end{cases} \qquad i = 0, 1
$$

then we get a corresponding Cauchy problem

$$
\begin{cases} \psi_{tt} + \alpha\psi_t - \beta\psi_{xx} = 0, & (x,t) \in \mathbb{R} \times \mathbb{R}_+ \\ \psi|_{t=0} = \psi_0(x), & \psi_t|_{t=0} = \psi_1(x). \end{cases} \tag{4.6}
$$

Thus, the solution of (4.6) can be represented in an integral form

$$
\psi(x,t) = K_0(t) * \psi_0 + K_1(t) * \psi_1, \tag{4.7}
$$

where $K_i(x,t)$ $(i=0,1)$ are the fundamental solutions of (4.6), that is,

$$K_{itt} + \alpha K_{it} - \beta K_{ixx} = 0, \quad i = 0,1 \tag{4.8}$$

with

$$\begin{cases} K_0(x,0) = \delta(x) \\ \frac{d}{dt}K_0(x,0) = 0 \end{cases} \quad \text{and} \quad \begin{cases} K_1(x,0) = 0 \\ \frac{d}{dt}K_1(x,0) = \delta(x), \end{cases} \tag{4.9}$$

where $\delta(x)$ is the Delta function. The asterisk $*$ means convolution, i.e., $K_i(t) * \psi_i = \int_{-\infty}^{\infty} K_i(x-y,t)\psi_i(y)dy$.

As in [12], let $R_i(\xi,t)$ be the Fourier transform of $K_i(x,t)$, $i=0,1$, then $R_i$ satisfies the following ODE

$$\frac{d^2}{dt^2}R_i + \alpha\frac{d}{dt}R_i + \beta\xi^2 R_i = 0, \quad i = 0,1 \tag{4.10}$$

with the initial data

$$\begin{cases} R_0(\xi,0) = 1 \\ \frac{d}{dt}R_0(\xi,0) = 0 \end{cases} \quad \text{and} \quad \begin{cases} R_1(\xi,0) = 0 \\ \frac{d}{dt}R_1(\xi,0) = 1 \end{cases} \tag{4.11}$$

respectively. By solving the previous two ODEs directly, we obtain the exact solutions as

$$R_1(\xi,t) = \begin{cases} \frac{2e^{-\alpha t/2}}{\sqrt{\alpha^2-4\beta\xi^2}} \sinh\left(\frac{\sqrt{\alpha^2-4\beta\xi^2}}{2}t\right), & |\xi| < \frac{\alpha}{2\sqrt{\beta}} \\ te^{-\alpha t/2}, & |\xi| = \frac{\alpha}{2\sqrt{\beta}} \\ \frac{2e^{-\alpha t/2}}{\sqrt{4\beta\xi^2-\alpha^2}} \sin\left(\frac{\sqrt{4\beta\xi^2-\alpha^2}}{2}t\right), & |\xi| > \frac{\alpha}{2\sqrt{\beta}} \end{cases} \tag{4.12}$$

and

$$R_0(\xi,t) = \frac{\alpha}{2}R_1(\xi,t) + R_2(\xi,t) \tag{4.13}$$

where

$$R_2(\xi,t) = \begin{cases} e^{-\alpha t/2} \cosh\left(\frac{\sqrt{\alpha^2-4\beta\xi^2}}{2}t\right), & |\xi| < \frac{\alpha}{2\sqrt{\beta}} \\ e^{-\alpha t/2}, & |\xi| = \frac{\alpha}{2\sqrt{\beta}} \\ e^{-\alpha t/2} \cos\left(\frac{\sqrt{4\beta\xi^2-\alpha^2}}{2}t\right), & |\xi| > \frac{\alpha}{2\sqrt{\beta}}. \end{cases} \tag{4.14}$$

Thus, we see that the fundamental solutions $K_j(x,t)$ $(j=0,1)$ can be given by making use of the inverse Fourier transform to $R_j(\xi,t)$ $(j=0,1)$:

$$K_j(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ix\xi}R_j(\xi,t)d\xi.$$

