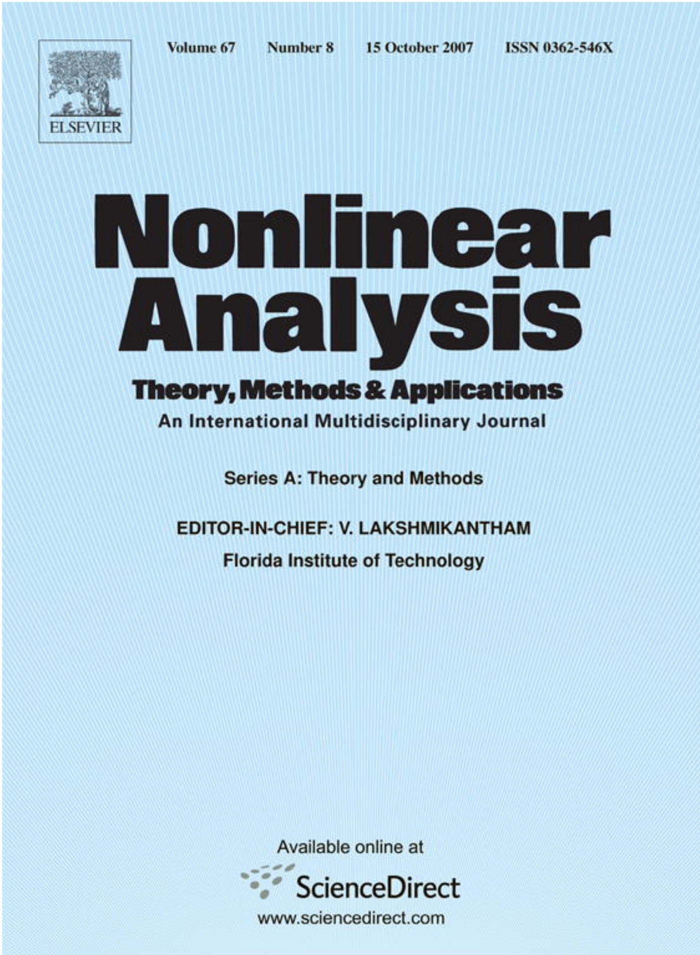
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# Asymptotic behavior of solutions to the Rosenau–Burgers equation with a periodic initial boundary

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**Abstract**

This study focuses on the Rosenau–Burgers equation $u_t + u_{xxxxt} - \alpha u_{xx} + f(u)_x = 0$ with a periodic initial boundary condition. It is proved that with smooth initial value the global solution uniquely exists. Furthermore, for $\alpha > 0$, the global solution converges time asymptotically to the average of the initial value in an exponential form, and the convergence rate is optimal; while for $\alpha = 0$, the unique solution oscillates around the initial average all the time. Finally, the numerical simulations are reported to confirm the theoretical results.


*MSC:* 35Q53; 35B40; 76B15

*Keywords:* Rosenau–Burgers equation; Periodic boundary condition; Convergence; Oscillation

## 1. Introduction and main results

In the study of the dynamics of dense discrete systems, the cases of wave–wave and wave–wall interactions cannot be treated using the well-known KdV equations. To overcome this shortcoming, Rosenau [17] proposed the so-called Rosenau equation

$$u_t + u_{xxxxt} + u_x + uu_x = 0.$$

Since then, much work has been done on the solution existence and uniqueness, as well as numerical schemes by the Galerkin method; cf. [3–6,8,13–16], and the references therein. On the other hand, for the further consideration of the dissipation in space for the dynamic system, such as the phenomenon of bore propagation and the water waves, the viscous term $-\alpha u_{xx}$ needs to be included:

$$u_t + u_{xxxxt} - \alpha u_{xx} + u_x + uu_x = 0,$$

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with $\alpha > 0$. This equation is usually called the Rosenau–Burgers equation, because its dissipative effect is the same as that in the Burgers equation

$$u_t - \alpha u_{xx} + u_x + uu_x = 0.$$

The asymptotic behavior of the solution for the Cauchy problem to the Rosenau–Burgers equation, in particular, the stability of traveling waves and diffusion waves, have been well studied in [7,10,11].

Subsequently, this study focuses on the periodic initial–boundary value problem for the generalized Rosenau–Burgers equation:

$$\begin{cases} u_t + u_{xxxxt} - \alpha u_{xx} + f(u)_x = 0, & x \in \mathbf{R},\ t \in \mathbf{R}_+, \\ u|_{t=0} = u_0(x) & x \in \mathbf{R}, \\ u_0(x) = u_0(x + 2L), & x \in \mathbf{R}, \end{cases} \tag{1.1}$$

where $f(u)$ is a smooth function of $u$, $L > 0$, and $2L$ is the period of the initial value $u_0(x)$. The main purpose is to investigate the asymptotic behavior of the solution $u(x,t)$ to the periodic IBVP (1.1).

On the basis of the $2L$-periodicity, the solution $u(x,t)$ on the whole number line, $-\infty < x < \infty$, can be regarded as a $2L$-periodic extension of $u(t,x)$ on $[0, 2L]$. Therefore, the investigation concentrates on Eq. (1.1) on the bounded interval $[0, 2L]$. Integrating (1.1) with respect to $x$ over the interval $[0, 2L]$, and noticing the periodicity of $u(x,t)$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{2L} u(t,x)\mathrm{d}x = 0,$$

which gives

$$\int_0^{2L} u(t,x)\mathrm{d}x = \int_0^{2L} u_0(x)\mathrm{d}x. \tag{1.2}$$

Let $m_0$ be the average of the initial value $u_0(x)$ over the interval $[0, 2L]$

$$m_0 := \frac{1}{2L}\int_0^{2L} u_0(x)\mathrm{d}x. \tag{1.3}$$

Then (1.2) with (1.3) implies

$$\int_0^{2L} [u(t,x) - m_0]\mathrm{d}x = 0. \tag{1.4}$$

For the asymptotic profile of the solution $u(t,x)$, Eq. (1.1) is linearized around $m_0$:

$$U_t + U_{xxxxt} - \alpha U_{xx} + f'(m_0)U_x = 0, \tag{1.5}$$

where $U(t,x) := u(t,x) - m_0$. It is known that (1.5) admits a solution in the form

$$U(t,x) = A\mathrm{e}^{\lambda t + \mathrm{i}\omega x}, \tag{1.6}$$

where $A$ is a constant, $\lambda$ is the wave frequency (which may be complex), and $\omega$ is the wavenumber satisfying the periodic condition

