Journal of Differential Equations **159**, 138–185 (1999)
Article ID jdeq.1999.3640, available online at http://www.idealibrary.com on IDEAL®

Convergence to Traveling Waves with Decay Rates for Solutions of the Initial Boundary Problem to a Relaxation Model

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Received December 19, 1997; revised January 20, 1999

In this paper we consider a 2×2 relaxation hyperbolic system of conservation laws with a boundary effect, and we show that the solutions of this initial boundary problem tend to the traveling wave solutions of the corresponding Cauchy problem time-asymptotically. In particular, we give the algebraic and exponential decay rates by using the weighted energy method. The location of a shift for the traveling wave, to overcome the difficulty in the boundary, plays a key role in this paper.

Contents.

1. INTRODUCTION

Relaxation phenomenon often arise in many physical situations, for example, gases not in thermodynamic equilibrium, kinetic theory, chromatography, river flows, traffic flows, and more general waves, cf. [32, 5].

0022-0396/99 $30.00

The general 2×2 relaxation hyperbolic system of conservation laws in the form

$$\begin{cases} u_t + f(u,v)_x = 0 \\ v_t + g(u,v)_x = h(u,v) \end{cases} \tag{1.1}$$

was first analyzed by T.-P. Liu [16] to justify some nonlinear stability criteria for diffusion waves, expansion waves and traveling waves in the Cauchy problem case. After then, the stability of traveling waves with decay rates for the Cauchy problem and the stability theory but without decay rate for the initial boundary problem were studied by Zingano [34] and Nishibata [29], respectively. The problem on the convergence to the diffusion waves was given by Chern [4], too. Related results on the relaxation time limit can be found in Chen and T.-P. Liu [3], Chen *et al.* [2], Natalini [28], and Marcati and Rubino [21].

In this paper, we investigate the simplest relaxation model in the half line $x\in\mathbb{R}_+=(0,+\infty)$

$$\begin{cases} u_t + v_x = 0, \\ v_t + au_x = \dfrac{(f(u)-v)}{\varepsilon}, \qquad (x,t)\in\mathbb{R}_+\times\mathbb{R}_+ \end{cases} \tag{1.2}$$

with the initial boundary conditions

$$\begin{cases} (u,v)|_{t=0} = (u_0,v_0)(x) \to (u_+,v_+), \qquad \text{as} \quad x\to+\infty \\ v|_{x=0} = g(t), \end{cases} \tag{1.3}$$

where (u_+,v_+) satisfying $v_+=f(u_+)$ is one of the end constant states of the traveling wave solutions $(U,V)(x-st)$ of (1.2) corresponding to the Cauchy problem, with other end states given as (u_-,v_-) and satisfying $v_-=f'(u_-)$. The unknowns u, v belong to $\mathbb{R}$. We give $v_0(0)=g(0)$ as the compatibility condition. The boundary function $g(t)$ is given in $[v_+,v_-]$. The function $f(u)$ is smooth (say, $f\in C^2$) and can be in general nonconvex. a is a positive constant satisfying

$$-\sqrt{a} < f'(u) < \sqrt{a}, \qquad \text{for all } u \text{ under consideration}, \tag{1.4}$$

which is the subcharacteristic condition introduced in [16]. In this paper, we always assume the relaxation time $\varepsilon=1$ without loss of generality, because we can scale the variable (x,t) to a new one $(\varepsilon x,\varepsilon t)$, then we have Eqs. (1.2) with $\varepsilon=1$.

The model (1.2) was first introduced by Jin and Xin [9] for numerical analysis interest. The stability of traveling wave solutions, $(u,v)(x,t)=(U,V)(x-st)$ with $(U,V)(\pm\infty)=(u_\pm,v_\pm)$, for the Cauchy problem associated to (1.2), was studied by H. L. Liu *et al.* [12, 13], Mascia and Natalini [22], and finally Mei and Yang [27]. The authors [27] improved the algebraic decay rates shown in [12] to the optimal one and also contributed an exponential decay rate when the initial perturbation decays in a spatial exponential form. The stability of front waves in the higher space dimensions was shown by Luo and Xin [20] recently. For the convergence theory of the corresponding rarefaction waves the reader is referred to Luo [19] and Mascia and Natalini [23]. The convergence to the traveling wave solutions, as the relaxation time goes to zero, was recently considered by Jin and H. L. Liu [8]. Furthermore, the numerical computation and the properties of entropy solution for the model (1.2) were shown by Aregba-Driollet and Natalini [1] (see also [6, 7]). On the other hand, the asymptotic limit of relaxation time for (1.2) with a boundary effect was given by Wang and Xin [31] (with a different choice of the boundary condition, $u_{|x=0}=0$) and by Yong [33] for the case of smooth solutions. Regarding the boundary layer behaviors for some hyperbolic systems of conservation laws, the reader is referred to J.-G. Liu and Xin [14, 15]. However, in the initial boundary value problem (*IBVP* from now on) case, there is no work on the decay rate convergence to the traveling wave solution, even for the scalar viscous conservation law

$$\begin{cases} u_t + f(u)_x = u_{xx}, & x<0,\ t>0 \\ u|_{t=0} = u_0(x), & u|_{x=0} = u_+ \end{cases} \tag{1.5}$$

studied by T.-P. Liu and Yu [18] and T.-P. Liu and Nishihara [17]. In fact both of them have to choose the shift to be a time-function $d(t)$ depending on the solution $u(x, t)$ of (1.5), and this arises a difficulty to yield a decay rate. Therefore, it should be significant to show the decay rate to the traveling wave for the IBVP (1.2) and (1.3), both in the senses of the scalar case and the system case. This is our purpose in the present paper.

It seems to be interesting to compare our problem with Nishibata's problem [29]. That paper is in fact a pioneer in considering the IBVP for a relaxation model and the author examines there the general case of systems of the form (1.1). Therein, he put the boundary layer on the traveling waves, $(u, v)|_{x=0} = (U, V)(-st)$, which corresponds to the case when there is no perturbation on the boundary layer: this case, even more important, should be a bit special; somehow it is like the corresponding Cauchy problem by cutting off the other side $x<0$. Moreover he proved, under the convex assumption on f in the sense of the simplest model (1.2), that the solutions of (1.1) approach the corresponding traveling waves as $t \to +\infty$, but without decay rate. On the other hand in our problem, if we put $g(t) = V(-st)$ the same as Nishibata's problem, we can have the convergence to the wave with the exponential and algebraic decay rates. If we let $g(t)$

be so closed to the wave $V(-st)$, the same convergence theory can be expected, too.

The really interesting case is $g(t)=v_-$ or v_+ which will be our main purpose in this paper. First of all, we will make an effort to the case $g(t)=v_\mp$ with $s\neq 0$ in Section 4 and to the case $g(t)=v_\mp$ with $s=0$ in Section 5. In these cases we have, for the special relaxation system (1.2), a boundary perturbation $(v-V)|_{x=0}=v_\mp-V(-st)$, as a consequence of the boundary condition $g(t)=v_-$ or v_+ contained in (1.3). The other boundary perturbation $(u-U)|_{x=0}$ is not known and can be controlled automatically by Eqs. (1.2). Under this background, we are going to prove the asymptotic behavior of the solutions of the IBVP (1.2) and (1.3), with some kinds of decay rates like $O(t^{-\alpha/2})$ and $O(e^{-\theta t/2})$ for some constants α, $\theta>0$, to the traveling waves for the general nonconvex $f(u)$. To treat the nonconvexity, as in [27] we will introduce two weight functions. How to locate the shift for the traveling waves plays a key role in this paper.

To go to our goal in the cases $g(t)=v_-$ or v_+, here, we have two observations on the IBVP (1.2) and (1.3). First, in the front traveling wave case, i.e., $s>0$, or the back wave case $s<0$, since the first equation of (1.2) is a conservation law, by borrowing the idea of Matsumura and Mei [24] used in the case of the viscous p-system, we expect to determine the shift as an exact constant. This may be important to ensure the decay rates we can have. Since the shift may be a constant, the weighted energy method used in [11, 26, 25] (see also [12, 27]) is expected availably to deduce the same decay rates to [27] in the IBVP case. Second, in the stationary traveling wave case ($s=0$), since the wave $V(x)$ is a constant $V(x)\equiv v_\pm$, quite similarly to the case $g(t)=V(-st)$, we can also obtain the decay rates. However, we discover that, for some shift functions $d(t)>0$ including $c\log(1+t)$ or $(1+t)^c$ for some constant $c>0$, we can get the convergence as well as the algebraic and exponential decay rates to the shifted stationary traveling waves $(U,V)(x+d(t))$ even for the shock conditions $f'(u_+)<s=0\leqslant f'(u_-)$, since $x+d(t)>0$ and the wave $U(x+d(t))$ does not go to the end state u_-. We remark that the shift $d(t)=c\log(1+t)$ is the one considered by T.-P. Liu and Yu [18] for the Burger equation (1.5) with $f=u^2/2$ in the case $s=0$ by using pointwise estimate technic. In T.-P. Liu and Nishihara [17], it remained an open problem the convergence in the case $s=0$ for the general viscous equation (1.5) via energy method analysis.

This paper is organized as follows. After stating some notations in the following, in Section 2, we give some preliminaries on the traveling wave solutions of the corresponding Cauchy problem, then we discuss the easier case $g(t)=V(-st)$ for the convergence theory in Section 3. Section 4 is devoted to considering the case $g(t)=v_\mp$ with $s\neq 0$. We will prove that the solutions $(u,v)(x,t)$ converge, with some algebraic and exponential decay rates, to the traveling waves as t goes to infinity. Section 5 treats the stationary traveling wave for $g(t) = v_{\mp}$. We will prove that the stationary waves without any shift function on the time t are stable for the IBVP (1.2) and (1.3) when the initial perturbations are small. Moreover, the convergence to the shifted stationary waves $(U, V)(x + d(t))$ will be obtained with the algebraic and exponential decay rates, for some shift functions $d(t)$, both in the nondegenerate shock case $(f'(u_+) < s = 0 < f'(u_-))$ and the degenerate case $(f'(u_+) < s = 0 = f'(u_-))$. For the general case of $g(t)$ in Section 6, when the boundary layer $g(t)$ is closed to the wave $V(-st)$ in the space $W^{3,1}(\mathbb{R}_+)$, we discuss the convergence of the solutions $(u, v)(x, t)$ of the IBVP (1.2) and (1.3) time-asymptotically toward the corresponding traveling waves $(U, V)(x - st)$ with some decay rates. Finally, we give some concluding remarks on the unsolved problems in this paper in Subsection 6.1.

Notations. L^2 denotes the space of measurable functions on $\mathbb{R}$ or $\mathbb{R}_+$ which are square integrable, with the norm

$$\|f\| = \left(\int |f(x)|^2 \, dx \right)^{1/2}.$$

H^l $(l \geqslant 0)$ denotes the Sobolev space of L^2-functions f on $\mathbb{R}$ or $\mathbb{R}_+$ whose derivatives $\partial_x^j f$, $j = 1, ..., l$, are also L^2-functions, with the norm

$$\|f\|_l = \left(\sum_{j=0}^{l} \|\partial_x^j f\|^2 \right)^{1/2}.$$

L_w^2 denotes the space of measurable functions on $\mathbb{R}$ or $\mathbb{R}_+$ which satisfy $w(x)^{1/2} f \in L^2$, where $w(x) > 0$ is a so-called weight function, with the norm

$$|f|_w = \left(\int w(x) \, |f(x)|^2 \, dx \right)^{1/2}.$$

H_w^l $(l \geqslant 0)$ denotes the weighted Sobolev space of L_w^2-functions f on $\mathbb{R}$ whose derivatives $\partial_x^j f$, $j = 1, ..., l$, are also L_w^2-functions, with the norm

$$|f|_{l,w} = \left(\sum_{j=0}^{l} |\partial_x^j f|_w^2 \right)^{1/2}.$$

Denoting $\langle x \rangle = \sqrt{1 + x^2}$ and

$$\langle x \rangle_+ = \begin{cases} \sqrt{1 + x^2}, & \text{if} \quad x \geqslant 0, \\ 1, & \text{if} \quad x < 0, \end{cases}$$

we will make use of the spaces $L^2_{\langle x \rangle_+}$ and $H^l_{\langle x \rangle_+}$ $(l = 1, 2)$ later. If $w(x) = \langle x \rangle^\alpha$, we denote $L_w^2 = L_\alpha^2$. The weighted space L_w^2 for such weight function $w=\langle x\rangle^{\alpha}\langle x\rangle_{+}$ is denoted as $L^2_{\alpha\langle x\rangle_{+}}$, and the corresponding norm is $|\cdot|_{\alpha\langle x\rangle_{+}}$. Since we consider $x\in\mathbb{R}_{+}$, sometimes we mean $\langle x\rangle=\langle x\rangle_{+}$. We denote also $f(x)\sim g(x)$ as $x\to x_0$ when $C^{-1}g\leqslant f\leqslant Cg$ in a neighborhood of x_0, and $|(f_1, f_2, f_3)|_X\sim|f_1|_X+|f_2|_X+|f_3|_X$, where $|\cdot|_X$ is the norm of space X. Without any ambiguity, we denote several constants by C_i, or c_i, $i=1, 2, ...$, or by C. When $C^{-1}\leqslant w(x)\leqslant C$ for $x\in\mathbb{R}$, we note that $L^2=H^0=L^2_w=H^0_w$ and $\|\cdot\|=\|\cdot\|_0\sim|\cdot|_w=|\cdot|_{0,w}$.