For the IBVP (4.3)–(4.5), due to the odd symmetry $\psi(-x,t) = -\psi(x,t)$ and $\psi_i(-x) = -\psi_i(x)$ $(i=0,1)$, we obtain the expression of the solution as follows, for all $x \geq 0$,

$$\phi(x,t) = \psi(x,t) = \int_{-\infty}^{\infty} K_0(x-y,t)\psi_0(y)dy + \int_{-\infty}^{\infty} K_1(x-y,t)\psi_1(y)dy$$

$$
\begin{aligned}
&= \Big( \int_0^\infty + \int_{-\infty}^0 \Big) K_0(x-y,t)\psi_0(y)dy \\
&\quad + \Big( \int_0^\infty + \int_{-\infty}^0 \Big) K_1(x-y,t)\psi_1(y)dy \\
&= \int_0^\infty [K_0(x-y,t) - K_0(x+y,t)]\phi_0(y)dy \\
&\quad + \int_0^\infty [K_1(x-y,t) - K_1(x+y,t)]\phi_1(y)dy. \qquad (4.15)
\end{aligned}
$$

Furthermore, for the linear equation with source term

$$
\phi_{tt} + \alpha\phi_t - \beta\psi_{xx} = g(x,t), \quad (x,t) \in \mathbb{R}_+ \times \mathbb{R}_+ \qquad (4.16)
$$

with the initial value condition (4.4) and the Dirichlet boundary value condition (4.5), by making an odd extension to the source term $g(-x,t) = -g(x,t)$, we can produce the expressions of solutions by the Duhamel's principle:

$$
\begin{aligned}
\phi(x,t) &= \int_0^\infty [K_0(x-y,t) - K_0(x+y,t)]\phi_0(y)dy \\
&\quad + \int_0^\infty [K_1(x-y,t) - K_1(x+y,t)]\phi_1(y)dy \\
&\quad + \int_0^t \int_0^\infty [K_1(x-y,t-\tau) - K_1(x+y,t-\tau)]g(y,\tau)dyd\tau, \qquad (4.17)
\end{aligned}
$$

In the Cauchy problem case, we note that Matsumura [12] got the following energy estimates.

**Lemma 4.1** ([12]). *If $f \in L^1(\mathbb{R}) \cap H^{j+k-1}(\mathbb{R})$, then*

$$
\left\| \partial_t^j \partial_x^k (K_1(t) * f) \right\|_{L^2(\mathbb{R})} \le C(1+t)^{-j-\frac{2k+1}{4}} (\|f\|_{L^1(\mathbb{R})} + \|f\|_{H^{j+k-1}(\mathbb{R})}). \qquad (4.18)
$$

*If $f \in L^1(\mathbb{R}) \cap H^{j+k}(\mathbb{R})$, then*

$$
\left\| \partial_t^j \partial_x^k (K_0(t) * f) \right\|_{L^2(\mathbb{R})} \le C(1+t)^{-j-\frac{2k+1}{4}} (\|f\|_{L^1(\mathbb{R})} + \|f\|_{H^{j+k}(\mathbb{R})}). \qquad (4.19)
$$

By using this lemma, we may prove the energy estimates for the linear IBVP.

**Lemma 4.2.** *If $f \in L^1(\mathbb{R}_+) \cap H^{j+k-1}(\mathbb{R}_+)$, then*

$$
\begin{aligned}
&\left\| \partial_t^j \partial_x^k \int_0^\infty [K_1(x-y,t) - K_1(x+y,t)]f(y)dy \right\|_{L^2(\mathbb{R}_+)} \\
&\le C(1+t)^{-j-\frac{2k+1}{4}} [\|f\|_{L^1(\mathbb{R}_+)} + \|f\|_{H^{j+k-1}(\mathbb{R}_+)}], \qquad (4.20)
\end{aligned}
$$