$$\mathrm{e}^{\mathrm{i}\omega x} = \mathrm{e}^{\mathrm{i}\omega(x+2L)}.$$

Substituting (1.6) into (1.5) yields

$$\lambda + (\mathrm{i}\omega)^4\lambda - \alpha(\mathrm{i}\omega)^2 + f'(m_0)\mathrm{i}\omega = 0,$$

which is solved as

$$\lambda = \frac{-\alpha\omega^2 - \mathrm{i}f'(m_0)\omega}{1 + \omega^4}.$$

The real part of $\lambda$ is

$$\mathrm{Re}(\lambda) = -\frac{\alpha\omega^2}{1+\omega^4}. \tag{1.7}$$

When $\alpha > 0$, $\mathrm{Re}(\lambda) < 0$, then $U(t,x)$ converges to zero as time $t$ approaches infinity:

$$|U(t,x)| \sim O(1)\mathrm{e}^{\mathrm{Re}(\lambda)t} = O(1)\mathrm{e}^{-\frac{\alpha\omega^2}{1+\omega^4}t}. \tag{1.8}$$

When $\alpha = 0$, $\mathrm{Re}(\lambda) = 0$, then

$$U(t,x) \sim O(1)\sin(\mathrm{Im}(\lambda)t + \omega x) + O(1)\cos(\mathrm{Im}(\lambda)t + \omega x),$$

where $\mathrm{Im}(\lambda) = f'(m_0)\omega/(1+\omega^4)$. Therefore, $U(t,x)$ oscillates around zero all the time.

This study, as shown later, concludes that similar results hold for the nonlinear case. Before the main results, some useful notation is introduced.

**Notations.** Let $L^2_{\mathrm{per}}(R)$ denote the space of square integrable $2L$-periodic functions on $R$

$$L^2_{\mathrm{per}}(R) = \{v(x) | v(x) = v(x+2L) \text{ for all } x \in R, \text{ and } v(x) \in L^2(0,2L) \text{ for } x \in [0,2L]\}$$

with the norm

$$\|v\|_{L^2_{\mathrm{per}}} = \left(\int_0^{2L} v^2(x)\mathrm{d}x\right)^{1/2}.$$

The inner product of $L^2_{\mathrm{per}}(R)$ is defined as

$$\langle\phi,\psi\rangle = \int_0^{2L} \phi(x)\psi(x)\mathrm{d}x.$$

$H^k_{\mathrm{per}}(R)$ $(k \geq 0)$ is the periodic Sobolev space of $L^2_{\mathrm{per}}$-functions $v(x)$ defined on $R$ whose derivatives $\partial^i_x v$ $(i = 1,\ldots,k)$ also belong to $L^2_{\mathrm{per}}(R)$, with the norm

$$\|v\|_{H^k_{\mathrm{per}}} = \left(\sum_{i=0}^{k}\int_0^{2L} |\partial^i_x v(x)|^2\mathrm{d}x\right)^{1/2}.$$

For $T > 0$ and a Banach space $\mathcal{B}$, $C^k(0,T;\mathcal{B})$ denotes the space of $\mathcal{B}$-valued $k$-times-continuously differentiable functions on $[0,T]$. The corresponding spaces of $\mathcal{B}$-valued functions on $[0,\infty)$ are defined similarly.

Now we are ready to state the main results.

**Theorem 1.1** (*Exponential Convergence*). *Let $\alpha > 0$ and $u_0(x) \in H^2_{\mathrm{per}}(R)$. For the periodic IBVP (1.1) there exists a unique and global solution satisfying*

$$u(t,x) - m_0 \in C(0,\infty; H^2_{\mathrm{per}}(R))$$

*and*

$$\|(u-m_0)(t)\|_{H^2_{\mathrm{per}}} \leq \frac{\sqrt{\pi^2+L^2}}{\pi}\|(u_0-m_0)\|_{H^2_{\mathrm{per}}}, \tag{1.9}$$

$$\sup_{x\in[0,2L]} |u(t,x) - m_0| = O(1)\mathrm{e}^{-\gamma t}, \tag{1.10}$$

*for all $t \in [0,\infty)$, where*

$$\gamma = \frac{\alpha\omega_1^2}{1+\omega_1^4} = \frac{\alpha\pi^2L^2}{\pi^4+L^4}, \tag{1.11}$$

*and $\omega_1 = \frac{\pi^2}{L^2}$ is the smallest wavenumber (eigenvalue) which satisfies $\mathrm{e}^{\mathrm{i}\omega x} = \mathrm{e}^{\mathrm{i}\omega(x+2L)}$ and $\int_0^{2L} \mathrm{e}^{\mathrm{i}\omega x}\mathrm{d}x = 0$.*

**Theorem 1.2** (*Oscillatory Divergence*). *Let $\alpha = 0$ and $u_0(x) \in H^2_{\text{per}}(R)$. For the periodic IBVP (1.1) there exists a unique and global solution satisfying*

$$u(t,x) - m_0 \in C(0,\infty; H^2_{\text{per}}(R))$$

*and*

$$\|(u-m_0)(t)\|^2_{L^2_{\text{per}}} + \|(u-m_0)_{xx}(t)\|^2_{L^2_{\text{per}}} = \|(u_0-m_0)\|^2_{L^2_{\text{per}}} + \|(u_0-m_0)_{xx}\|^2_{L^2_{\text{per}}} \tag{1.12}$$

*for all $t \in [0,\infty)$. In particular, if $u_0(x) \not\equiv m_0$, then the solution $u(t,x)$ oscillates around $m_0$ all the time.*

**Remark 1.3.** 1. By Sobolev's embedding theorem $H^2_{\text{per}} \hookrightarrow C^1_{\text{per}}$, the solution $u(t,x)$ in $C(0,\infty; H^2_{\text{per}}(R))$ is continuous in time $t$ and differentiable in space $x$, i.e., $u(t,x) \in C(0,\infty; C^1_{\text{per}}(R))$, but it is not a classical solution to Eq. (1.1). Such a solution is called a strong solution.

2. We show the global existence of the strong solution in Theorems 1.1 and 1.2 without the smallness assumption on the initial value, namely, we obtain the global existence of the solution for any "large" initial data.

3. Compared to the decay rate (1.8) of the linear case, the rate (1.10) obtained for the nonlinear case in Theorem 1.1 is optimal, which also matches the optimal convergence rate for the periodic initial–boundary value problem of the $2 \times 2$ system of the BBM–Burgers equations given by Bisognin et al. [2].