Let T and B be a positive constant and a Banach space, respectively. We denote $C^k(0, T; B)$ $(k\geqslant 0)$ as the space of B-valued k-times continuously differentiable functions on $[0, T]$, and L^2 $(0, T; B)$ as the space of B-valued L^2-functions on $[0, T]$. The corresponding spaces of B-valued function on $[0, \infty)$ are defined similarly.

2. TRAVELING WAVE SOLUTIONS

The traveling wave solution of system (1.2) for the corresponding Cauchy problem is such a solution $(U, V)(z)$, $(z=x-st)$, satisfying Eqs. (1.2) and $(U, V)(\pm\infty)=(u_{\pm}, v_{\pm})$ where $v_{\pm}=f(u_{\pm})$, namely,

$$\begin{cases} -sU_z+V_z=0, \\ -sV_z+aU_z=f(U)-V, \\ (U, V)(\pm\infty)=(u_{\pm}, v_{\pm}), \end{cases} \tag{2.1}$$

which implies

$$(a-s^2)\,U_z=f(U)-V. \tag{2.2}$$

Integrating the first equation of (2.1) over $(\pm\infty, z)$ and noting $(U, V)(\pm\infty)=(u_{\pm}, v_{\pm})$, we have

$$-sU+V=-su_{\pm}+v_{\pm}=-su_{\pm}+f(u_{\pm}). \tag{2.3}$$

Substituting (2.3) into the second equation of (2.1) we obtain

$$(a-s^2)\,U_z=f(U)-f(u_{\pm})-s(U-u_{\pm})\equiv: h(U). \tag{2.4}$$

From (2.3), we see that the speed s and the state constants $(u_{\pm}, v_{\pm})$ satisfy the so-called Rankine–Hugoniot condition

$$s=\frac{v_{+}-v_{-}}{u_{+}-u_{-}}=\frac{f(u_{+})-f(u_{-})}{u_{+}-u_{-}}. \tag{2.5}$$

It is well known that the ordinary differential equation (2.4) has a solution if and only if the R-H condition (2.5) and the Oleinik's entropy condition

$$h(u) = f(u) - f(u_\pm) - s(u - u_\pm) \begin{cases} <0, & u_+ < u_- \\ >0, & u_+ > u_- \end{cases} \tag{2.6}$$

hold. This entropy condition implies

$$f'(u_+) < s < f'(u_-) \tag{2.7}$$

or

$$f'(u_+) = s < f'(u_-) \qquad \text{or} \qquad f'(u_+) < s = f'(u_-) \qquad \text{or} \qquad f'(u_\pm) = s. \tag{2.8}$$

Condition (2.7) is the well-known Laxian shock condition. Here we will call this the *nondegenerate* shock condition and we will refer to each one of the possibilities in (2.8) as the *degenerate* shock condition, or the *contact* shock condition.

When $g(t) = V(-st)$, we say that all the shock cases in (2.6) are valid for the convergence theory, since this problem is like the corresponding Cauchy problem by cutting off another side $x < 0$.

When $g(t) = v_-$ with $s > 0$, we will restrict our focus in this paper to the cases

$$f'(u_+) \leqslant s < f'(u_-), \tag{2.9}$$

since other cases in (2.8) $f'(u_+) \leqslant s = f'(u_-)$ cannot yield a constant shift; see the Subsection 4.1.1 below. Therefore, it remains still an *open problem* to get the decay rate for $s = f'(u_-)$ in our problem (1.2) and (1.3).

When $g(t) = v_+$ with $s < 0$, we will consider only the cases

$$f'(u_+) < s \leqslant f'(u_-), \tag{2.10}$$

because we are not able to control the boundary integration in the cases $f'(u_+) = s \leqslant f'(u_-)$, and these shock cases are open too.

When $g(t) = v_\mp$ with $s = 0$, it means from (2.1) and (2.5) that

$$V(x) = v_\pm = f(u_\pm);$$

this problem is included in the case $g(t) = V(-st)$, so we can easily treat it. However, if we want to have a convergence to the shifted stationary waves $(U, V)(x+d(t))$ with some shift function $d(t)$ satisfying $d(t) \to +\infty$ as $t \to +\infty$, we have to restrict ourself on

$$f'(u_+) < s = 0 \leqslant f'(u_-). \tag{2.11}$$

The reason is that we cannot determine a suitable shift $d(t)$ for the stationary waves to have a convergence when $f'(u_+) = 0 = s$, which will be precisely stated in Section 5. So, unfortunately, the cases $f'(u_+) = s \leqslant f'(u_-)$ remain also open problems.

Furthermore, without loss of generality we assume in this paper $u_+ < u_-$, and for $s = f'(u_+)$ or $s = f'(u_-)$

$$f^{(n)}(u_\pm) = 0 \qquad \text{and} \qquad f^{(n+1)}(u_\pm) \neq 0 \qquad \text{for} \quad n \geqslant 2. \tag{2.12}$$

We now state the existence of the traveling wave solutions given in [13] by a similar proof in [25] for the scalar viscous conservation laws.

PROPOSITION 2.1 [13]. *Under Oleinik's shock condition* (2.6) *and the R-H condition* (2.5), *then there exists a traveling wave solution* $(U, V)(x-st)$ *of* (1.2) *with* $(U, V)(\pm\infty) = (u_\pm, v_\pm)$, *unique up to a shift, and the speed satisfies*

$$s^2 < a. \tag{2.13}$$

Moreover, it holds

$$(a-s^2)\, U_z = h(U) < 0 \qquad \text{for} \quad u_+ < u_- \tag{2.14}$$

and as $z = x - st \to \pm\infty$

$$\begin{cases} |h(U)| \sim |(U-u_\pm, V-v_\pm)(z)| \sim \exp(-c_\pm |z|), & f'(u_+) < s < f'(u_-) \\ |h(U)|^{1/(1+n)} \sim |(U-u_\pm, V-v_\pm)(z)| \sim |z|^{-1/n}, & f'(u_\pm) = s, \end{cases} \tag{2.15}$$

where $c_\pm = |f'(u_\pm) - s|/(a-s^2) > 0$.

Defining the following weight functions, cf. [27, 25],

$$w_1(U) = \frac{(U-u_+)(U-u_-)}{h(U)}, \qquad w_2(U) = -\frac{(U-u_+)^{1/2}\,(u_- - U)^{1/2}}{h(U)} \tag{2.16}$$

for $U \in (u_+, u_-)$, which are positive due to $u_+ < u_-$ and $h(U) < 0$, we recall the properties of the traveling wave solutions (U, V) given in [27] as follows.

LEMMA 2.2 [27]. *Let $(U, V)(x-st)$ be the traveling wave solutions of (1.2) for the corresponding Cauchy problem. Then it holds*

$$\begin{cases} w_1(U) \sim O(1),\ w_2(U) \sim e^{c_\pm |z|/2}, & \text{if } \ f'(u_+) < s < f'(u_-) \\ w_1(U) \sim \langle z \rangle_\pm, & \text{if } \ f'(u_\pm) = s \end{cases} \tag{2.17}$$

as $z \to \pm\infty$, and

$$(w_1 h)''(U) = 2, \qquad \left| \frac{w_i(U)_z}{w_i(U)} \right| = O(1)\, \frac{|u_+ - u_-|}{a - s^2}, \quad i = 1, 2, \tag{2.18}$$

$$-h(U)(w_2 h)''(U) = O(1)\, w_2(U), \qquad \text{for } \ f'(u_+) < s < f'(u_-). \tag{2.19}$$

3. CASE $g(t) = V(-st)$

In this section, we discuss the easier case $g(t) = V(-st)$, which means that there is no perturbation in the boundary $x = 0$. Such a problem is the same as Nishibata's problem [29]. We can easily prove the convergence with some decay rates to the traveling waves $(U, V)(x-st)$, since it can be treated somewhat like the corresponding Cauchy problem.

By using the result of Section 2 about traveling waves $(U, V)(x-st)$, we can assume $s > 0$ or $= 0$ or < 0, as well as we can consider any one of the shock cases $f'(u_+) \leqslant s \leqslant f'(u_-)$. We assume

$$\int_0^\infty [u_0(x) - U(x)]\, dx = 0. \tag{3.1}$$

From the first equation of (1.2) we have $(u - U)_t = -(v - V)_x$, and integrating it over $[0, +\infty) \times [0, t]$, using $g(t) = V(-st)$, $v|_{x=+\infty} = V|_{x=+\infty} = v_+$ and (3.1), we finally obtain

$$\begin{aligned} &\int_0^\infty [u(x,t) - U(x-st)]\, dx \\ &\quad = \int_0^\infty [u_0(x) - U(x)]\, dx - \int_0^t \int_0^\infty (v(x,\tau) - V(x - s\tau))_x\, dx\, d\tau \\ &\quad = \int_0^\infty [u_0(x) - U(x)]\, dx - \int_0^t (v(x,\tau) - V(x - s\tau))|_{x=0}^{\infty}\, d\tau \\ &\quad = \int_0^\infty [u_0(x) - U(x)]\, dx = 0. \end{aligned} \tag{3.2}$$

Let us consider

$$\phi(x,t) := -\int_x^\infty [u(y,t) - U(y-st)]\, dy, \qquad \psi(x,t) := v(x,t) - V(x-st), \tag{3.3}$$

which implies, by using (3.2), that

$$\phi|_{x=0} = 0, \tag{3.4}$$

then the original IBVP (1.2) and (1.3) can be transformed into the new IBVP

$$\begin{cases} \phi_t + \psi = 0, \\ \psi_t + a\phi_{xx} - f'(U)\,\phi_x + \psi = F, \qquad x > 0,\ t > 0, \\ (\phi,\psi)|_{t=0} = \left(-\int_x^\infty [u_0(y) - U(y)]\, dy,\ v_0(x) - V(x) \right) =: (\phi_0,\psi_0)(x), \\ \phi|_{x=0} = 0, \end{cases} \tag{3.5}$$

where $F = f(U + \phi_x) - f(U) - f'(U)\,\phi_x$.

Since the new IBVP (3.5) has a zero boundary layer, the convergence to the traveling waves is the same as the one studied by Nishibata [29]. The decay rates can be obtained, without any difficulty, by a similar argument used for the Cauchy problem (see [12, 27, 34]). The details may be omitted.