*If $f \in L^1(\mathbb{R}_+) \cap H^{j+k}(\mathbb{R}_+)$, then*

$$\begin{aligned}&\left\|\partial_t^j\partial_x^k\int_0^\infty[K_0(x-y,t)-K_0(x+y,t)]f(y)dy\right\|_{L^2(\mathbb{R}_+)}\\&\le C(1+t)^{-j-\frac{2k+1}{4}}[\|f\|_{L^1(\mathbb{R}_+)}+\|f\|_{H^{j+k}(\mathbb{R}_+)}],\end{aligned}\tag{4.21}$$

*Proof.* Firstly we are going to prove (4.20). Setting

$$\bar\phi(x,t):=\int_0^\infty[K_1(x-y,t)-K_1(x+y,t)]f(y)dy,$$

we know that $\bar\phi(x,t)$ is a solution of the IBVP as follows

$$\begin{cases}\bar\phi_{tt}+\alpha\bar\phi_t-\beta\bar\phi_{xx}=0, & (x,t)\in\mathbb{R}_+\times\mathbb{R}_+\\ \bar\phi|_{t=0}=0,\quad \bar\phi_t|_{t=0}=f(x), & x\in\mathbb{R}_+\\ \bar\phi|_{x=0}=0.\end{cases}$$

By an odd extension to the above IBVP

$$\bar\psi(x,t):=\begin{cases}\bar\phi(x,t), & x\ge 0\\ -\bar\phi(-x,t), & x<0,\end{cases}\qquad h(x):=\begin{cases}f(x), & x\ge 0\\ -f(-x), & x<0,\end{cases}$$

such that we may consider its corresponding Cauchy problem for $\bar\psi(x,t)$ with the initial data $(0,h(x))$, and note that

$$\int_{-\infty}^\infty\bar\psi^2(x,t)dx=2\int_0^\infty\bar\phi^2(x,t)dx,\quad \|h\|_{L^1(\mathbb{R})}=2\|f\|_{L^1(\mathbb{R}_+)},\quad \|h\|^2_{H^j(\mathbb{R})}=2\|f\|^2_{H^j(\mathbb{R}_+)}$$

due to the odd symmetry, and by using Matsumura's lemma (Lemma 4.1), then we obtain

$$\begin{aligned}&\left\|\partial_t^j\partial_x^k\int_0^\infty[K_1(x-y,t)-K_1(x+y,t)]f(y)dy\right\|_{L^2(\mathbb{R}_+)}\\&=\left\|\partial_t^j\partial_x^k\int_{-\infty}^\infty K_1(x-y,t)h(y)dy\right\|_{L^2(\mathbb{R}_+)}\\&=\frac{1}{\sqrt2}\left\|\partial_t^j\partial_x^k\int_{-\infty}^\infty K_1(x-y,t)h(y)dy\right\|_{L^2(\mathbb{R})}\\&\le C(1+t)^{-j-\frac{2k+1}{4}}[\|h\|_{L^1(\mathbb{R})}+\|h\|_{H^{j+k-1}(\mathbb{R})}]\\&=2C(1+t)^{-j-\frac{2k+1}{4}}[\|f\|_{L^1(\mathbb{R}_+)}+\|f\|_{H^{j+k-1}(\mathbb{R}_+)}].\end{aligned}\tag{4.22}$$

This proved (4.20).

Similarly, without any difficulty we can prove (4.21). Here we omit the details. □

We even state another auxiliary lemma which is used to prove the decay rates. The proof and a lot of applications of this lemma can be found in many bibliographies, for example in [19, 12, 13, 8] and the references therein.