4. Although the solution $u(t,x)$ in the case $\alpha = 0$ has been recognized to be oscillatory at all time around $m_0$, this does not mean that the solution is not stable. In fact, as shown in Section 4 numerically, we conjecture that $u(t,x)$ converges to its periodic traveling wave $\phi(x-ct)$. For such a wave-stability problem, this is still open at this moment.

## 2. Proof of Theorem 1.1

Let

$$v(t,x) = u(t,x) - m_0. \tag{2.1}$$

With (1.4), periodic IBVP (1.1) is reduced to

$$\begin{cases} v_t + v_{xxxxt} - \alpha v_{xx} + F(v)_x = 0, & x \in \mathbf{R},\ t \in \mathbf{R}_+, \\ v|_{t=0} = u_0(x) - m_0 =: v_0(x), & x \in \mathbf{R}, \\ v_0(x) = v_0(x+2L), & x \in \mathbf{R}, \\ \displaystyle\int_0^{2L} v_0(x)\mathrm{d}x = 0, \end{cases} \tag{2.2}$$

where

$$F(v) = f(v+m_0) - f(m_0). \tag{2.3}$$

Corresponding to Theorem 1.1, the equivalent result for the new periodic IBVP (2.2) is as follows.

**Theorem 2.1.** *Let $\alpha > 0$ and $v_0(x) \in H^2_{\text{per}}(R)$. For the periodic IBVP (2.2) there exists a unique and global solution satisfying*

$$v(t,x) \in C(0,\infty; H^2_{\text{per}}(R))$$

*and*

$$\|v(t)\|_{H^2_{\text{per}}} \le \frac{\sqrt{L^2+\pi^2}}{\pi}\|v_0\|_{H^2_{\text{per}}}, \tag{2.4}$$

$$\sup_{x\in[0,2L]} |v(t,x)| = O(1)\mathrm{e}^{-\gamma t} \tag{2.5}$$

*for all $t \ge 0$, where $\gamma$ is given in (1.11).*

When Theorem 2.1 is proved, Theorem 1.1 is obtained automatically. Therefore, the main objective in this section is to prove Theorem 2.1. The method of continuity extension with the $L^2$-energy estimates is adopted. The solution space is defined as

$$X_M(t_1,t_2)=\left\{v(t,x)|v(t,x)\in C(t_1,t_2;H^2_{\mathrm{per}}(R)),\quad \sup_{t\in[t_1,t_2]}\|v(t)\|_{H^2_{\mathrm{per}}}\le M\right\},$$

where $M>0$ and $t_2\ge t_1\ge 0$ are constants.

**Proposition 2.2** (*Local Existence*). *For $\alpha\ge 0$, consider the periodic IBVP at the initial time $\tau\ge 0$*

$$\begin{cases} v_t+v_{xxxxt}-\alpha v_{xx}+F(v)_x=0, & x\in\mathbf{R},\ t\in[\tau,\infty),\\ v|_{t=\tau}=v_\tau(x), & x\in\mathbf{R},\\ v_\tau(x)=v_\tau(x+2L), & x\in\mathbf{R},\\ \displaystyle\int_0^{2L}v_\tau(x)\mathrm{d}x=0. \end{cases} \tag{2.6}$$

*Let $v_\tau(x)\in H^2_{\mathrm{per}}(R)$ and $M>0$ be such that $\|v_\tau\|_{H^2_{\mathrm{per}}}\le M$. Then there exists a number $t_0=t_0(M)>0$ such that the periodic IBVP (2.6) has a unique solution $v(t,x)$ in $X_{2M}(\tau,\tau+t_0)$.*

This proposition can be proved by a standard iteration method; cf. [9,12]. Therefore the detail is omitted here.

The following Poincaré inequality is needed in the *a priori* estimates for the solution.

**Lemma 2.3** (*Poincaré Inequality*). *Consider (2.2) with $\alpha\ge 0$. Let $T>0$ and $v(t,x)\in C(0,T;H^2_{\mathrm{per}}(R))$ be a solution of (2.2). Then*

$$\frac{\pi}{L}\|\partial_x^k v(t)\|_{L^2_{\mathrm{per}}}\le\|\partial_x^{k+1}v(t)\|_{L^2_{\mathrm{per}}},\quad k=0,1. \tag{2.7}$$

**Proof.** First, an orthonormal basis is constructed for the space whose functions satisfy the periodic condition

$$\Phi=\left\{\phi|\phi(x)=\phi(x+2L),\int_0^{2L}\phi(x)\mathrm{d}x=0\right\}. \tag{2.8}$$

Consider the eigenvalue problem

$$\begin{cases} -\phi''=\omega^2\phi,\\ \phi(x)=\phi(x+2L),\\ \displaystyle\int_0^{2L}\phi(x)\mathrm{d}x=0. \end{cases} \tag{2.9}$$

The eigenvalues and the corresponding normalized eigenfunctions are solved in the form

$$\begin{cases} \text{eigenvalues: } \omega_k=\dfrac{k\pi}{L},\\ \text{eigenfunctions: } \begin{cases} \phi_{1,k}=\dfrac{1}{\sqrt{L}}\cos\omega_k x,\\ \phi_{2,k}=\dfrac{1}{\sqrt{L}}\sin\omega_k x, \end{cases} \end{cases} \qquad k=1,2,\ldots, \tag{2.10}$$

with inner products

$$\langle\phi_{i,k},\phi_{j,l}\rangle=\int_0^{2L}\phi_{i,k}(x)\phi_{j,l}(x)\mathrm{d}x=\begin{cases}1, & \text{for } i=j,k=l,\\ 0, & \text{otherwise.}\end{cases} \tag{2.11}$$

Thus, the sequence $\{\phi_{i,k}\}$ ($i=1,2$ and $k=1,2,3,\ldots$) forms the orthonormal basis of the space $\Phi$ defined in (2.8).