THEOREM 1.3 (Algebraic Rates). *Under the assumption* (3.1), *let a be suitably large or, fixed $a > 0$, $|u_+ - u_-|$ be suitably small.*

(i) *Case $f'(u_+) < s < f'(u_-)$. Suppose that $(\phi_0,\psi_0)(x) \in L^2_\alpha \cap H^2$ for some $\alpha > 0$ holds. Then there exists a constant $\delta_1 > 0$ such that if $|(\phi_0,\psi_0)|_\alpha + \|(\phi_0,\psi_0)\|_2 < \delta_1$, then the system* (1.2) *and* (1.3) *has a unique global solution $(u,v)(x,t)$ satisfying*

$$\sup_{x\in\mathbb{R}_+} |(u,v)(x,t) - (U,V)(x-st)| \leqslant C(1+t)^{-\alpha/2} \left(|(\phi_0,\psi_0)|_\alpha + \|(\phi_0,\psi_0)\|_2 \right).$$

(ii) *Case $f'(u_+) = s < f'(u_-)$. Suppose that $(\phi_0,\psi_0)(x) \in L^2_{\alpha\langle x\rangle_+} \cap H^2$ for some $0 < \alpha < 2/n$ holds. Then there exists a constant $\delta_2 > 0$ such that, if $|(\phi_0,\psi_0)|_{\alpha\langle x\rangle_+} + \|(\phi_0,\psi_0)\|_2 < \delta_2$, then the system* (1.2) *and* (1.3) *has a unique global solution $(u,v)(x,t)$ satisfying*

$$\sup_{x\in\mathbb{R}_+} |(u,v)(x,t) - (U,V)(x-st)| \leqslant C(1+t)^{-\alpha/4} \left(|(\phi_0,\psi_0)|_{\alpha\langle x\rangle_+} + \|(\phi_0,\psi_0)\|_2 \right).$$

THEOREM 3.2 (Exponential Rates). *Under the assumption* (3.1), *let* a *be suitably large, or* $|u_+ - u_-|$ *be suitably small for any given* $a>0$. *If* $f'(u_+) < s < f'(u_-)$ *and* $\phi_0 \in H^3_{w_2(U)}$, $\psi_0 \in H^2_{w_2(U)}$, *then there exist constants* $\delta_3 > 0$ *and* $\theta = \theta(|u_+ - u_-|, a) > 0$ *such that if* $|(\phi_0, \psi_0)|_{2, w_2} \leqslant \delta_3$, *the IBVP* (1.2) *and* (1.3) *has a unique global solution* $(u, v)(x, t)$ *satisfying*

$$\sup_{x \in \mathbb{R}_+} |(u, v)(x, t) - (U, V)(x - st)| \leqslant Ce^{-\theta t/2} |(\phi_0, \psi_0)|_{2, w_2}. \tag{3.6}$$

4. CASE $g(t) = v_\mp$, $s \neq 0$

This section is devoted to studying convergence for the solutions of the IBVP (1.2) and (1.3) to the corresponding traveling waves $(U, V)(x - st)$. In particular, as discussed in Section 2, here we need to restrict to the shock cases $f'(u_+) \leqslant s < f'(u_-)$ for the front traveling waves ($s > 0$) and boundary condition $g(t) = v_-$, and to $f'(u_+) < s \leqslant f'(u_-)$ for the back traveling waves ($s < 0$) and boundary condition $g(t) = v_+$. We will obtain results of algebraic and exponential decay rates under the hypothesis of small initial-boundary perturbations. We will develop the details only for the case $s > 0$ and $g(t) = v_-$, summarizing in a last subsection the corresponding results for the case $s < 0$ and $g(t) = v_+$.

4.1. *Case* $s > 0$, $g(t) = v_-$

4.1.1. *Determination of the Shift.* Here, we share Matsumura and Mei's idea in [24] to determine the shift as a constant. Assume the initial data $(u_0, v_0)(x)$ of the fixed front traveling waves $(U, V)(x - st)$ located in a neighborhood of the traveling wave solutions $(U, V)(x - x_1)$. Then we try to make an heuristic argument to determine which of the shifted front waves $(U, V)(x - st + x_0 - x_1)$ the solutions tend toward.

Denote $(U, V) = (U, V)(x - st + x_0 - x_1)$. From the first equation of (1.2), we have

$$(u - U)_t = -(v - V)_x. \tag{4.1}$$

Integrating (4.1) over $\mathbb{R}_+$ with respect to x, and noting $v|_{x=0} = v_-$ and $v|_{x=+\infty} = V|_{x=+\infty} = v_+$, it yields

$$\begin{aligned} &\frac{d}{dt} \int_0^\infty [u(x, t) - U(x - st + x_0 - x_1)]\, dx \\ &\qquad = -(v - V)|_{x=0}^{\infty} = v_- - V(-st + x_0 - x_1). \end{aligned} \tag{4.2}$$

By integration on the time t, we have

$$\int_0^\infty [u(x,t)-U(x-st+x_0-x_1)]\,dx$$
$$=\int_0^\infty [u_0(x)-U(x+x_0-x_1)]\,dx+\int_0^t [v_- - V(-s\tau+x_0-x_1)]\,d\tau. \tag{4.3}$$

If we assume

$$\int_0^\infty [u(x,t)-U(x-st+x_0-x_1)]\,dx\to 0 \qquad \text{as} \quad t\to+\infty \tag{4.4}$$

for some x_0, i.e., the right hand side of (4.3) must go to zero as $t\to+\infty$. Hence, if we set

$$I(x_0):=\int_0^\infty [u_0(x)-U(x+x_0-x_1)]\,dx+\int_0^\infty [v_- - V(-st+x_0-x_1)]\,dt, \tag{4.5}$$

the shift x_0 must be determined so that $I(x_0)=0$. Differentiating $I(x_0)$ with respect to x_0 gives

$$\begin{aligned} I'(x_0) &= -\int_0^\infty U'(x+x_0-x_1)\,dx-\int_0^\infty V'(-st+x_0-x_1)\,dt \\ &= -[u_+-U(x_0-x_1)]+\frac{1}{s}[v_- - V(x_0-x_1)] \\ &= u_- - u_+, \end{aligned} \tag{4.6}$$

where we used formula (2.3). Hence, by integration of (4.6) it follows $I(x_0)=I(0)+(u_- - u_+)\,x_0$. Thus, the shift $x_0=x_0(x_1,u_0)$ should be determined explicitly by $I(0)+(u_- - u_+)\,x_0=I(x_0)=0$, that is,

$$x_0:=\frac{1}{u_+-u_-}\left\{\int_0^\infty [u_0(x)-U(x-x_1)]\,dx+\int_0^\infty [v_- - V(-st-x_1)]\,dt\right\}. \tag{4.7}$$

In order to conclude we must show that the right hand side of (4.7) makes sense. In fact, we have $u_0(x)$ in a neighborhood of $U(x-x_1)$ as hypotheses and we may be more precise assuming that

$$\left|\int_0^\infty [u_0(x)-U(x-x_1)]\,dx\right|<+\infty.$$

Moreover, since we restrict ourself to the shock case of $s>0$ and $f'(u_+)\leqslant s<f'(u_-)$ as stated in (2.9), we have

$$\left|\int_0^\infty [v_- - V(-st-x_1)]\,dt\right|$$

$$\leqslant \int_0^\infty |v_- - V(-st-x_1)|\,dt = O(1)\int_0^\infty e^{-c_-(st+x_1)}\,dt \leqslant C, \quad (4.8)$$

and this assures us that the fundamental assumption (4.4) is verified when x_0 is fixed as in (4.7). So, x_0 defined in (4.7) should be an exact constant. Let us remark that, when $s=f'(u_-)$, since $|v_- - V(-st-x_1)| = O(1)(st+x_1)^{-1/n}$ we have $|\int_0^\infty [v_- - V(-st-x_1)]\,dt| = +\infty$, so with this technique we cannot choose the shift x_0 to be a constant and this will remain still an open problem as we mentioned above.

Thus, it follows from (4.3) and $I(x_0)=0$ that

$$\int_0^\infty [u(x,t) - U(x-st+x_0-x_1)]\,dx$$

$$= \int_0^\infty [u_0(x) - U(x+x_0-x_1)]\,dx + \int_0^t [v_- - V(-s\tau+x_0-x_1)\,d\tau$$

$$= I(x_0) - \int_t^\infty [v_- - V(-s\tau+x_0-x_1)\,d\tau$$

$$= -\int_t^\infty [v_- - V(-s\tau+x_0-x_1)\,d\tau \to 0, \qquad \text{as} \quad t\to+\infty.$$

Let

$$(u_0(x) - U(x-x_1),\, v_0(x) - V(x-x_1)) \in H^1 \quad (4.9)$$

and

$$(\Phi_0, \Psi_0)(x) := \left(-\int_x^\infty (u_0(y) - U(y-x_1)\,dy,\, v_0(x) - V(x-x_1)\right) \in L^2, \quad (4.10)$$

then we have an asymptotic property of the constant shift x_0 as follows; we omit its proof since a similar one can be found in [24].

LEMMA 4.1. *With the previous hypotheses, it holds* $(\Phi_0, \Psi_0)\in H^2$ *and* $|x_0|\to 0$ *as* $\|(\Phi_0,\Psi_0)\|_2 \to 0$ *and* $x_1\to+\infty$.

4.1.2. *Main Theorems.* Due to $x_1 \gg |x_0|$ (see the Lemma 4.1), we define

$$w_{0,2}(x) = \begin{cases} e^{c_+(x+x_0-x_1)}, & \text{if } x \geqslant x_1 - x_0, \\ 1, & \text{if } 0 \leqslant x < x_1 - x_0, \end{cases} \tag{4.11}$$

then $w_{0,2}(x) \sim w_2(U(x+x_0-x_1))$ by Lemma 2.2. Setting

$$(\phi_0, \psi_0)(x) := \left(-\int_x^\infty (u_0(y) - U(y+x_0-x_1)\, dy, v_0(x) - V(x+x_0-x_1) \right), \tag{4.12}$$

we state our main theorems as follows.

THEOREM 4.2 (Convergence). *Under the assumptions* (1.4), (2.5), (2.6), *and* (2.9), *let $a > 0$ be a suitably large but fixed constant.*

(i) *Case $f'(u_+) < s < f'(u_-)$. Suppose $\phi_0 \in H^2$ and $\psi_0(x) \in H^1$ hold. Then there exists a constant $\varepsilon_1 > 0$ such that if $a(\|\phi_0\|_2 + \|\psi_0\|_1 + x_1^{-1}) < \varepsilon_1$, then the IBVP* (1.2) *and* (1.3) *has a unique global solution*

$$u - U \in C^0([0, +\infty); H^1) \cap L^2([0, +\infty); H^1)$$
$$v - V \in C^0([0, +\infty); H^1) \cap L^2([0, +\infty); H^1)$$

satisfying

$$\sup_{x \in R_+} |(u, v)(x,t) - (U, V)(x - st + x_0 - x_1)| \to 0, \qquad \text{as} \quad t \to +\infty. \tag{4.13}$$

(ii) *Case $f'(u_+) = s < f'(u_-)$. Suppose $\phi_0 \in L^2_{\langle x \rangle_+} \cap H^2$ and $\psi_0 \in L^2_{\langle x \rangle_+} \cap H^1$ hold. Then there exists a constant $\varepsilon_2 > 0$ such that if $a(|(\phi_0, \psi_0)|_{\langle x \rangle_+} + \|\phi_0\|_2 + \|\psi_0\|_1 + x_1^{-1}) < \varepsilon_2$, then the system* (1.2) *and* (1.3) *has a unique global solution*

$$u - U \in C^0([0, +\infty); L^2_{w_1} \cap H^1) \cap L^2([0, +\infty); H^1)$$
$$v - V \in C^0([0, +\infty); L^2_{w_1} \cap H^1) \cap L^2([0, +\infty); L^2_{w_1} \cap H^1)$$

satisfying the asymptoticity condition (4.13).

THEOREM 4.3 (Exponential Rate). *Under the assumptions of* (1.4), (2.5), (2.6), *and* (2.9), *let $a > 0$ be a suitably large but fixed constant. If $f'(u_+) < s < f'(u_-)$ and $\phi_0 \in H^2_{w_{0,2}}$, $\psi_0 \in H^1_{w_{0,2}}$, then there exist constants $\varepsilon_3 > 0$ and*

$\theta=\theta(|u_+-u_-|, a)>0$ *such that if* $a(|\phi_0|_{2, w_{0,2}}+|\psi_0|_{1, w_{0,2}}+x_1^{-1})\leqslant\varepsilon_3$, *then the IBVP* (1.2) *and* (1.3) *has a unique global solution* $(u, v)(x, t)$ *satisfying*

$$u-U\in C^0(0, \infty; H^1_{w_2})\cap L^2(0, \infty; H^1_{w_2})$$
$$v-V\in C^0(0, \infty; H^1_{w_2})\cap L^2(0, \infty; H^1_{w_2})$$

and

$$\sup_{x\in R_+}|(u, v)(x, t)-(U, V)(x-st+x_0-x_1)|$$
$$\leqslant Ce^{-\theta t/2}(|\phi_0|_{2, w_{0,2}}+|\psi_0|_{1, w_{0,2}}+e^{-c_- x_1/4}). \tag{4.14}$$

THEOREM 4.4 (Algebraic Rates). *Let us assume the hypotheses of Theorem* 4.2.