**Lemma 4.3.** *Let $a > 0$ and $b > 0$ be constants. If* $\max(a, b) > 1$*, then*

$$\int_0^t (1+t-s)^{-a}(1+s)^{-b}ds \le C(1+t)^{-\min(a,b)}. \tag{4.23}$$

*If* $\max(a, b) = 1$*, then*

$$\int_0^t (1+t-s)^{-a}(1+s)^{-b}ds \le C(1+t)^{-\min(a,b)}\ln(2+t). \tag{4.24}$$

*If* $\max(a, b) < 1$*, then*

$$\int_0^t (1+t-s)^{-a}(1+s)^{-b}ds \le C(1+t)^{1-a-b}. \tag{4.25}$$

*Proof of Theorem 3.2.* Firstly, we prove the optimal decay rates for $\|\partial_x^k V(t)\|_{L^2}$ ($k = 0, 1, 2$), namely, (3.13).

By noticing (4.17), we obtain an equivalent integral equation of the IBVP (4.1) as follows

$$\begin{aligned} V(x,t) = &\int_0^\infty [K_0(x-y,t) - K_0(x+y,t)]\bar{V}_0(y)dy \\ &+ \int_0^\infty [K_1(x-y,t) - K_1(x+y,t)]\bar{z}_0(y)dy \\ &+ \int_0^t \int_0^\infty [K_1(x-y,t-\tau) - K_1(x+y,t-\tau)]G(y,\tau)dyd\tau. \end{aligned} \tag{4.26}$$

By differentiating (4.26) $k$-times ($k = 0, 1, 2$) with respect to $x$, and by taking its $L^2(\mathbb{R}_+)$-norm, we obtain

$$\begin{aligned} &\|\partial_x^k V(t)\|_{L^2(\mathbb{R}_+)} \\ &\le \left\|\partial_x^k \int_0^\infty [K_0(x-y,t) - K_0(x+y,t)]\bar{V}_0(y)dy\right\|_{L^2(\mathbb{R}_+)} \\ &+ \left\|\partial_x^k \int_0^\infty [K_1(x-y,t) - K_1(x+y,t)]\bar{z}_0(y)dy\right\|_{L^2(\mathbb{R}_+)} \\ &+ \int_0^t \left\|\partial_x^k \int_0^\infty [K_1(x-y,t-\tau) - K_1(x+y,t-\tau)]G(y,\tau)dy\right\|_{L^2(\mathbb{R}_+)} d\tau. \end{aligned} \tag{4.27}$$

Since $\bar{V}_0 \in L^1(\mathbb{R}_+) \cap H^3(\mathbb{R}_+)$ and $\bar{z}_0 \in L^1(\mathbb{R}_+) \cap H^2(\mathbb{R}_+)$, we apply Lemma 4.2 then to have

$$\begin{aligned} &\left\|\partial_x^k \int_0^\infty [K_0(x-y,t) - K_0(x+y,t)]\bar{V}_0(y)dy\right\|_{L^2(\mathbb{R}_+)} \\ &\le C[\|\bar{V}_0\|_{L^1(\mathbb{R}_+)} + \|\bar{V}_0\|_{H^3(\mathbb{R}_+)}](1+t)^{-(2k+1)/4}, \end{aligned} \tag{4.28}$$

and

$$\begin{aligned}&\left\|\partial_x^k\int_0^\infty[K_1(x-y,t)-K_1(x+y,t)]\bar{z}_0(y)dy\right\|_{L^2(\mathbb{R}_+)}\\&\le C[\|\bar{z}_0\|_{L^1(\mathbb{R}_+)}+\|\bar{z}_0\|_{H^2(\mathbb{R}_+)}](1+t)^{-(2k+1)/4}\end{aligned}\tag{4.29}$$

for $k=0,1,2$.