Notice that the solution $v(t,x) \in C(0,T; H^2_{\mathrm{per}})$ satisfies $v(t,x) = v(t, x+2L)$ and $\int_0^{2L} v(t,x)\mathrm{d}x = 0$. Therefore, $v(t,x) \in \Phi$, and can be expressed in the form

$$v(t,x) = \sum_{k=1}^{\infty}[a_{1,k}(t)\phi_{1,k}(x) + a_{2,k}(t)\phi_{2,k}(x)], \tag{2.12}$$

where $a_{i,k}(t)$ ($i = 1,2$ and $k = 1,2,3,\ldots$) are the so-called Fourier coefficients determined as

$$a_{i,k}(t) = \langle v(t,x), \phi_{i,k}(x)\rangle = \int_0^{2L} v(t,x)\phi_{i,k}(x)\mathrm{d}x, \quad i = 1,2,\ k = 1,2,\ldots. \tag{2.13}$$

Differentiating both sides of (2.12) with respect to $x$ yields

$$v_x(t,x) = \sum_{k=1}^{\infty}\omega_k[-a_{1,k}(t)\phi_{2,k}(x) + a_{2,k}(t)\phi_{1,k}(x)]. \tag{2.14}$$

Now, taking the inner product of $v(t,x)$ with itself and using (2.11), we then have

$$\begin{aligned}
\|v(t)\|^2_{L^2_{\mathrm{per}}} &= \langle v(t,x), v(t,x)\rangle \\
&= \left\langle \sum_{k=1}^{\infty}[a_{1,k}(t)\phi_{1,k}(x) + a_{2,k}(t)\phi_{2,k}(x)], \sum_{l=1}^{\infty}[a_{1,l}(t)\phi_{1,l}(x) + a_{2,l}(t)\phi_{2,l}(x)]\right\rangle \\
&= \sum_{k=1}^{\infty}\sum_{l=1}^{\infty}[a_{1,k}(t)a_{1,l}(t)\langle\phi_{1,k},\phi_{1,l}\rangle + a_{1,k}(t)a_{2,l}(t)\langle\phi_{1,k},\phi_{2,l}\rangle \\
&\quad + a_{2,k}(t)a_{1,l}(t)\langle\phi_{2,k},\phi_{1,l}\rangle + a_{2,k}(t)a_{2,l}(t)\langle\phi_{2,k},\phi_{2,l}\rangle] \\
&= \sum_{k=1}^{\infty}[(a_{1,k}(t))^2 + (a_{2,k}(t))^2].
\end{aligned} \tag{2.15}$$

Similarly, from (2.14) we obtain

$$\|v_x(t)\|^2_{L^2_{\mathrm{per}}} = \sum_{k=1}^{\infty}\omega_k^2[(a_{1,k}(t))^2 + (a_{2,k}(t))^2]. \tag{2.16}$$

Since $\omega_k > \omega_1 = \frac{\pi}{L}$ for $k = 2,3,\ldots$, (2.15) and (2.16) imply

$$\begin{aligned}
\|v_x(t)\|^2_{L^2_{\mathrm{per}}} &= \sum_{k=1}^{\infty}\omega_k^2[(a_{1,k}(t))^2 + (a_{2,k}(t))^2] \\
&\ge \omega_1^2\sum_{k=1}^{\infty}[(a_{1,k}(t))^2 + (a_{2,k}(t))^2] \\
&= \frac{\pi^2}{L^2}\|v(t)\|^2_{L^2}.
\end{aligned} \tag{2.17}$$

Similarly, we can prove

$$\|v_{xx}(t)\|^2_{L^2_{\mathrm{per}}} \ge \frac{\pi^2}{L^2}\|v_x(t)\|^2_{L^2_{\mathrm{per}}}. \tag{2.18}$$

Hence (2.7) is proved. □

**Proposition 2.4** (*A Priori Estimate*). *Let $T > 0$ and $M > 0$ be arbitrary fixed constants, and $v(t,x) \in X_M(0,T)$ be a solution of (2.2). Then*

$$\|v(t)\|_{H^2_{\mathrm{per}}} \le \frac{\sqrt{L^2+\pi^2}}{\pi}\|v_0\|_{H^2_{\mathrm{per}}}, \quad t \in [0,T], \tag{2.19}$$

$$\sup_{x\in[0,2L]} |v(t,x)| = O(1)\mathrm{e}^{-\gamma t}, \quad t \in [0,T]. \tag{2.20}$$

**Proof.** Let $v(t,x) \in C(0,T; H^2_{\mathrm{per}}(R))$ be a solution of (2.2). The *a priori* estimates (2.19) and (2.20) will be proved by the $L^2$-energy method. Since the strong solution $v(t,x)$ lacks the regularity for $v_{xxxxt}$ since $v_0(x) \in H^2_{\mathrm{per}}$, $v(t,x)$ cannot be treated directly in the form of differential equation (2.2). In order to overcome this shortcoming, the periodic IBVP (2.2) is first investigated with a smooth enough initial value. The regularity of the solution $v(t,x)$ depends on the smoothness of its initial value $v_0(x)$. That is to say: the smoother the initial value, the smoother the solution.

Smooth the initial value $v_0(x)$ as

$$v_0^\varepsilon(x) = \int_{-\infty}^{\infty} J_\varepsilon(x-y)v_0(y)\mathrm{d}y,$$

where $\varepsilon > 0$ is a constant, $J_\varepsilon(x) \in C_0^\infty(R)$ is the mollifier satisfying $J_\varepsilon(x) = 0$ for $|x| \geq \varepsilon$ and $\int_{-\infty}^{\infty} J_\varepsilon(x)\mathrm{d}x = 1$. It is known that $v_0^\varepsilon(x) \in C^\infty(R)$ and

$$\lim_{\varepsilon\to 0^+} v_0^\varepsilon(x) = v_0(x) \tag{2.21}$$

for all $x \in R$; cf. [1]. The periodic conditions still hold:

$$\begin{aligned} v_0^\varepsilon(x+2L) &= \int_{-\infty}^{\infty} J_\varepsilon(x+2L-y)v_0(y)\mathrm{d}y \quad \text{[by change of variables: } z = y-2L] \\ &= \int_{-\infty}^{\infty} J_\varepsilon(x-z)v_0(z+2L)\mathrm{d}z \quad \text{[by periodicity: } v_0(x+2L) = v_0(z)] \\ &= \int_{-\infty}^{\infty} J_\varepsilon(x-z)v_0(z)\mathrm{d}z \\ &= v_0^\varepsilon(x), \end{aligned} \tag{2.22}$$

and

$$\begin{aligned} \int_0^{2L} v_0^\varepsilon(x)\mathrm{d}x &= \int_0^{2L} \left(\int_{-\infty}^{\infty} J_\varepsilon(x-y)v_0(y)\mathrm{d}y\right)\mathrm{d}x \\ &= \int_0^{2L} \left(\int_{-\infty}^{\infty} J_\varepsilon(y)v_0(x-y)\mathrm{d}y\right)\mathrm{d}x \\ &= \int_{-\infty}^{\infty} J_\varepsilon(y)\left(\int_0^{2L} v_0(x-y)\mathrm{d}x\right)\mathrm{d}y \\ &= \int_{-\infty}^{\infty} J_\varepsilon(y)\left(\int_y^{2L+y} v_0(z)\mathrm{d}z\right)\mathrm{d}y \quad \left[\text{by periodicity: } \int_y^{2L+y} v_0(z)\mathrm{d}z = 0\right] \\ &= \int_{-\infty}^{\infty} J_\varepsilon(y)\cdot 0\,\mathrm{d}y = 0. \end{aligned} \tag{2.23}$$