(i) *Case* $f'(u_+)<s<f'(u_-)$. *Suppose* $\phi_0\in L^2_\alpha\cap H^2$, $\psi_0\in L^2_\alpha\cap H^1$ *for some* $\alpha>0$. *When* (ϕ_0, ψ_0) *is small enough in* $(L^2_\alpha\cap H^2)\times(L^2_\alpha\cap H^1)$, *then*

$$\sup_{x\in\mathbb{R}_+}|(u, v)(x, t)-(U, V)(x-st+x_0-x_1)|$$
$$\leqslant C(1+t)^{-\alpha/2}(|(\phi_0, \psi_0)|_\alpha+\|\phi_0\|_2+\|\psi_0\|_1+e^{-c_- x_1/4}). \tag{4.15}$$

(ii) *Case* $f'(u_+)=s<f'(u_-)$. *Suppose the initial data* $(\phi_0, \psi_0)\in(L^2_{\alpha\langle x\rangle_+}\cap H^2)\times(L^2_{\alpha\langle x\rangle_+}\cap H^1)$, *for some* $0<\alpha<2/n$ *and that they are suitably small. Then*

$$\sup_{x\in\mathbb{R}_+}|(u, v)(x, t)-(U, V)(x-st+x_0-x_1)|$$
$$\leqslant C(1+t)^{-\alpha/4}(|(\phi_0, \psi_0)|_{\alpha\langle x\rangle_+}+\|\phi_0\|_2+\|\psi_0\|_1+e^{-c_- x_1/4}). \tag{4.16}$$

Remark 4.5. (1) The restriction of $a\gg 1$ but without $|u_+-u_-|\ll 1$ is the same used by H. L. Liu *et al.* [12, 13] for the Cauchy problem. This means that we need a stronger diffusion effect for the convergence. Regarding the Cauchy problem the restriction $a\gg 1$ was recently substituted with $|u_+-u_-|\ll 1$ in Mei and Yang [27]. Unfortunately here we still need $a\gg 1$ to overcome the difficulty arising on the boundary.

(2) The algebraic decay rates both in the nondegenerate and degenerate cases seem to be optimal compared with the corresponding Cauchy problem studied in [27].

4.1.3. *Reformulation of the Original Problem.* Let us define the new unknowns as

$$\begin{cases} \phi(x,t) := -\int_x^\infty [u(y,t) - U(y - st + x_0 - x_1)]\, dy \\ \psi(x,t) := v(x,t) - V(x - st + x_0 - x_1), \end{cases} \tag{4.17}$$

then the original system (1.2) can be reduced to

$$\begin{cases} \phi_{xt} + \psi_x = 0, \\ \psi_t + a\phi_{xx} = f(\phi_x + U) - f(U) - \psi, \end{cases} \tag{4.18}$$

which we can rewrite as

$$\begin{cases} \phi_t + \psi = 0, \\ \psi_t + a\phi_{xx} - f'(U)\,\phi_x + \psi = F, \end{cases} \tag{4.19}$$

where $F := f(\phi_x + U) - f(U) - f'(U)\,\phi_x$ satisfies $|F| \leqslant C\,|\phi_x|^2$.

The initial boundary conditions (1.3) will be now transformed in

$$\phi|_{t=0} = \phi_0(x), \qquad \psi|_{t=0} = \psi_0(x), \tag{4.20}$$

$$\begin{aligned} \phi|_{x=0} &= -\int_0^\infty [u(x,t) - U(x - st + x_0 - x_1)]\, dx \\ &= \int_t^\infty [v_- - V(-s\tau + x_0 - x_1]\, d\tau =: A(t), \end{aligned} \tag{4.21}$$

$$\psi|_{x=0} = -\phi_t|_{x=0} = v_- - V(-st + x_0 - x_1) = -A'(t). \tag{4.22}$$

Here, since $|v_- - V(-st + x_0 - x_1)| = O(1)\exp\{-c_-(st + x_1)\}$ by (2.15) and $|x_0| \ll x_1$, we have

$$|A(t)| \sim |A'(t)| \sim |A''(t)| \sim |A'''(t)| \sim O(1)\exp\{-c_-(st + x_1)\}. \tag{4.23}$$

Substituting $\psi = -\phi_t$ into the second equation of (4.19), we have

$$\begin{cases} L(\phi) := \phi_{tt} + \phi_t - a\phi_{xx} + f'(U)\,\phi_x = -F, & x > 0,\ t > 0 \\ (\phi, \phi_t)|_{t=0} = (\phi_0, \phi_1)(x), & x > 0 \\ (\phi, \phi_t)|_{x=0} = (A, A')(t), & t > 0. \end{cases} \tag{4.24}$$

We reformulate the theorems corresponding to the main Theorems 4.2–4.4 as follows.

THEOREM 4.6 (Convergence). *Under the assumptions in Theorem 4.2, then the IBVP (4.24) has a unique global solution $\phi(x,t)$ satisfying the following properties*:

(i) *Case* $f'(u_+) < s < f'(u_-)$.

$$\phi \in C^0([0, +\infty); H^2), \qquad \phi_x \in L^2([0, +\infty); H^1)$$
$$\phi_t \in C^0([0, +\infty); H^1) \cap L^2([0, +\infty); H^1)$$

and, if $M_0^2 := e^{-c_- x_1/2} + \|\phi_0\|_2^2 + \|\phi_1\|_1^2$,

$$\|\phi(t)\|_2^2 + \|\phi_t(t)\|_1^2 + \int_0^t \|(\phi_x, \phi_t)(\tau)\|_1^2 \, d\tau \leqslant CM_0^2, \tag{4.25}$$

which implies

$$\sup_{x \in \mathbb{R}_+} |(\phi_x, \phi_t)(x, t)| \to 0, \qquad \text{as} \quad t \to +\infty. \tag{4.26}$$

(ii) *Case* $f'(u_+) = s < f'(u_-)$.

$$\phi \in C^0([0, +\infty); L^2_{w_1} \cap H^2), \qquad \phi_x \in L^2([0, +\infty); L^2_{w_1} \cap H^1)$$
$$\phi_t \in C^0([0, +\infty); L^2_{w_1} \cap H^1) \cap L^2([0, +\infty); L^2_{w_1} \cap H^1)$$

and, if $\bar{M}_0^2 := e^{-c_- x_1/2} + |(\phi_0, \phi_1)|^2_{\langle x \rangle_+} + \|\phi_0\|_2^2 + \|\phi_1\|_1^2$,

$$|(\phi, \phi_t)(t)|^2_{w_1} + \|\phi_x(t)\|_1^2 + \|\phi_{xt}(t)\|^2 + \int_0^t [|(\phi_x, \phi_t)(\tau)|^2_{w_1} + \|(\phi_{xx}, \phi_{xt})(\tau)\|^2] \, d\tau \leqslant C\bar{M}_0^2, \tag{4.27}$$

which in particular implies (4.26).

THEOREM 4.7 (Exponential Rate). *Under the assumptions in Theorem* 4.3, *then the IBVP* (4.24) *has a unique global solution satisfying*

$$\phi \in C^0(0, \infty; H^2_{w_2}) \cap L^2(0, \infty; H^2_{w_2}), \qquad \phi_t \in C^0(0, \infty; H^1_{w_2}) \cap L^2(0, \infty; H^1_{w_2})$$

and, if $\hat{M}_0^2 := e^{-c_- x_1/2} + |\phi_0|^2_{2, w_{0,2}} + |\phi_1|^2_{1, w_{0,2}}$,

$$|\phi(t)|^2_{2, w_2} + |\phi_t(t)|^2_{1, w_2} + \theta \int_0^t [|\phi(\tau)|^2_{2, w_2} + |\phi_t(\tau)|^2_{1, w_2}] \, d\tau \leqslant C\hat{M}_0^2, \tag{4.28}$$

namely,

$$|\phi(t)|^2_{2, w_2} + |\phi_t(t)|^2_{1, w_2} \leqslant C\hat{M}_0^2 e^{-\theta t}. \tag{4.29}$$

THEOREM 4.8 (Algebraic Rates). *Under the assumptions in Theorem* 4.4, *denoting* $M_1^2 := |(\phi_0, \phi_1)|^2_\alpha + M_0^2$, $\bar{M}_1^2 := |(\phi_0, \phi_1)|^2_{\alpha\langle x \rangle_+} + \bar{M}_0^2$, *then the solution* $\phi(x, t)$ *of* (4.24) *satisfies*

$$\sup_{x\in\mathbb{R}_+} |(\phi, \phi_x, \phi_t)(x,t)| \leqslant CM_1(1+t)^{-\alpha/2}, \qquad \textit{for} \quad f'(u_+)<s<f'(u_-), \tag{4.30}$$

$$\sup_{x\in\mathbb{R}_+} |(\phi, \phi_x, \phi_t)(x,t)| \leqslant C\bar{M}_1(1+t)^{-\alpha/4}, \qquad \textit{for} \quad f'(u_+)=s<f'(u_-). \tag{4.31}$$

These theorems can be shown by the continuity argument dependent on the local existence result together with the *a priori* estimates. We will omit here the local existence result since it is standard, while the *a priori* estimates will be shown in the following two subsections.

4.1.4. *Convergence to Front Waves and Exponential Decay Rat*e. In this subsection, we are going to prove Theorems 4.6 and 4.7. We focus on the nondegenerate case $f'(u_+)<s<f'(u_-)$, because the degenerate case $f'(u_+)=s<f'(u_-)$ can be similarly treated.

Define the solution spaces of (4.24) as

$$X_1=\{\phi\in C^0(0,T;H^2):\phi_x\in L^2(0,T;H^1), \phi_t\in C^0(0,T;H^1)\cap L^2(0,T;H^1)\},$$

$$X_2=\{\phi\in C^0(0,T;H^2_{w_2})\cap L^2(0,T;H^2_{w_2}):\phi_t\in C^0(0,T;H^1_{w_2})\cap L^2(0,T;H^1_{w_2})\},$$

and let

$$N_1(T)=\sup_{0\leqslant t\leqslant T}\{\|\phi(t)\|_2^2+\|\phi_t(t)\|_1^2\},$$

$$N_2(T)=\sup_{0\leqslant t\leqslant T}\{|\phi(t)|^2_{2,w_2}+|\phi_t(t)|^2_{1,w_2}\},$$

for $T\in[0,+\infty]$. In order to obtain *a priori* estimates, in what follows we will assume to have $N_i(t)$ $(i=1,2)$ small enough. To say this we will use the notation $N_i(t)\ll 1$ $(i=1,2)$. Before proving the basic energy estimate, we need the following estimates for the boundary in lower order differential form.

LEMMA 4.9. *Let us assume* $N_i(t)$ *bounded. It holds for* $i=1,2$

$$\int_0^t \left(|aw_i\phi\phi_x|+\frac{a}{2}|w_{ix}\phi^2|+\frac{1}{2}|w_if'\phi^2|+|2aw_i\phi_t\phi_x|\right)\Big|_{x=0} d\tau \leqslant aCe^{-c_-x_1/2}, \tag{4.32}$$

where $C>0$ *depends on* $N_i(t)$, $i=1,2$ *but it is independent from* a *and* x_1.

Proof. We have $|\phi_{x}|_{x=0}| \leqslant CN_i(t) \leqslant C$ by the Sobolev's inequality, and $|\phi|_{x=0}| = |A(t)| = O(1)\, e^{-c_- x_1} e^{-c_- st}$, $w_1(U) = O(1)$, $w_2(U)|_{x=0} = O(1)\, e^{c_- x_1/2} e^{c_- st/2}$, by (4.23) and (2.17). As a consequence $|(w_1(U)\,\phi)|_{x=0}| = O(1)\, e^{-c_- x_1} e^{-c_- st}$ and $|(w_2(U)\,\phi)|_{x=0}| = O(1)\, e^{-c_- x_1/2} e^{-c_- st/2}$ hold, so we obtain the estimate

$$\int_0^t |(aw_i \phi\phi_x)|_{x=0}|\, d\tau$$

$$\leqslant aCN_i(t) \int_0^t |[w_i(U)\,\phi]|_{x=0}|\, d\tau \leqslant aCe^{-c_- x_1/2}, \qquad i = 1, 2. \tag{4.33}$$

On the other hand from (2.18) we have $|w_{ix}| = O(1)^{(|u_+ - u_-|/(a - s^2))}\, w_i$. So, since $|\phi_t(0, t)| = |A'(t)| \sim A(t)$, and $|f'(U)|$ is bounded, by the same way in (4.33), we can prove

$$\int_0^t \left(\left| \frac{a}{2} w_{ix} \phi^2 \right| + \frac{1}{2} |w_i f' \phi^2| + |2aw_i \phi_t \phi_x| \right) \bigg|_{x=0} d\tau \leqslant aCe^{-c_- x_1/2}, \qquad i = 1, 2. \tag{4.34}$$

Combining (4.33) and (4.34), we have completed the proof of the lemma. ∎

Lemma 4.10 (Basic Energy Estimate). *Let us assume the solution $\phi \in X_1(0, T)$ for a fixed $T > 0$. Then*

$$\|(\phi, \sqrt{a}\phi_x, \phi_t)(t)\|^2 + \int_0^t [a\, \|(\phi_x(\tau)\|^2 + \|\phi_t(\tau)\|^2]\, d\tau$$

$$\leqslant aC\{e^{-c_- x_1/2} + \|(\phi_0, \phi_{0,x}, \phi_1)\|^2\}, \tag{4.35}$$

holds for $t \in [0, T]$, provided $N_1(T) \ll 1$, where $C > 0$ is independent of a, x_1 and (ϕ_0, ϕ_1).