Now we are going to estimate the last term in (4.27). By Taylor's expansion, and by noticing (4.2) and (3.7), we have

$$|G|\sim O(1)\{|\bar{\bar{v}}_x\bar{\bar{v}}_t|+|\bar{\bar{v}}_{xt}|+|((\bar{\bar{v}}-v_+)V_x)_x|+|\hat{v}_x|+|(V_x^2)_x|\},\tag{4.30}$$

$$|\partial_x^kG|\sim O(1)\{|\partial_x^k(\bar{\bar{v}}_x\bar{\bar{v}}_t)|+|\partial_x^k\bar{\bar{v}}_{xt}|+|\partial_x^k((\bar{\bar{v}}-v_+)V_x)_x|+|\partial_x^k\hat{v}|+|\partial_x^k(V_x^2)_x|\}.\tag{4.31}$$

From (3.1), (2.20), (2.21) and (3.10), and by Hölder's inequality $\|fg\|_{L^1}\le\|f\|_{L^2}\|g\|_{L^2}$, then the $L^1$-norm for $G$ can be estimated as follows

$$\begin{aligned}&\|G(t)\|_{L^1(\mathbb{R}_+)}\\&\le C\{\|\bar{\bar{v}}_t(t)\|_{L^2(\mathbb{R}_+)}\|\bar{\bar{v}}_x(t)\|_{L^2(\mathbb{R}_+)}+\|\bar{\bar{v}}_{xt}(t)\|_{L^1(\mathbb{R}_+)}\\&\quad+\|(\bar{\bar{v}}-v_+)(t)\|_{L^2(\mathbb{R}_+)}\|V_{xx}(t)\|_{L^2(\mathbb{R}_+)}+\|(\bar{\bar{v}}-v_+)_x(t)\|_{L^2(\mathbb{R}_+)}\|V_x(t)\|_{L^2(\mathbb{R}_+)}\\&\quad+\|\hat{v}_x(t)\|_{L^1(\mathbb{R}_+)}+\|V_x(t)\|_{L^2(\mathbb{R}_+)}\|V_{xx}(t)\|_{L^2(\mathbb{R}_+)}\}\\&\le C\{(1+t)^{-1-\frac14-\frac34}+(1+t)^{-\frac32}+(1+t)^{-\frac14-\frac22}\\&\quad+(1+t)^{-\frac34-\frac12}+e^{-\alpha t}+(1+t)^{-\frac12-\frac22}\}\\&\le C(1+t)^{-5/4}.\end{aligned}\tag{4.32}$$

Similarly, we can also prove

$$\|G(t)\|_{H^k(\mathbb{R}_+)}\le C(1+t)^{-3/2}.\tag{4.33}$$

By noting (4.32), (4.33) and $3/2>5/4\ge(2k+1)/4$ for $k=0,1,2$, and applying Lemmas 4.2 and 4.3, we obtain optimal rates for the last term of (4.27) as follows

$$\begin{aligned}&\int_0^t\left\|\partial_x^k\int_0^\infty[K_1(x-y,t-\tau)-K_1(x+y,t-\tau)]G(y,\tau)dy\right\|_{L^2(\mathbb{R}_+)}d\tau\\&\le C\int_0^t(1+t-\tau)^{-(2k+1)/4}[\|G(\tau)\|_{L^1(\mathbb{R}_+)}+\|G(\tau)\|_{H^{k-1}(\mathbb{R}_+)}]d\tau\\&\le C\int_0^t(1+t-\tau)^{-(2k+1)/4}[(1+\tau)^{-5/4}+(1+\tau)^{-3/2}]d\tau\\&\le C(1+t)^{-(2k+1)/4},\qquad\text{for}\quad k=0,1,2.\end{aligned}\tag{4.34}$$

Applying (4.28), (4.29) and (4.34) to (4.27), we prove (3.13) for $k=0,1,2$.