Let $v^\varepsilon(t,x)$ be the local solution of the periodic IBVP

$$\begin{cases} v_t^\varepsilon + v_{xxxxt}^\varepsilon - \alpha v_{xx}^\varepsilon + F(v^\varepsilon)_x = 0, \quad x \in \mathbf{R},\ t \in [0,T], \\ v^\varepsilon|_{t=0} = v_0^\varepsilon(x), \quad x \in \mathbf{R}, \\ v_0^\varepsilon(x) = v_0^\varepsilon(x+2L), \quad x \in \mathbf{R}, \\ \displaystyle\int_0^{2L} v_0^\varepsilon(x)\mathrm{d}x = 0. \end{cases} \tag{2.24}$$

Since the initial value $v_0^\varepsilon(x) \in C^\infty_{\mathrm{per}}(R)$, the solution $v^\varepsilon(t,x)$ has good enough regularities in $x$ and $t$, and

$$\lim_{\varepsilon\to 0^+} v^\varepsilon(t,x) = v(t,x) \tag{2.25}$$

for all $(t,x) \in [0,T] \times R$. The convergence (2.25) can be proved like in [9,12], and the details are omitted here.

We now prove the estimates (2.19) and (2.20) for $v^\varepsilon(t,x)$. Multiplying the first equation of (2.24) by $v^\varepsilon(x,t)$, we obtain

$$\frac{1}{2}\{(v^\varepsilon)^2+(v^\varepsilon_{xx})^2\}_t+\alpha(v^\varepsilon_x)^2+\{v^\varepsilon v^\varepsilon_{xxxt}-v^\varepsilon_x v^\varepsilon_{xxt}-\alpha v^\varepsilon v^\varepsilon_x+G(v^\varepsilon)\}_x=0, \tag{2.26}$$

where $G(v^\varepsilon)$ is an antiderivative of $F'(v^\varepsilon)v^\varepsilon$, i.e., $G'(v^\varepsilon)=F'(v^\varepsilon)v^\varepsilon$. Integrating (2.26) over $[0,2L]\times[0,t]$ with respect to $x$ and $t$, and using the periodicity of $v^\varepsilon$ such that the last term of the left-hand side of (2.26) disappears after integration with respect to $x$, we then have

$$\|v^\varepsilon(t)\|^2_{L^2_{\mathrm{per}}}+\|v^\varepsilon_{xx}(t)\|^2_{L^2_{\mathrm{per}}}+2\alpha\int_0^t\|v^\varepsilon_x(\tau)\|^2_{L^2_{\mathrm{per}}}\mathrm{d}\tau=\|v^\varepsilon_0\|^2_{L^2_{\mathrm{per}}}+\|v^\varepsilon_{0,xx}\|^2_{L^2_{\mathrm{per}}},\quad t\in[0,T]. \tag{2.27}$$

Applying the Poincaré inequalities (2.7), we obtain

$$\begin{aligned}\|v^\varepsilon(t)\|^2_{L^2_{\mathrm{per}}}+\|v^\varepsilon_{xx}(t)\|^2_{L^2_{\mathrm{per}}}&=\|v^\varepsilon(t)\|^2_{L^2_{\mathrm{per}}}+\frac{L^2}{\pi^2+L^2}\|v^\varepsilon_{xx}(t)\|^2_{L^2_{\mathrm{per}}}+\frac{\pi^2}{\pi^2+L^2}\|v^\varepsilon_{xx}(t)\|^2_{L^2_{\mathrm{per}}}\\&\ge\|v^\varepsilon(t)\|^2_{L^2_{\mathrm{per}}}+\frac{L^2}{\pi^2+L^2}\cdot\frac{\pi^2}{L^2}\|v^\varepsilon_x(t)\|^2_{L^2_{\mathrm{per}}}+\frac{\pi^2}{\pi^2+L^2}\|v^\varepsilon_{xx}(t)\|^2_{L^2_{\mathrm{per}}}\\&=\|v^\varepsilon(t)\|^2_{L^2_{\mathrm{per}}}+\frac{\pi^2}{\pi^2+L^2}\|v^\varepsilon_x(t)\|^2_{L^2_{\mathrm{per}}}+\frac{\pi^2}{\pi^2+L^2}\|v^\varepsilon_{xx}(t)\|^2_{L^2_{\mathrm{per}}}\\&\ge\frac{\pi^2}{\pi^2+L^2}\|v^\varepsilon(t)\|^2_{H^2_{\mathrm{per}}},\quad t\in[0,T].\end{aligned} \tag{2.28}$$

Substituting (2.28) into (2.27) and dropping the positive term $2\alpha\int_0^t\|v^\varepsilon_x(\tau)\|^2_{L^2_{\mathrm{per}}}\mathrm{d}\tau$ implies (2.19) for the smooth solution $v^\varepsilon(x,t)$, i.e.,

$$\|v^\varepsilon(t)\|^2_{H^2_{\mathrm{per}}}\le\frac{\pi^2+L^2}{\pi^2}\|v^\varepsilon_0\|^2_{H^2_{\mathrm{per}}},\quad t\in[0,T]. \tag{2.29}$$