Moreover, if we assume the solution $\phi \in X_2(0, T)$, then there exists a constant $\theta_1 > 0$ such that

$$|(\phi, \sqrt{a}\phi_x, \phi_t)(t)|^2_{w_2} + \theta_1 \int_0^t \|(\phi, \sqrt{a}\phi_x, \phi_t)(\tau)\|^2_{w_2}]\, d\tau$$

$$\leqslant aC\{e^{-c_- x_1/2} + |(\phi_0, \phi_{0,x}, \phi_1)|^2_{w_2}\}, \tag{4.36}$$

holds, provided $N_2(T) \ll 1$, where $C > 0$ is independent of a, x_1, and (ϕ_0, ϕ_1).

Proof. As in [13, 12, 27], we multiply (4.24) by $2w_i(U)\,\phi$ and $2w_i\phi_t$, $i=1, 2$, respectively, to have

$$2w_i(U)\,\phi\cdot L(\phi)=-2w_i(U)\,\phi F, \qquad i=1,2, \tag{4.37}$$

$$2w_i(U)\,\phi_t\cdot L(\phi)=-2w_i(U)\,\phi_t F, \qquad i=1,2. \tag{4.38}$$

Combining $(4.37)\times\frac{1}{2}+(4.38)$, by a simple but tedious computation, we have

$$\{E_1(\phi,\phi_t)+E_2(\phi_x)\}_t+E_3(\phi_x,\phi_t)+E_4(\phi)-\{\cdots\}_x=-2Fw_i[\tfrac{1}{2}\phi+\phi_t], \tag{4.39}$$

where

$$E_1(\phi,\phi_t)=w_i\left[\phi_t^2+\phi\phi_t+\frac{1}{2}\left(1+s\frac{w_{ix}}{w_i}\right)\phi^2\right], \tag{4.40}$$

$$E_2(\phi_x)=aw_i\phi_x^2, \tag{4.41}$$

$$E_3(\phi_x,\phi_t)=w_i\left[\left(1+s\frac{w_{ix}}{w_i}\right)\phi_t^2+2\left(f'+a\frac{w_{ix}}{w_i}\right)\phi_x\phi_t+a\left(1+s\frac{w_{ix}}{w_i}\right)\phi_x^2\right], \tag{4.42}$$

$$E_4(\phi)=-\frac{1}{2}\,(w_ih)''\,(U)\;U_x\phi^2, \tag{4.43}$$

$$\{\cdots\}=\left\{aw_i\phi\phi_x-\frac{1}{2}\,w_{ix}\phi^2-\frac{1}{2}\,w_if'\phi^2+2aw_i\phi_t\phi_x\right\}. \tag{4.44}$$

When $a>0$ is suitably large, namely, $|w_{ix}/w_i|=O(1)^{(|u_+-u_-|/(a-s^2))}\ll 1$, by using the subcharacteristic condition (1.4), with a similar argument in [13, 27], we conclude

$$D_1:\equiv 1-4\times\frac{1}{2}\left(1+s\frac{w_{ix}}{w_i}\right)=-\left(1+2s\frac{w_i(U)_x}{w_i(U)}\right)\leqslant -C<0, \qquad i=1,2, \tag{4.45}$$

$$D_3:\equiv 4\left(f'(U)+a\frac{w_i(U)_x}{w_i(U)}\right)^2-4a\left(1+s\frac{w_i(U)_x}{w_i(U)}\right)^2\leqslant -4C<0, \qquad i=1,2, \tag{4.46}$$

for some constant $C>0$, where D_1 and D_3 denote the discriminants of E_1 and E_3, respectively. Thus, we have

$$E_1(\phi, \phi_t) \geqslant c_1 w_i \phi^2 + c_2 w_i \phi_t^2, \qquad i = 1, 2, \tag{4.47}$$

$$E_3(\phi_x, \phi_t) \geqslant c_3 w_i \phi_x^2 + c_4 w_i \phi_t^2, \qquad i = 1, 2, \tag{4.48}$$

for some positive constants c_j, $j = 1, 2, 3, 4$.

On the other hand, from (2.18) and (2.19) in Lemma 2.2, we have

$$E_4(\phi) = \begin{cases} |U_x| \, \phi^2, & \text{for} \quad w_1(U), \\ O(1) \, w_2(U) \, \phi^2, & \text{for} \quad w_2(U). \end{cases} \tag{4.49}$$

Integrating (4.39) over $\mathbb{R}_+ \times [0, t]$, using (4.47), (4.48), (4.49) and the boundary estimate (4.32) into it, since the nonlinear term can be controlled as

$$\left| \int_0^t \int_0^{+\infty} F w_i (\phi + 2\phi_t) \, dx \, d\tau \right| \leqslant C N_i(T) \int_0^t |\phi_x(\tau)|_{w_i}^2 \, d\tau$$

due to $|F| \leqslant C \phi_x^2$, we prove the basic estimate (4.35) provided $N_1(T) \ll 1$ and the estimate (4.36) with some constant $\theta_1 > 0$ provided $N_2(T) \ll 1$. ∎

The next step is to do a bit of effort for the boundary in the higher order case.

LEMMA 4.11. *It holds for* $i = 1, 2$,

$$\int_0^t \left(|a w_i \phi_x \phi_{xx}| + \frac{a}{2} |w_{i,x} \phi_x^2| + \frac{1}{2} |w_i f' \phi_x^2| \right) \Big|_{x=0} d\tau$$

$$\leqslant a C (e^{-c_- x_1/2} + |\phi_0|_{1, w_i}^2 + |\phi_1|_{w_i}^2), \tag{4.50}$$

$$\left| \int_0^t (a w_i \phi_{xt} \phi_{xx})|_{x=0} \, d\tau \right|$$

$$\leqslant C \, |\phi_{xx}(t)|_{w_i}^2 + a C (e^{-c_- x_1/2} + |\phi_0|_{1, w_i}^2 + |\phi_1|_{w_i}^2), \tag{4.51}$$

where $C > 0$ *is independent of* a, x_1, *and* (ϕ_0, ϕ_1).

Proof. First, we prove

$$\int_0^t (w_i \phi_x^2)|_{x=0} \, d\tau \leqslant a C (e^{-c_- x_1/2} + |\phi_0|_{1, w_i}^2 + |\phi_1|_{w_i}^2), \qquad i = 1, 2, \tag{4.52}$$

$$\int_0^t |(a w_i \phi_x \phi_{xx})_{x=0}| \, d\tau \leqslant a C (e^{-c_- x_1/2} + |\phi_0|_{1, w_i}^2 + |\phi_1|_{w_i}^2), \qquad i = 1, 2. \tag{4.53}$$

Since $2w_i\phi_x \cdot L(\phi) = -2w_i\phi_x F$, we have

$$-\{aw_i\phi_x^2\}_x = -\{2w_i\phi_x\phi_t\}_t + 2w_{it}\phi_x\phi_t + \{w_i\phi_t^2\}_x - w_{ix}\phi_t^2$$
$$-2w_i\phi_x\phi_t - aw_{ix}\phi_x^2 - 2w_i f'\phi_x^2 - 2w_i\phi_x F. \qquad (4.54)$$

Integrating (4.54) on $[0, t] \times \mathbb{R}_+$, using $|\frac{1}{s}w_{it}| = |w_{ix}| = O(1)^{(|u_+ - u_-|/(a-s^2))} w_i$, the boundary estimate (4.32), the basic estimates (4.35) for $i=1$, and (4.36) for $i=2$, we have

$$\begin{aligned}
&a\int_0^t [w_i\phi_x^2]|_{x=0}\,d\tau \\
&\quad\leqslant 2\int_0^\infty w_i\,|\phi_x\phi_t|\,dx + 2\int_0^\infty w_{i0}\,|\phi_{0,x}\phi_0|\,dx + 2C\int_0^t\int_0^\infty w_i\,|\phi_x\phi_t|\,dx\,d\tau \\
&\qquad + \int_0^t [w_i\phi_t^2]|_{x=0}\,d\tau + C\int_0^t\int_0^\infty w_i\phi_t^2\,dx\,d\tau \\
&\qquad + C\int_0^t\int_0^\infty w_i\phi_x^2\,dx\,d\tau + CN_i(t)\int_0^t |\phi_x(\tau)|^2_{\omega_i}\,d\tau \\
&\quad\leqslant C\Big\{|\phi_x(t)|^2_{w_i} + |\phi_t(t)|^2_{w_i} + |\phi_{0,x}|^2_{w_i} + |\phi_1|^2_{w_i} \\
&\qquad + \int_0^t [|\phi_x(\tau)|^2_{w_i} + |\phi_t(\tau)|^2_{w_i}]\,d\tau + e^{-c_- x_1/2}\Big\} \\
&\quad\leqslant aC(e^{-c_- x_1/2} + |\phi_0|^2_{1,w_i} + |\phi_1|^2_{w_i}).
\end{aligned}$$

This proves (4.52). To prove (4.53), let us use (4.24) to write

$$a\phi_{xx}|_{x=0} = [\phi_{tt} + \phi_t + f(U+\phi_x) - f(U)]|_{x=0}$$
$$= A''(t) + A'(t) + O(1)\,\phi_x|_{x=0}. \qquad (4.55)$$

Then using (4.52) and (4.55) we can easily prove (4.53) as

$$\begin{aligned}
&\int_0^t |(aw_i\phi_x\phi_{xx})|_{x=0}|\,d\tau \\
&\quad = \int_0^t |(w_i\phi_x)_{x=0}\,(A''(\tau) + A'(\tau) + (\pm O(1)\,\phi_x|_{x=0})|\,d\tau \\
&\quad\leqslant CN_i(T)\int_0^t w_i|_{x=0}(|A''(\tau)| + |A'(\tau)|)\,d\tau + C\int_0^t (w_i\phi_x^2)|_{x=0}\,d\tau \\
&\quad\leqslant aC(e^{-c_- x_1/2} + |\phi_0|^2_{w_i} + |\phi_1|^2_{w_i}), \qquad i = 1, 2.
\end{aligned}$$

Therefore, by (4.52) and (4.53), and using $|w_{ix}| = O(1)^{(|u_+ - u_-|/(a-s^2))} w_i(U)$ and $|f'(U)| \leqslant C$, we complete the proof of (4.50).