Now, we are going to prove (3.14). It is well known that

$$
\begin{aligned}
z(x,t) &= V_t(x,t)\\
&= \partial_t \int_0^\infty [K_0(x-y,t) - K_0(x+y,t)]\bar V_0(y)dy\\
&\quad + \partial_t \int_0^\infty [K_1(x-y,t) - K_1(x+y,t)]\bar z_0(y)dy\\
&\quad + \int_0^t \partial_t \int_0^\infty [K_1(x-y,t-\tau) - K_1(x+y,t-\tau)]G(y,\tau)dyd\tau\\
&\quad + \int_0^\infty [K_1(x-y,0) - K_1(x+y,0)]G(y,t)dy. \qquad (4.35)
\end{aligned}
$$

By making use of the fashion as before, then Lemmas 4.2 and 4.3 help us to reach the goal

$$
\begin{aligned}
&\|z(t)\|_{L^2(\mathbb{R}_+)} = \|V_t(t)\|_{L^2(\mathbb{R}_+)}\\
&\le \left\| \partial_t \int_0^\infty [K_0(x-y,t) - K_0(x+y,t)]\bar V_0(y)dy\right\|_{L^2(\mathbb{R}_+)}\\
&\quad + \left\|\partial_t \int_0^\infty [K_1(x-y,t) - K_1(x+y,t)]\bar z_0(y)dy\right\|_{L^2(\mathbb{R}_+)}\\
&\quad + \int_0^t \left\| \partial_t \int_0^\infty [K_1(x-y,t-\tau) - K_1(x+y,t-\tau)]G(y,\tau)dy\right\|_{L^2(\mathbb{R}_+)} d\tau\\
&\quad + \left\| \int_0^\infty [K_1(x-y,0) - K_1(x+y,0)]G(y,t)dy\right\|_{L^2(\mathbb{R}_+)}\\
&\le C(\|\bar V_0\|_{L^1(\mathbb{R}_+)} + \|\bar V_0\|_{H^3(\mathbb{R}_+)}(1+t)^{-5/4}\\
&\quad + C(\|\bar z_0\|_{L^1(\mathbb{R}_+)} + \|\bar z_0\|_{H^2(\mathbb{R}_+)}(1+t)^{-5/4}\\
&\quad + C\int_0^t (1+t-\tau)^{-5/4}(\|G(\tau)\|_{L^1(\mathbb{R}_+)} + \|G(\tau)\|_{H^2(\mathbb{R}_+)})d\tau\\
&\quad + C(\|G(t)\|_{L^1(\mathbb{R}_+)} + \|G(t)\|_{H^2(\mathbb{R}_+)})\\
&\le C(\|\bar V_0\|_{L^1(\mathbb{R}_+)} + \|\bar V_0\|_{H^3(\mathbb{R}_+)})(1+t)^{-5/4}\\
&\quad + C(\|\bar z_0\|_{L^1(\mathbb{R}_+)} + \|\bar z_0\|_{H^2(\mathbb{R}_+)})(1+t)^{-5/4}\\
&\quad + C\int_0^t (1+t-\tau)^{-5/4}((1+\tau)^{-5/4} + (1+\tau)^{-3/2})d\tau\\
&\quad + C((1+t)^{-5/4} + (1+t)^{-3/2})\\
&\le C(1+t)^{-5/4}. \qquad (4.36)
\end{aligned}
$$

This proved (3.14). □

## 5. Remark

This section is devoted to Marcati–Mei's IBVP paper [10]

$$\begin{cases} v_t - u_x = 0 \\ u_t + p(v)_x = -\alpha u, \quad (x,t) \in \mathbb{R}_+ \times \mathbb{R}_+ \\ (v,u)(x,0) = (v_0,u_0)(x) \to (v_+,u_+) \ \text{ as } x \to \infty \\ v(0,t) = g(t) \to v_+ \ \text{ as } t \to \infty, \end{cases} \tag{5.1}$$

where

$$|g(t) - v_+| = O(1)|v_+ - v_+|(1+t)^{-\gamma_1}, \qquad \gamma_1 > 3/4 \tag{5.2}$$

and the compatibility condition $g(0) = v_0(0)$ holds. Let $\bar{v}(x,t) = \phi(x/\sqrt{1+t})$ be the self-similar solution to the parabolic equation in whole space $x \in (-\infty,\infty)$

$$\alpha\phi_t + p(\phi)_{xx} = 0, \quad (x,t) \in \mathbb{R} \times \mathbb{R}_+,$$