On the other hand, again, applying the Poincaré inequality (2.7), we have

$$\|v^\varepsilon_{xx}(t)\|^2_{L^2_{\mathrm{per}}}\ge\frac{\pi^2}{L^2}\|v^\varepsilon_x(t)\|^2_{L^2_{\mathrm{per}}} \tag{2.30}$$

and

$$\begin{aligned}2\alpha\int_0^t\|v^\varepsilon_x(\tau)\|^2_{L^2_{\mathrm{per}}}\mathrm{d}\tau&=2\alpha\int_0^t\left\{\frac{\pi^4}{\pi^4+L^4}\|v^\varepsilon_x(\tau)\|^2_{L^2_{\mathrm{per}}}+\frac{L^4}{\pi^4+L^4}\|v^\varepsilon_x(\tau)\|^2_{L^2_{\mathrm{per}}}\right\}\mathrm{d}\tau\\&\ge2\alpha\int_0^t\left\{\frac{\pi^4}{\pi^4+L^4}\|v^\varepsilon_x(\tau)\|^2_{L^2_{\mathrm{per}}}+\frac{L^4}{\pi^4+L^4}\cdot\frac{\pi^2}{L^2}\|v^\varepsilon(\tau)\|^2_{L^2_{\mathrm{per}}}\right\}\mathrm{d}\tau\\&=\frac{2\alpha\pi^2L^2}{\pi^4+L^4}\int_0^t\left\{\frac{\pi^2}{L^2}\|v^\varepsilon_x(\tau)\|^2_{L^2_{\mathrm{per}}}+\|v^\varepsilon(\tau)\|^2_{L^2_{\mathrm{per}}}\right\}\mathrm{d}\tau\\&=2\gamma\int_0^t\left\{\|v^\varepsilon(\tau)\|^2_{L^2_{\mathrm{per}}}+\frac{\pi^2}{L^2}\|v^\varepsilon_x(\tau)\|^2_{L^2_{\mathrm{per}}}\right\}\mathrm{d}\tau,\end{aligned} \tag{2.31}$$

where $\gamma=\frac{\alpha\pi^2L^2}{\pi^4+L^4}$. Substituting (2.30) and (2.31) into (2.27) yields

$$\|v^\varepsilon(t)\|^2_{L^2_{\mathrm{per}}}+\frac{\pi^2}{L^2}\|v^\varepsilon_x(t)\|^2_{L^2_{\mathrm{per}}}+2\gamma\int_0^t\left\{\|v^\varepsilon(\tau)\|^2_{L^2_{\mathrm{per}}}+\frac{\pi^2}{L^2}\|v^\varepsilon_x(\tau)\|^2_{L^2_{\mathrm{per}}}\right\}\mathrm{d}\tau\le\|v^\varepsilon_0\|^2_{H^2_{\mathrm{per}}},\quad t\in[0,T]. \tag{2.32}$$

Applying Gronwall's inequality to (2.32), we have

$$\|v^\varepsilon(t)\|^2_{L^2_{\mathrm{per}}}+\frac{\pi^2}{L^2}\|v^\varepsilon_x(t)\|^2_{L^2_{\mathrm{per}}}\le\|v^\varepsilon_0\|^2_{H^2_{\mathrm{per}}}\mathrm{e}^{-2\gamma t},\quad t\in[0,T],$$

which implies

$$\|v^{\varepsilon}(t)\|^2_{H^1_{\mathrm{per}}} \le C_1\|v_0^{\varepsilon}\|^2_{H^2_{\mathrm{per}}} \mathrm{e}^{-2\gamma t}, \quad t \in [0, T], \tag{2.33}$$

where $C_1 = 1/\min\{1, \pi^2/L^2\}$. Using Sobolev's embedding inequality $H^1 \hookrightarrow C^0$, we have

$$\sup_{x\in[0,2L]} |v^{\varepsilon}(x,t)| \le C\|v_0^{\varepsilon}\|_{H^2_{\mathrm{per}}} \mathrm{e}^{-\gamma t}, \quad t \in [0, T], \tag{2.34}$$

which proves (2.20) for $v^{\varepsilon}(t, x)$.

With (2.25), i.e., $\lim_{\varepsilon\to 0^+} v^{\varepsilon}(t,x) = v(t,x)$, (2.29) and (2.34) imply (2.19) and (2.20) for the strong solution $v(t, x)$. The proof is complete. □

**Proof of Theorem 2.1.** For any given $2L$-periodic initial value $v_0(x) \in H^2_{\mathrm{per}}(R)$, we choose $M$ such that

$$\|v_0\|_{H^2_{\mathrm{per}}} \le \frac{M\pi}{\sqrt{\pi^2+L^2}} \le M, \quad \text{i.e.,} \ \frac{\pi^2+L^2}{\pi^2}\|v_0\|^2_{H^2_{\mathrm{per}}} \le M^2. \tag{2.35}$$

By Proposition 2.2 with $\tau = 0$, for the periodic IBVP (2.2) there exists a unique local solution $v(t, x)$ in $X_{2M}(0, t_0)$ for the time determined, $t_0 = t_0(M) > 0$. For such a local solution, applying Proposition 2.4, from (2.35) we have

$$\|v(t)\|^2_{H^2_{\mathrm{per}}} \le \frac{\pi^2+L^2}{\pi^2}\|v_0\|^2_{H^2_{\mathrm{per}}} \le M^2, \quad t \in [0, t_0] \tag{2.36}$$

and

$$\sup_{x\in[0,2L]} |v(t,x)| = O(1)\mathrm{e}^{-\gamma t}, \quad t \in [0, t_0].$$

Therefore, $v(t, x) \in X_M(0, t_0)$. Since $\|v(t_0)\|_{H^2_{\mathrm{per}}} \le M$ (see (2.36)), we apply Proposition 2.2 with $\tau = t_0$ to obtain the solution $v$ in $X_{2M}(t_0, 2t_0)$, namely, we extend the existence interval of the solution $v(t, x)$ from $[0, t_0]$ to $[0, 2t_0]$. For $v(t, x) \in X_{2M}(0, 2t_0)$, again we use Proposition 2.4 to obtain

$$\|v(t)\|^2_{H^2} \le \frac{\pi^2+L^2}{\pi^2}\|v_0\|^2_{H^2_{\mathrm{per}}} \le M^2, \quad t \in [0, 2t_0]$$

and

$$\sup_{x\in[0,2L]} |v(t,x)| = O(1)\mathrm{e}^{-\gamma t}, \quad t \in [0, 2t_0].$$

We prove $v(t, x) \in X_M(0, 2t_0)$.

Repeating the above procedure, we prove that $v(t, x) \in X_M(0, +\infty)$ and

$$\|v(t)\|^2_{H^2_{\mathrm{per}}} \le \frac{\pi^2+L^2}{\pi^2}\|v_0\|^2_{H^2_{\mathrm{per}}} \le M^2, \quad t \in [0, +\infty)$$

and

$$\sup_{x\in[0,2L]} |v(t,x)| = O(1)\mathrm{e}^{-\gamma t}, \quad t \in [0, +\infty).$$

The proof of Theorem 2.1 is complete. □

## 3. Proof of Theorem 1.2

With $v(t, x) = u(t, x) - m_0$, the periodic IBVP (1.1) with $\alpha = 0$ is reduced to

$$\begin{cases} v_t + v_{xxxxt} + F(v)_x = 0, \quad x \in \mathbf{R},\ t \in \mathbf{R}_+, \\ v|_{t=0} = u_0(x) - m_0 =: v_0(x), \quad x \in \mathbf{R}, \\ v_0(x) = v_0(x + 2L), \quad x \in \mathbf{R}, \\ \displaystyle\int_0^{2L} v_0(x)\mathrm{d}x = 0, \end{cases} \tag{3.1}$$

where $F(v) = f(v + m_0) - f(m_0)$ as in (2.3).