Now we are going to prove (4.51). Noting (4.55) and using the integration by part with respect to t, we have

$$\begin{aligned}
&\left| \int_0^t (aw_i\phi_{xt}\phi_{xx})_{x=0}\,d\tau \right| \\
&\quad = \left| \int_0^t (w_i\phi_{xt})|_{x=0}\,\{A''(\tau)+A'(\tau)+[f(U+\phi_x)-f(U)]|_{x=0}\}\,d\tau \right| \\
&\quad \leqslant \left| \left\{ ([w_i\phi_x]|_{x=0}\cdot[A''(\tau)+A'(\tau)])|_0^t \right.\right. \\
&\qquad - \int_0^t [w_i\phi_x]|_{x=0}\cdot[A'''(\tau)+A''(\tau)]\,d\tau \\
&\qquad \left.\left. - \int_0^t [w_{it}\phi_x]|_{x=0}\cdot[A''(\tau)+A'(\tau)]\,d\tau \right\} \right| \\
&\qquad + \left| \int_0^t [w_i\phi_{xt}]|_{x=0}\cdot[f(U+\phi_x)-f(U)]|_{x=0}]\,d\tau \right| \\
&\quad =: I_1 + I_2.
\end{aligned} \tag{4.56}$$

Since $|A''(t)| \sim |A'(t)| \sim e^{-c_-(st+x_1)}$, $w_1(U) \sim O(1)$, $w_2(U)|_{x=0} \sim e^{c_-(st+x_1)/2}$, we have $w_i(U)|_{x=0}\cdot(|A''(t)|+|A'(t)|) \leqslant Ce^{-c_-(st+x_1)/2}$ for all $t \geqslant 0$, and by the Sobolev inequality $|\phi_x(0,t)| \leqslant CN_i(T) \leqslant C$, we get

$$|([w_i\phi_x]|_{x=0}\cdot[A''(\tau)+A'(\tau)])|_0^t| \leqslant Ce^{-c_-x_1/2}. \tag{4.57}$$

On the other hand, by the facts $\int_0^t w_i|_{x=0}(|A'''(\tau)|+|A''(\tau)|+|A'(\tau)|)\,d\tau \leqslant Ce^{-c_-x_1/2}$, and $|w_{it}| = s\,|w_{ix}| = O(1)^{(s\,|u_+ - u_-|/(a-s^2))}\,w_i$, we obtain

$$\begin{aligned}
&\int_0^t |[w_i\phi_x]|_{x=0}\cdot[A'''(\tau)+A''(\tau)]| + |[w_{it}\phi_x]|_{x=0}\cdot[A''(\tau)+A'(\tau)]|\,d\tau \\
&\quad \leqslant Ce^{-c_-x_1/2}.
\end{aligned} \tag{4.58}$$

Hence, thanks to (4.57) and (4.58), we proved

$$I_1 \leqslant Ce^{-c_-x_1/2}. \tag{4.59}$$

To control I_2, let us define

$$q(t) := \int_0^{\phi_x(0,t)} [f(U+\eta)-f(U)]\,d\eta. \tag{4.60}$$

We can easily check that

$$q_t(t) = ([f(U+\phi_x) - f(U)]\,\phi_{xt})|_{x=0} + U_t|_{x=0} \cdot \int_0^{\phi_x(0,\,t)} [f'(U+\eta) - f'(U)]\,d\eta, \tag{4.61}$$

and, by using $|f'(U+\bar{\eta})| \leqslant C$,

$$|q(t)| = \left|\tfrac{1}{2}\int_0^{\phi_x(0,\,t)} f'(U+\bar{\eta})\,d\eta^2\right| \leqslant C\phi_x(0,t)^2, \tag{4.62}$$

where $\bar{\eta} \in (0, \phi_x(0,t))$.

By the Sobolev inequality and the basic estimates in Lemma 4.10 for $|\phi_x(t)|^2_{w_i}$, we have

$$|\sqrt{w_i}\phi_x|^2 \leqslant C\,|\phi_x(t)|^2_{1,\,w_i} \leqslant C\,|\phi_{xx}|^2_{w_i} + aC(e^{-c_-x_1/2} + |\phi_0|^2_{1,\,w_i} + |\phi_1|^2_{w_i}). \tag{4.63}$$

Hence, from (4.60)–(4.63) and (4.52) we can control I_2 as

$$\begin{aligned} I_2 &= \left|\int_0^t [w_i\phi_{xt}]|_{x=0} \cdot \{f(U+\phi_x) - f(U)\}|_{x=0}\,d\tau\right| \\ &= \left|\int_0^t w_i|_{x=0} \cdot q_t(\tau)\,d\tau - \int_0^t (w_i U_t)|_{x=0} \right. \\ &\qquad \left. \cdot \int_0^{\phi_x(0,\,\tau)} [f'(U+\eta) - f'(U)]\,d\eta\,d\tau\right| \\ &\leqslant \left|(w_i|_{x=0}\,q(\tau))|_0^t - \int_0^t w_{it}|_{x=0} \cdot q(\tau)\,d\tau\right| \\ &\qquad + C\int_0^t |(w_i U_t \phi_x)|_{x=0}|\,d\tau \\ &\leqslant (w_i\phi_x^2)|_{\{x=0,\,\tau=t\}} + (w_i\phi_x^2)|_{\{x=0,\,\tau=0\}} \\ &\qquad + C\frac{s\,|u_+ - u_-|}{a - s^2}\int_0^t (w_i\phi_x^2)|_{x=0}\,d\tau + sCN_i(t)\int_0^t (w_i\,|U_x|)|_{x=0}\,d\tau \\ &\leqslant C\,|\phi_{xx}(t)|^2_{w_i} + aC(e^{-c_-x_1/2} + |\phi_0|^2_{1,\,w_i} + |\phi_1|^2_{w_i}). \end{aligned} \tag{4.64}$$

Thus, substituting (4.59) and (4.64) into (4.56), we have completed the proof of (4.51). ∎

LEMMA 4.12. *Let us assume the solution* $\phi \in X_i(0, T)$, $i=1, 2$. *Then, for some constant* $\theta_2 > 0$,

$$|(\phi_x, \phi_{xt})(t)|^2_{w_i} + a\, |\phi_{xx}(t)|^2_{w_i} + \theta_2 \int_0^t [a\, |(\phi_{xx}(\tau)|^2_{w_2} + |(\phi_x, \phi_{xt})(\tau)|^2_{w_2}]\, d\tau$$

$$\leqslant aCM^2 \tag{4.65}$$

holds, provided $N_i(T) \ll 1$, $i=1, 2$. *Here* $M = M_0$ *or* $\hat{M}_0$ *when* $i=1$ *or* 2.

Proof. Since

$$\int_0^t \int_0^\infty w_i(\phi_x + 2\phi_{xt}) \cdot \partial_x L(\phi)\, dx\, d\tau = -\int_0^t \int_0^\infty w_i(\phi_x + 2\phi_{xt})\, F\, dx\, d\tau$$

using the basic estimate Lemma 4.10, and the boundary estimate Lemma 4.11 we obtain, by using again the argument used in Lemma 4.10,

$$|(\phi_x, \phi_{xt})(t)|^2_{w_i} + (a - C)\, |\phi_{xx}(t)|^2_{w_i} + \theta_2' \int_0^t [a\, |(\phi_{xx}(\tau)|^2_{w_i} + |(\phi_x, \phi_{xt})(\tau)|^2_{w_i}]\, d\tau$$

$$\leqslant aCM^2. \tag{4.66}$$

Since $a \gg 1$, the above estimate (4.66) implies (4.65) for some $\theta_2 > 0$, which completes the proof. ∎

Combining Lemmas 4.10 and 4.12, we prove Theorem 4.6 and Theorem 4.7. Therein we take $\theta = \min\{\theta_1, \theta_2\}$ for Theorem 4.7.

4.1.5. *Algebraic Decay Rate.* In this subsection, we are going to prove the algebraic decay rates. First, we pay our attention to the nondegenerate case $f'(u_+) < s < f'(u_-)$. Let us define $\bar{u} := (u_+ + u_-)/2$. Since U is strictly decreasing in $\mathbb{R}$, there exists a unique number $z^* \in R$ such that $U(z^*) = \bar{u}$. Denote $K(x, t) = (1+t)^\gamma \langle (z - z^*)/a \rangle^\beta\, w_1(U)$, $\bar{K}(x, t) = (1+t)^\gamma \langle (z - z^*)/a \rangle^\beta$, i.e., $K(x, t) = \bar{K}(x, t)\, w_1(U)$, where $U = U(z)$, $z = x - st + x_0 - x_1$. Multiplying Eq. (4.24) by $2K(x, t)\, \phi$ and $2K(x, t)\, \phi_t$, respectively, yields

$$2K(x, t)\, \phi \cdot L(\phi) = -2K(x, t)\, \phi F, \tag{4.67}$$

$$2K(x, t)\, \phi_t \cdot L(\phi) = -2K(x, t)\, \phi_t F. \tag{4.68}$$

Combining $(4.67) \times \frac{1}{2} + (4.68)$, by a straightforward but tedious calculation as in [12], we obtain

$$\left\{K\phi_t^2 + K\phi_t\phi + \frac{1}{2}(K + sK_x)\phi^2 + aK\phi_x^2\right\}_t$$

$$-\frac{\gamma}{1+t}\left[K\phi_t^2 + K\phi\phi_t + \frac{1}{2}(K+sK_x)\phi^2 + aK\phi_x^2\right]$$

$$+(K+sK_x)\phi_t^2 + 2(f'(U)K + aK_x)\phi_x\phi_t + a(K+sK_x)\phi_x^2$$

$$+(a-s^2)\bar{K}_x w_1(U)\phi\phi_x + \frac{1}{2}P_\beta\phi^2 + \{\text{Boundary}\}_x$$

$$= -K(\phi + 2\phi_t)F, \tag{4.69}$$

where

$$\{\text{Boundary}\} := \{a\bar{K}w_1\phi\phi_x - \tfrac{1}{2}a(\bar{K}_x w_1 + \bar{K}w_{1x})\phi^2 - \tfrac{1}{2}\bar{K}w_1 f'(U)\phi^2 + 2a\bar{K}w_1\phi_t\phi_x + \tfrac{1}{2}(a-s^2)\bar{K}_x w_1\phi^2\}, \tag{4.70}$$

while $P_\beta(z) := -\bar{K}_x(w_1 h)' - \bar{K}(w_1 h)'' U_x$ satisfies the following lemma proved in [12].

Lemma 4.13 [12]. *Let α be a given positive number. For $\beta \in [0, \alpha]$, there exists a constant $c_0 > 0$ independent of β such that*

$$P_\beta(z) \geqslant c_0\beta(1+t)^\gamma \langle (z - z^*)/a\rangle^{\beta-1} \qquad \text{for any} \quad z \in R. \tag{4.71}$$

Since $\bar{K}(x,t)|_{x=0} = (1+t)^\gamma \langle(-st + x_0 - x_1)/a\rangle^\beta$, $\bar{K}_x(x,t)|_{x=0} = 2\beta(1+t)^\gamma \langle(-st+x_0-x_1)/a\rangle^{\beta-2}(-st+x_0-x_1)/a$, $w_1(U) = O(1)$ and $|w_{1x}(U)| \leqslant C^{(|u_+ - u_-|/(a-s^2))} w_1(U)$, we get

$$\int_0^t [\bar{K}(0,\tau) + |\bar{K}_x(0,\tau)|]\, e^{-c_-(s\tau + x_1)}\, d\tau \leqslant Ce^{-c_- x_1/2}$$

for all $t \geqslant 0$, so, the boundary integration can be controlled as follows by a similar fashion as in Lemma 4.9. Here, we omit the details.

Lemma 4.14. *It holds*

$$\int_0^t |\{\textit{Boundary}\}_{|x=0}|\, d\tau \leqslant aCe^{-c_- x_1/2}. \tag{4.72}$$

Since

$$\left|\frac{K_x}{K}\right| = \left|\frac{\bar{K}w_{ix} + \bar{K}_x w_i}{\bar{K}w_i}\right| = \left|\frac{w_{ix}}{w_1} + \frac{\bar{K}_x}{\bar{K}}\right| \leqslant \left|\frac{w_{ix}}{w_1}\right| + \left|\frac{\beta}{a}\frac{(z-z^*)/a}{\langle(z-z^*)/a\rangle^2}\right| \leqslant \frac{C}{a} \ll 1$$

for $a \gg 1$, denoting by D_5 and D_6 the discriminantes of E_5 and E_6, respectively, we have

$$D_5 = -1 - 2sK_x/K < 0,$$
$$D_6 = 4[(f' + aK_x/K)^2 - a(1 + sK_x/K)^2] < 0.$$

Thus, we get

$$E_5 := K\phi_t^2 + K\phi_t\phi + \tfrac{1}{2}(K + sK_x)\,\phi^2 \geqslant CK(\phi^2 + \phi_t^2), \tag{4.73}$$

$$E_6 := (K + sK_x)\,\phi_t^2 + 2(f'(U)\,K + aK_x\phi_x\phi_t + a(K + sK_x)\,\phi_x^2 \geqslant CK(\phi_x^2 + \phi_t^2) \tag{4.74}$$

for some $C > 0$.

After integrating (4.69) over $\mathbb{R}_+ \times [0, t]$, using (4.71)–(4.74), it yields

$$\begin{aligned}
&(1+t)^\gamma\,|(\phi, \phi_x, \phi_t)(t)|_\beta^2 + \beta \int_0^t (1+\tau)^\gamma\,|\phi(\tau)|_{\beta-1}^2\,d\tau \\
&\quad + \int_0^t (1+\tau)^\gamma\,|(\phi_x, \phi_t)(\tau)|_\beta^2\,d\tau \\
&\leqslant C\left\{a\,|(\phi, \phi_x, \phi_t)(0)|_\beta^2 + ae^{-c_- x_1/2}\right. \\
&\quad + \gamma \int_0^t (1+\tau)^{\gamma-1}\,|(\phi, \phi_x, \phi_t)(\tau)|_\beta^2\,d\tau + (a - s^2) \int_0^t \int_0^\infty |\bar{K}_x|\,|\phi\phi_x|\,dx\,d\tau \\
&\quad \left. + \int_0^t \int_0^\infty K(x, \tau)\,|(\phi + 2\phi_t)\,F|\,dx\,d\tau\right\},
\end{aligned} \tag{4.75}$$

where $|\cdot|_\beta = |\cdot|_{\langle z\rangle^\beta}$, $z = x - st + x_0 - x_1$.