Marcati and Mei in [10] selected a shifted nonlinear diffusion wave

$$(\bar{v},\bar{u})(x+d(t),t) := \Big(\phi, -\frac{1}{\alpha}p(\phi)_x\Big)\Big(\frac{x+d(t)}{\sqrt{1+t}}\Big)$$

as the asymptotic profile of (5.1), where the shift $d(t)$ in $C^3(\mathbb{R}_+)$ satisfies

$$d(t) > 0, \quad \text{for all } \ t \geq 0, \tag{5.3}$$

$$\exp\Big\{ -\alpha c_0\Big(\frac{d(t)}{\sqrt{t+1}}\Big)^2\Big\} \leq O(1)(2+t)^{-\gamma_2}, \qquad \gamma_2 > 3/4, \tag{5.4}$$

$$d'(t)\exp\Big\{ -\alpha c_0\Big(\frac{d(t)}{\sqrt{t+1}}\Big)^2\Big\} \leq O(1)(1+t)^{-(\gamma_2+\frac{1}{2})}\sqrt{\log(2+t)}. \tag{5.5}$$

Under some smoothness and smallness restrictions on the initial data, they proved in [10] the stability as follows

$$\|\partial_x^k(v-\bar{v})(t)\|_{L^2(\mathbb{R}_+)} = O(1)(1+t)^{-(k+1)/2}, \quad k = 0,1,2, \tag{5.6}$$

$$\|\partial_x^k(u-\bar{u})(t)\|_{L^2(\mathbb{R}_+)} = O(1)(1+t)^{-(k+2)/2}, \quad k = 0,1, \tag{5.7}$$

$$\|(v-\bar{v})(t)\|_{L^p(\mathbb{R}_+)} = O(1)(1+t)^{-(\frac{3}{4}-\frac{1}{2p})}, \quad 2 \leq p \leq \infty. \tag{5.8}$$

These rates are not sharp and can be improved as follows.

**Theorem 5.1.** *Suppose that* $v_0 - v_+ \in L^1(\mathbb{R}_+)$, $(V_0,z_0)(x) := \Big(-\int_x^\infty[v_0(y)-\bar{v}(y+d(0),0)]dy, u_0(x)-\bar{u}(x+d(0),0)\Big) \in (H^3(\mathbb{R}_+)\cap L^1(\mathbb{R}_+))\times(H^2(\mathbb{R}_+)\cap L^1(\mathbb{R}_+))$ *and that* $\|v_0-v_+\|_{L^1(\mathbb{R}_+)}+\|V_0\|_{H^3(\mathbb{R}_+)}+\|z_0\|_{H^2(\mathbb{R}_+)}+\|V_0\|_{L^1(\mathbb{R}_+)}+\|z_0\|_{L^1(\mathbb{R}_+)}+|u_+| \ll 1$ *hold. Then there exists a unique time-global solution* $(v,u)(x,t)$ *of the IBVP* (5.1) *such that*

$$v-\bar{v} \in C^k(0,\infty;H^{2-k}(\mathbb{R}_+)), \ \ k = 0,1,2, \quad u-\bar{u} \in C^k(0,\infty;H^{1-k}(\mathbb{R}_+)), \ \ k = 0,1$$

*and*

$$\|\partial_x^k(v-\bar{v})(t)\|_{L^2(\mathbb{R}_+)} = O(1)(1+t)^{-(2k+3)/4}, \quad k=0,1, \tag{5.9}$$

$$\|(v-\bar{v})(t)\|_{L^p(\mathbb{R}_+)} = O(1)(1+t)^{-(2p-1)/(2p)}, \quad 2\le p\le +\infty, \tag{5.10}$$

$$\|(u-\bar{u})(t)\|_{L^2(\mathbb{R}_+)} = O(1)(1+t)^{-5/4}. \tag{5.11}$$

By using the Fourier transform and the energy method as before, we can similarly prove Theorem 5.1. The details are omitted.

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