The corresponding oscillatory divergence around zero of the solution $v(t, x)$ to (3.1) is as follows.

**Theorem 3.1.** *Let $\alpha = 0$ and $v_0(x) \in H^2_{\text{per}}(R)$. For the periodic IBVP (3.1) there exists a unique and global solution satisfying*

$$v(t, x) \in C(0, \infty; H^2_{\text{per}}(R))$$

*and*

$$\|v(t)\|^2_{L^2_{\text{per}}} + \|v_{xx}(t)\|^2_{L^2_{\text{per}}} = \|v_0\|^2_{L^2_{\text{per}}} + \|v_{0,xx}\|^2_{L^2_{\text{per}}} \tag{3.2}$$

*for all $t \in [0, \infty)$. In particular, if $v_0(x) \not\equiv 0$, then the solution $v(t, x)$ oscillates around zero at all time.*

Once Theorem 3.1 is proved, Theorem 1.2 can be easily obtained. As shown in the previous section, the local existence of the solution has been given in Proposition 2.2 for the case $\alpha = 0$, Theorem 3.1 can then be proved similarly by using the energy method based on the local existence and the following *a priori* estimates.

**Proposition 3.2** (*A Priori Estimate*). *Let $T > 0$ and $M > 0$ be arbitrary fixed constants, and $v(t, x) \in X_M(0, T)$ be a solution of (3.1), where the solution space $X_M(0, T)$ is defined as in the previous section. Then*

$$\|v(t)\|^2_{L^2_{\text{per}}} + \|v_{xx}(t)\|^2_{L^2_{\text{per}}} = \|v_0\|^2_{L^2_{\text{per}}} + \|v_{0,xx}\|^2_{L^2_{\text{per}}}, \quad t \in [0, T], \tag{3.3}$$

$$\|v(t)\|_{H^2_{\text{per}}} \le \frac{\sqrt{L^2 + \pi^2}}{\pi} \|v_0\|_{H^2_{\text{per}}}, \quad t \in [0, T]. \tag{3.4}$$

**Proof.** As shown in Propositions 2.4 and 3.2 can be treated similarly. Let $v(t, x) \in X_M(0, T)$ be a solution of (3.1). Although it lacks regularity, a mollifier may be introduced to smooth the equation as shown in the proof of Proposition 2.4. For the sake of simplicity, the regularity of $v$ is neglected. Multiplying Eq. (3.1) by $v$ yields

$$\frac{1}{2}(v^2 + v_{xx})_t + \{v_{xxxt} v - v_{xxt} v_x + G(v)\}_x = 0,$$

where $G'(v) = F'(v)v$. Integrating the above equation over $[0, 2L] \times [0, t]$ with respect to $x$ and $t$ implies (3.3)

$$\|v(t)\|^2_{L^2_{\text{per}}} + \|v_{xx}(t)\|^2_{L^2_{\text{per}}} = \|v_0\|^2_{L^2_{\text{per}}} + \|v_{0,xx}\|^2_{L^2_{\text{per}}}, \quad t \in [0, T].$$

Like for (2.28), using the Poincaré inequality (2.7) one deduces (3.4) from (3.3). □

**Proof of Theorem 3.1.** As shown in the previous section, the global existence of the solution to the periodic IBVP (3.1) can be similarly proved by the continuity argument. Its detail is omitted here.

Now, let us show the oscillatory divergence of $v(t, x)$. If $v_0(x) \not\equiv 0$, by the periodicity of $v(t, x)$

$$v(t, x) = v(t, x + 2L), \qquad \int_0^{2L} v(t, x)\mathrm{d}x = 0,$$

we know that $v(t, x) \not\equiv 0$ and $v(t, x)$ is oscillatory. To see the divergence, if the assertion is not true, i.e., $v(t, x)$ is convergent to 0 as $t \to \infty$, then we have

$$\|v(t)\|^2_{L^2_{\text{per}}} \to 0 \quad \text{and} \quad \|v_{xx}(t)\|^2_{L^2_{\text{per}}} \to 0 \quad \text{as } t \to \infty,$$

which implies from (3.2)

$$\|v_0\|_{L^2_{\text{per}}} + \|v_{0,xx}\|_{L^2_{\text{per}}} \to 0 \quad \text{as } t \to \infty.$$

This is a contradiction! Therefore, $v(t, x)$ has oscillatory divergence. □

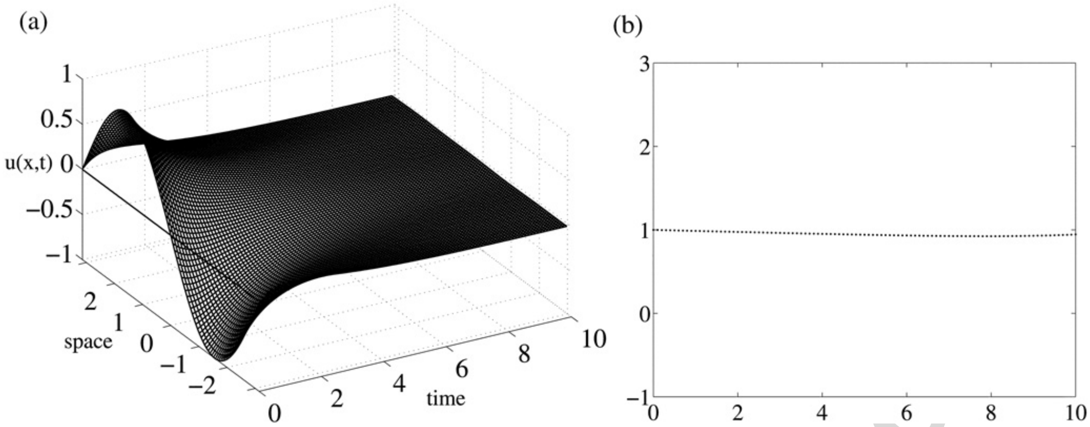


Fig. 4.1. (a) $u(t,x)$ with $\alpha = 1$, $u_0(x) = \sin x$; (b) $\frac{\sup_{x\in[0,2\pi]}|u(t,x)-m_0|}{\mathrm{e}^{-t/2}}$, with $u(t,x)$ from (a).