Making a similar estimate for

$$C \int_0^t \int_0^\infty |\bar{K}_x\phi\phi_x|\,dx\,d\tau \leqslant \tfrac{1}{2} \int_0^t |\phi_x(\tau)|_\beta^2\,d\tau + \beta C \int_0^t \|\phi_x(\tau)\|^2\,d\tau \tag{4.76}$$

as shown in [12], and controlling the nonlinear term by a usual way as

$$\int_0^t \int_0^\infty K(x, \tau)\,|(\phi + 2\phi_t)\,F|\,dx\,d\tau \leqslant CN_1(t) \int_0^t (1+\tau)^\gamma\,|(\phi_x, \phi_t)(\tau)|_\beta^2\,d\tau, \tag{4.77}$$

then applying (4.76) and (4.77) into (4.75), we proved

$$(1+t)^{\gamma}\,|(\phi,\phi_x,\phi_t)(t)|_{\beta}^2+\frac{\beta}{2}\int_0^t(1+\tau)^{\gamma}\,|\phi(\tau)|_{\beta-1}^2\,d\tau$$

$$+\left(\frac{1}{2}-CN_1(t)\right)\int_0^t(1+\tau)^{\gamma}\,|(\phi_x,\phi_t)(\tau)|_{\beta}^2\,d\tau$$

$$\leqslant C\left\{a\,|(\phi,\phi_x,\phi_t)(0)|_{\beta}^2+ae^{-c_-x_1/2}\right.$$

$$\left.+\gamma\int_0^t(1+\tau)^{\gamma-1}\,|(\phi,\phi_x,\phi_t)(\tau)|_{\beta}^2\,d\tau+\beta\int_0^t(1+\tau)^{\gamma}\,\|\phi(\tau)\|^2\,d\tau\right\},$$

which implies the following estimates.

Lemma 4.15. *The following estimates hold for* $t\in[0,T]$, *provided* $N_1(t)\ll 1$,

$$(1+t)^{\gamma}\,|(\phi,\phi_x,\phi_t)(t)|_{\beta}^2+\int_0^t\{\beta(1+\tau)^{\gamma}\,|\phi(\tau)|_{\beta-1}^2+(1+\tau)^{\gamma}\,|(\phi_x,\phi_t)(\tau)|_{\beta}^2\}\,d\tau$$

$$\leqslant C\left\{|(\phi,\phi_x,\phi_t)(0)|_{\beta}^2+e^{-c_-x_1/2}+\gamma\int_0^t(1+\tau)^{\gamma-1}\,|(\phi,\phi_x,\phi_t)(\tau)|_{\beta}^2\,d\tau\right.$$

$$\left.+\beta\int_0^t(1+\tau)^{\gamma}\,\|\phi_x(\tau)\|^2\,d\tau\right\}\tag{4.78}$$

for any $\gamma\geqslant 0$ *and* $\beta\in[0,\alpha]$,

$$(1+t)^{\gamma}\,|(\phi,\phi_x,\phi_t)(t)|_{\alpha-\gamma}^2+(\alpha-\gamma)\int_0^t(1+\tau)^{\gamma}\,|\phi(\tau)|_{\alpha-\gamma-1}^2\,d\tau$$

$$+\int_0^t(1+\tau)^{\gamma}\,|(\phi_x,\phi_t)(\tau)|_{\alpha-\gamma}^2\,d\tau$$

$$\leqslant C(|(\phi,\phi_x,\phi_t)(0)|_{\alpha}^2+e^{-c_-x_1/2})\tag{4.79}$$

for γ *integer in* $[0,\alpha]$.

The estimate (4.79) can be derived from (4.78), with a similar argument used in the case of the Cauchy problem in [12] (for the original proof see also [10]). Based on this lemma, as in [27] (see the Lemma 5.2 therein) or in [30] for the Burger's equation, we may immediately get the following optimal decay rate without any difficulty.

LEMMA 4.16. *It holds for any* $\varepsilon > 0$

$$(1+t)^{\alpha} \|(\phi, \phi_x, \phi_t)(t)\|^2 + (1+t)^{-\varepsilon} \int_0^t (1+\tau)^{\alpha+\varepsilon} \|(\phi_x, \phi_t)(\tau)\|^2 \, d\tau$$
$$\leqslant C(|(\phi, \phi_x, \phi_t)(0)|_{\alpha}^2 + e^{-c_- x_1/2}). \tag{4.80}$$

For the higher derivatives of the solution, since the same boundary estimates in (4.50) and (4.51) are bounded due to the estimate (4.65), by a similar procedure in Lemma 4.15, we can have the estimates as follows.

LEMMA 4.17. *It holds for any* $\varepsilon > 0$

$$(1+t)^{\alpha} \|\partial_x(\phi, \phi_x, \phi_t)(t)\|^2 + (1+t)^{-\varepsilon} \int_0^t (1+\tau)^{\alpha+\varepsilon} \|\partial_x(\phi_x, \phi_t)(\tau)\|^2 \, d\tau$$
$$\leqslant C(\|(\phi, \phi_x, \phi_t)(0)\|_2^2 + |(\phi, \phi_x, \phi_t)(0)|_{\alpha}^2 + e^{-c_- x_1/2}). \tag{4.81}$$

Combining Lemmas 4.16 and 4.17, we have completed the proof of Theorem 4.8 in the nondegenerate case $f'(u_+) < s < f'(u_-)$.

For the degenerate case $f'(u_+) = s < f'(u_-)$, since the boundary perturbations can be well controlled like the nondegenerate case $f'(u_+) < s < f'(u_-)$, taking a similar fashion as before, again available for the Cauchy problem case in [12, 25], we can prove the last part of Theorem 4.8. The details are omitted here.

4.2. *Case* $s < 0$, $g(t) = v_+$

Let $x_0 > 0$ be any given large constant. Our essential assumption in this subsection is

$$\int_0^{\infty} [u_0(x) - u_+] \, dx = 0. \tag{4.82}$$

By denoting $(U, V) = (U, V)(x - st + x_0)$, to determine the shift, we use that

$$\int_0^{\infty} [u(x,t) - U(x - st + x_0)] \, dx \to 0 \qquad \text{as} \quad t \to +\infty \tag{4.83}$$

for all x_0. We are going to reformulate the original IBVP (1.2) and (1.3).

Similar to Subsection 4.1.3, let us define the new unknowns as

$$\begin{cases} \phi(x,t) := -\displaystyle\int_x^{\infty} [u(y,t) - U(y - st + x_0)] \, dy \\ \psi(x,t) := v(x,t) - V(x - st + x_0), \end{cases} \tag{4.84}$$

then the original problem (1.2) and (1.3) can be reduced to

$$\begin{cases} \phi_t + \psi = 0, \\ \psi_t + a\phi_{xx} - f'(U)\,\phi_x + \psi = F, \end{cases} \tag{4.85}$$

where $F := f(\phi_x + U) - f(U) - f'(U)\,\phi_x$, with the initial boundary conditions

$$(\phi, \psi)|_{t=0} = \left(-\int_x^\infty [u_0(y) - U(y + x_0)]\, dy,\ v_0(x) - V(x + x_0) \right)$$

$$=: (\phi_0, \psi_0)(x), \tag{4.86}$$

$$\phi|_{x=0} = -\int_0^\infty [u(x, t) - U(x - st + x_0)]\, dx$$

$$= \int_t^\infty [v_+ - V(-s\tau + x_0]\, d\tau =: A_1(t), \tag{4.87}$$

$$\psi|_{x=0} = -\phi_t|_{x=0} = v_+ - V(-st + x_0) = -A_1'(t). \tag{4.88}$$

Substituting $\psi = -\phi_t$ into the second equation of (4.85), we get

$$\begin{cases} L(\phi) := \phi_{tt} + \phi_t - a\phi_{xx} + f'(U)\,\phi_x = -F, & x > 0,\ t > 0 \\ (\phi, \phi_t)|_{t=0} = (\phi_0, \phi_1)(x), & x > 0 \\ (\phi, \phi_t)|_{x=0} = (A_1, A_1')(t), & t > 0. \end{cases} \tag{4.89}$$

By a similar fashion to Subsection 4.1, we can also prove the following theorems. The details of the proof are omitted.

THEOREM 4.18 (Convergence). *Assume $f'(u_+) < s \leqslant f'(u_-)$ and (4.82). Let a be a large but fixed constant and suppose that $(\phi_0, \psi_0) \in H^2 \times H^1$. There exists a constant $\eta_1 > 0$ such that if $a(\|\phi_0\|_2 + \|\psi_0\|_1 + x_0^{-1}) < \eta_1$, then (4.89) has a unique global solution $\phi(x, t)$ satisfying*

$$\sup_{x \in \mathbb{R}_+} |(\phi_x, \phi_t)(x, t)| \to 0, \qquad as \quad t \to +\infty. \tag{4.90}$$

THEOREM 4.19 (Exponential Rate). *Assume $f'(u_+) < s \leqslant f'(u_-)$ and (4.82). Let a be a large but fixed constant and suppose that $(\phi_0, \psi_0) \in H^2_{w_2} \times H^1_{w_2}$. There exist constants $\eta_2 > 0$ and $\theta > 0$ such that if $a(|\phi_0|_{2, w_2} + |\psi_0|_{1, w_2} + x_0^{-1}) < \eta_2$, then (4.89) has a unique global solution $\phi(x, t)$ satisfying*

$$|(\phi(t)|^2_{2, w_2} + |\phi_t)(t)|^2_{1, w_2} \leqslant Ce^{-\theta t}. \tag{4.91}$$

THEOREM 4.20 (Algebraic Rates). *Assume* $f'(u_+) < s \leqslant f'(u_-)$ *and* (4.82). *Let* a *be a large but fixed constant and suppose* $(\phi_0, \psi_0) \in (L^2_\alpha \cap H^2) \times (L^2_\alpha \cap H^1)$ *for some* $\alpha > 0$. *There exists a constant* $\eta_3 > 0$ *such that if* $a(|\phi_0|_\alpha + |\psi_0|_\alpha + \|\phi_0\|_2 + \|\psi_0\|_1 + x_0^{-1}) < \eta_2$, *then* (4.89) *has a unique global solution* $\phi(x, t)$ *satisfying*

$$\sup_{x \in \mathbb{R}_+} |(\phi, \phi_x, \phi_t)(x, t)| \leqslant C\bar{M}_1(1+t)^{-\alpha/2}. \tag{4.92}$$

Remark 4.21. Since $s < 0$, $x \geqslant 0$ and $x_0 > 0$, namely, $x - st + x_0 > 0$, the back waves $(U, V)(x - st + x_0)$ do not go to the end state (u_-, v_-), i.e., $U \in [u_+, U(x_0)]$, which implies $w_1(U) \sim O(1)$, $w_2(U) \sim |U - u_+|^{-1/2}$ for both shock cases $f'(u_+) < s < f'(u_-)$ and $f'(u_+) < s = f'(u_-)$. Therefore, Theorems 4.18–4.20 hold for degenerate and nondegenerate shock cases.

5. CASE $g(t) = v_\mp$, $s = 0$

This section is devoted to the convergence toward the stationary waves. Since $V(x) \equiv v_\pm = f'(u_\pm)$, see (2.5) and (2.3), this is a special case of Section 3, so the convergence with decay rates to the stationary waves $(U, V)(x) = (U(x), v_+)$ can be well understood in Section 3. However, now we want to answer to the following question: Can we have the possibility to consider the convergence to the waves $(U, V)(x + d(t)) = (U(x + d(t)), v_+)$ for some shift function dependent on the time t? More precisely, can we find shift functions $d(t)$ such that the solutions of (1.2) and (1.3) converge to the shifted waves time-asymptotically when the initial and boundary perturbations to the shifted stationary waves $(U(x + d(t)), v_+)$ are small? The answer is positive and will be our main effort in this section. In order to do it, we have to restrict ourself on the shock cases $f'(u_+) < s = 0 \leqslant f'(u_-)$. To this end will be essential to choose a suitable shift function and to reformulate the original problem. As we shown before, to treat the nonconvexity of $f(u)$, the weight functions $w_i(U)$ $(i = 1, 2)$ are valid for the convergence theory in the case $s = 0$. But to show the algebraic decay, the weight $w_1(U)$ is now not sufficient, so we have to choose another new one. For the details, let us see (5.47) and (5.51) below.