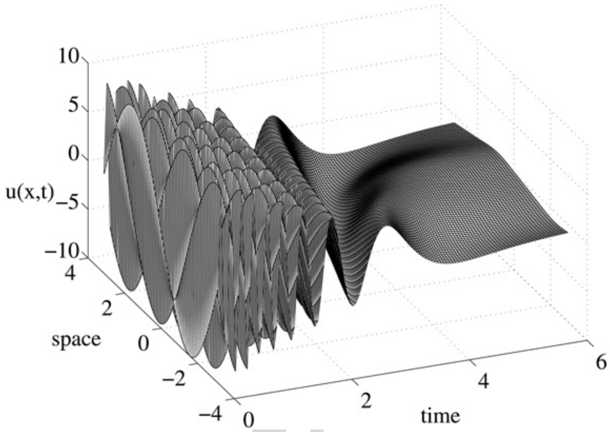


Fig. 4.2. $u(t,x)$ with $\alpha = 1$ and $u_0(x) = 10\sin x$.

## 4. Numerical simulations

In this section, the numerical simulations are reported to confirm the theoretical results: Theorems 1.1 and 1.2. Here, the nonlinear function is $f(u) = u^3/3$ and the period $2L = 2\pi$.

**Case A.** $\alpha > 0$. Without loss of generality, let $\alpha = 1$. The small initial value is chosen to be $u_0(x) = \sin x$, and its average over $[0, 2\pi]$ is $m_0 = \int_0^{2\pi} \sin x \mathrm{d}x = 0$. The numerical results in Fig. 4.1(a) and (b) show convergence of the solution $u(t,x)$ to its initial average $m_0 = 0$. In particular, Fig. 4.1(b) shows

$$\frac{\sup\limits_{x\in[0,2\pi]} |u(t,x) - m_0|}{\mathrm{e}^{-t/2}} \approx 1,$$

which indicates that $u(t,x)$ converges to $m_0$ at the optimal rate $\mathrm{e}^{-\gamma t} = \mathrm{e}^{-t/2}$ (see (1.11)), namely, $\gamma = \frac{\alpha\pi^2 L^2}{\pi^4+L^4} = \frac{1\cdot\pi^2\cdot\pi^2}{\pi^4+\pi^4} = \frac{1}{2}$. This confirms Theorem 1.1.

The large initial value is chosen to be $u_0(x) = 10\sin x$. The initial average is still $m_0 = 0$, but the perturbation of $u_0(x)$ around the initial average $m_0$ is large, $\sup|u_0(x) - m_0| = 10$. For this case, convergence is also obtained; see Fig. 4.2, which confirms the convergence of Theorem 1.1 in the case of the "large" initial value. Unlike the case with small initial perturbation around its average $m_0$ shown in Fig. 4.1 in which the solution $u(t,x)$ decays smoothly to $m_0 = 0$, Fig. 4.2 exhibits a lot of irregular and sharp oscillations within the initial short time from 0 to 2.5, then converging smoothly to the initial average $m_0$.

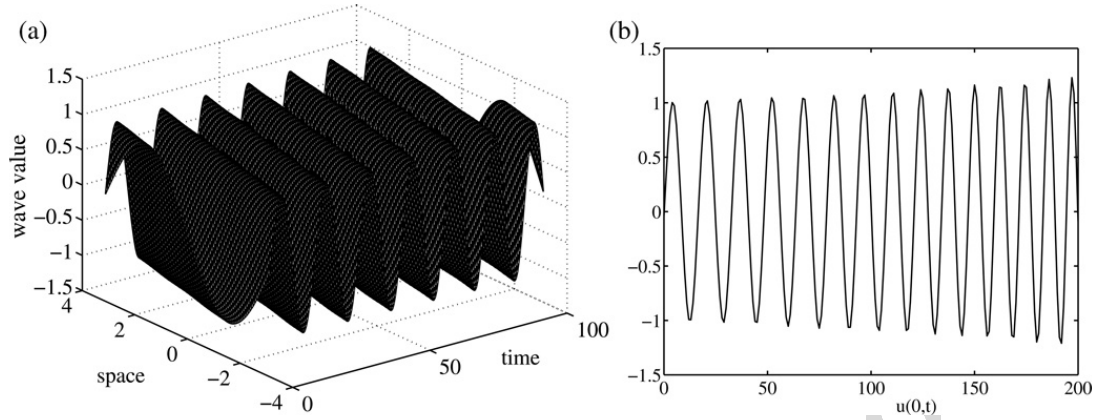


Fig. 4.3. (a) $u(t, x)$ with $\alpha = 0$ and $u_0(x) = \sin x$; (b) $u(t, 0)$ with $\alpha = 0$ and $u_0(x) = \sin x$.

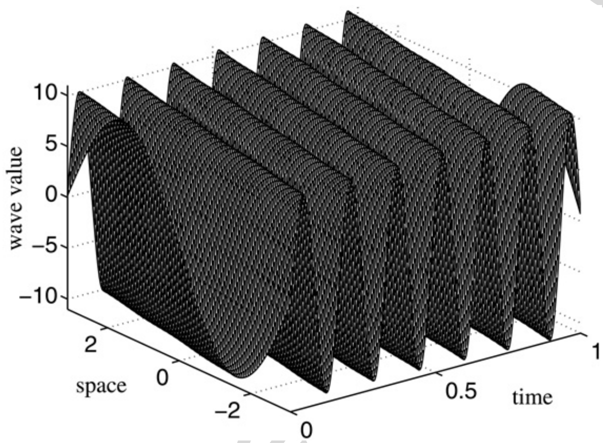


Fig. 4.4. $u(t, x)$ with $\alpha = 0$ and $u_0(x) = 10 \sin x$.

**Case B.** $\alpha = 0$. Like for Case A, the small initial value is set as $u_0(x) = \sin x$, whose average over $[0, 2\pi]$ is $m_0 = 0$. The numerical results in Fig. 4.3 (a) and (b) show the oscillatory divergence of the solution $u(t, x)$ around its initial average $m_0 = 0$ at all time. The numerical experiment presented here validates Theorem 1.2.

A similar numerical result for the large initial value is presented in Fig. 4.4, in which the initial value is $u_0(x) = 10 \sin x$.

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