5.1. *Shift Function and Main Theorems*

For any given constants $x_1 > 0$ and $d_0 > 0$ (x_1 may be taken large), we want to choose some smooth shift function, say $d(t) > 0$ in C^2 satisfying $d(0) = d_0$, such that the solutions $(u, v)(x, t)$ approach, as t goes to infinity, to the shifted stationary traveling waves $(U, V)(x + d(t) + x_1)$ with some decay rate, under the hypothesis of small initial perturbations, where we remember that $V(x)$ is a constant

$$V(x) \equiv v_+ = v_- = f(u_\pm), \tag{5.1}$$

see (2.3) and (2.5).

As in Subsection 4.2, here the hypothesis (4.82) will be essential. From the first equation of (1.2), we have

$$(u - U(x + d(t) + x_1))_t = -d'(t)\, U_x(x + d(t) + x_1)) - (v - V)_x, \tag{5.2}$$

where $d(t)$ is expected in C^2 and $d(t) > 0$, $d(0) = d_0$, $d(t) \to +\infty$ as $t \to +\infty$. Integrating (5.2) over $[0, +\infty)$ with respect to x and noting $v|_{x=+\infty} = v_+$, $v|_{x=0} = v_+$ and $V \equiv v_+ = v_-$, we have

$$\frac{d}{dt} \int_0^\infty [u(x, t) - U(x + d(t) + x_1)]\, dx = -d'(t)[u_+ - U(d(t) + x_1)],$$

which implies

$$\begin{aligned} &\int_0^\infty [u(x, t) - U(x + d(t) + x_1)]\, dx \\ &\quad = \int_0^\infty [u_0(x) - U(x + d_0 + x_1)]\, dx - \int_0^t d'(\tau)[u_+ - U(d(\tau) + x_1)]\, d\tau. \end{aligned} \tag{5.3}$$

As usual, to determine the shift we assume now

$$\int_0^\infty [u(x, t) - U(x + d(t) + x_1)]\, dx \to 0, \qquad \text{as} \quad t \to +\infty. \tag{5.4}$$

In other words, we expect that the right hand side of (5.3) tends to zero as $t \to +\infty$, namely,

$$\begin{aligned} I_d := &\int_0^\infty [u_0(x) - U(x + d_0 + x_1)]\, dx - \int_0^\infty d'(\tau)[u_+ - U(d(\tau) + x_1)]\, d\tau \\ = &\ 0. \end{aligned} \tag{5.5}$$

In fact, under the essential condition (4.82), for some shift $d(t)$ satisfying $d(t) > 0$, $d(0) = d_0$, $d(t) \to +\infty$ as $t \to +\infty$, and the following estimates hold

$$\begin{cases} \int_0^\infty e^{-c_+ d(t)}[1 + |d'(t) - d''(t)| + |d'(t)|^2]\, dt \leqslant C, \\ |d'(t)| \leqslant C, \end{cases} \tag{5.6}$$

for some constant $C>0$. We will see that (5.5) is true. Of course, (5.6) ensures the last integration in (5.5) is possible, since $|u_+ - U(d(t)+x_1)| \sim e^{-c_+(d(t)+x_1)}$ as $t \to +\infty$, so the definition (5.5) makes sense and

$$
\begin{aligned}
I_d &= \int_0^\infty [u_0(x) - U(x+d_0+x_1)]\,dx - \int_0^\infty d'(\tau)[u_+ - U(d(\tau)+x_1)]\,d\tau \\
&= \int_0^\infty [u_0(x) - U(x+d_0+x_1)]\,dx - \int_0^\infty [u_+ - U(y+d_0+x_1)]\,dy \\
&= \int_0^\infty [u_0(x) - u_+]\,dy = 0
\end{aligned}
$$

by using again the change of variable $y = d(t) - d_0$ previously used.

The shift function $d(t)$ satisfying (5.6) may include many examples. Two kinds of them are $d(t) = d_0 + b_1 \log(1+t)$ for all $b_1 > 1/c_+$, and $d(t) = d_0(1+t)^{b_2}$ for all $0 < b_2 \leqslant 1$. Especially, the first kind of examples, i.e., $d(t) = d_0 + b_1 \log(1+t)$, is just T.-P. Liu and Yu's shift function for the Burger's equation (1.5) with $f = u^2/2$ and $s = 0$ in [18].

We claim that when the shift function $d(t)$ satisfies (5.6), and the initial-boundary perturbation dealing with $d(t)$ is suitably small, we can prove the convergence of the solutions $(u, v)(x, t)$ to the shifted stationary waves $(U, V)(x + d(t) + x_1)$. However, to get the algebraic decay rate, instead of the condition (5.6), we need a stronger condition on $d(t)$ as

$$
\begin{cases}
\displaystyle\int_0^\infty (1+t)^\alpha e^{-c_+ d(t)}[1 + |d'(t) - d''(t)| + |d'(t)|^2]\,dt \leqslant C, \\
|d'(t)| \leqslant C
\end{cases}
\tag{5.7}
$$

for some constants $C>0$ and $\alpha>0$. Examples of functions satisfying (5.7) are again $d(t) = d_0 + b_1 \log(1+t)$ and $d(t) = d_0(1+t)^{b_2}$ but now respectively with $b_1 > \alpha/c_+$, and $0 < b_2 \leqslant 1$. To get the exponential decay rate, the restriction on the shift $d(t)$ is

$$
\begin{cases}
\displaystyle\int_0^\infty e^{-c_+ d(t)/2}[1 + |d'(t) - d''(t)| + |d'(t)|^2]\,dt \leqslant C, \\
|d^{(k)}(t)| \leqslant C, \qquad k = 1, 2,
\end{cases}
\tag{5.8}
$$

where C is some positive constant. Shift functions $d(t)$ satisfying the conditions (5.8) includes again as examples $d(t) = d_0 + b_1 \log(1+t)$ and $d(t) = d_0(1+t)^{b_2}$ but respectively with $b_1 > 2/c_+$ and $0 < b_2 \leqslant 1$.

Let us define

$$(\phi, \psi)(x, t) := \left(-\int_x^\infty [u(y, t) - U(y + d(t) + x_1)]\, dy, v(x, t) - v_+ \right), \quad (5.9)$$

Then the original equations (1.2) can be reduced to

$$\begin{cases} \phi_{xt} + d'(t)\, U_x(x + d(t) + x_1) + \psi_x = 0, \\ \psi_t + a\phi_{xx} = f(\phi_x + U) - f(U) - \psi, \end{cases} \quad (5.10)$$

and, after the integration $\int_x^\infty (5.10)_1\, dy$ we get

$$\begin{cases} \phi_t - d'(t)[u_+ - U(x + d(t) + x_1)] + \psi = 0, \\ \psi_t + a\phi_{xx} = f(\phi_x + U) - f(U) - \psi. \end{cases} \quad (5.11)$$

Substituting $\psi = -\phi_t + d'(t)[u_+ - U(x + d(t) + x_1)]$ in the second equation of (5.11), we have

$$L(\phi) := \phi_{tt} + \phi_t - a\phi_{xx} + f'(U)\, \phi_x = -F_1 - F_2, \qquad x > 0, t > 0, \quad (5.12)$$

where

$$F_1 = f(U + \phi_x) - f(U) - f'(U)\, \phi_x, \quad (5.13)$$

$$F_2 = -d''(t)[u_+ - U] + d'(t)^2\, U_x - d'(t)[u_+ - U]. \quad (5.14)$$

The initial values can be given as

$$\begin{aligned} (\phi, \psi)|_{t=0} &= \left(-\int_x^\infty [u_0(y) - U(y + d_0 + x_1)]\, dy, v_0(x) - v_+ \right) \\ &=: (\phi_0, \psi_0)(x). \end{aligned} \quad (5.15)$$

We also have, from the first equation of (5.10) and (5.15), that

$$\phi_t|_{t=0} = -\psi_0(x) + d'(0)[u_+ - U(x + d_0 + x_1)] =: \phi_1(x). \quad (5.16)$$

By (5.3) and (5.5), the boundary values are given in the form

$$\begin{aligned} \phi|_{x=0} &= -\int_0^\infty [u(x, t) - U(x + d(t) + x_2)]\, dx \\ &= -I_d - \int_t^\infty d'(\tau)[u_+ - U(d(\tau) + x_1)]\, d\tau \\ &= -\int_{d(t)}^\infty [u_+ - U(y + x_1)]\, dy \\ &=: B(t), \end{aligned} \quad (5.17)$$

by taking the variable transform $y = d(t)$ in the third step of (5.17), and

$$\phi_t|_{x=0} = d'(t)[u_+ - U(d(t) + x_1)] = B'(t). \tag{5.18}$$

Since $|u_+ - U(y)| = O(1)\, e^{-c_+ y}$ as $y \to +\infty$, and $d(t) > 0$, $x_1 > 0$, we have

$$\begin{cases} |B(t)| \sim e^{-c_+ x_1} e^{-c_+ d(t)}, & \text{as} \quad t \to +\infty, \\ |B'(t)| \sim e^{-c_+ x_1}\, |d'(t)|\, e^{-c_+ d(t)}, & \text{as} \quad t \to +\infty. \end{cases} \tag{5.19}$$

We now state our main results as follows.

THEOREM 5.1 (Convergence). *Assume $f'(u_+) < s \leqslant f'(u_-)$ and (4.82) hold and let a be a suitably large but fixed constant. Suppose that $\phi_0(x) \in H^2$, $\phi_1(x) \in H^1$, and $d(t)$ satisfies (5.6). Then there exists a constant $\varepsilon_4 > 0$ such that, if $a(\|\phi_0\|_2 + \|\phi_1\|_1 + x_1^{-1}) < \varepsilon_4$, then the IBVP (5.12), (5.15), (5.16), and (5.17) has a unique global solution $\phi(x, t)$ such that*

$$\phi \in C^0([0, +\infty); H^2), \qquad \phi_x \in L^2([0, +\infty); H^1),$$
$$\phi_t \in C^0([0, +\infty); H^1) \cap L^2([0, +\infty); H^1)$$

and the following estimates holds,

$$\|\phi(t)\|_2^2 + \|\phi_t(t)\|_1^2 + \int_0^t \|(\phi_x, \phi_t)(\tau)\|_1^2\, d\tau \leqslant C M_0^2, \tag{5.20}$$

where $M_0^2 := e^{-c_- x_1/2} + \|\phi_0\|_2^2 + \|\phi_1\|_1^2$, which implies

$$\sup_{x \in R_+} |(\phi_x, \phi_t)(x, t)| \to 0, \qquad \text{as} \quad t \to +\infty. \tag{5.21}$$

THEOREM 5.2 (Exponential Rate). *Assume $f'(u_+) < s \leqslant f'(u_-)$, the essential assumption of (4.82), and let $a \gg 1$, $|u_+ - u_-| \ll 1$. Suppose that $\phi_0(x) \in H^2_{w_2}$, $\phi_1(x) \in H^1_{w_2}$, and $d(t)$ satisfies (5.8). Then there exist constants $\varepsilon_5 > 0$ and $\theta > 0$ such that if $a(|\phi_0|_{2, w_2} + |\phi_1|_{1, w_2} + x_1^{-1}) < \varepsilon_5$, then the IBVP (5.12), (5.15), (5.16), and (5.17) has a unique global solution $(\phi, \phi_t)(x, t)$ in*

$$\phi \in C^0([0, +\infty); H^2_{w_2}), \cap L^2([0, +\infty); H^1_{w_2}),$$
$$\phi_t \in C^0([0, +\infty); H^1_{w_2}) \cap L^2([0, +\infty); H^1_{w_2})$$

and the following estimate holds,

$$|\phi(t)|_{2, w_2}^2 + |\phi_t(t)|_{1, w_2}^2 + \theta \int_0^t [\,|\phi_x(\tau)|_{2, w_2}^2 + |\phi_t)(\tau)|_{1, w_2}^2\, d\tau \leqslant C \bar{M}_0^2, \tag{5.22}$$

where $\bar{M}_0^2 := e^{-c_- x_1/2} + |\phi_0|^2_{2,w_2} + |\phi_1|^2_{1,w_2}$, *which implies*

$$|\phi(t)|^2_{2,w_2} + |\phi_t(t)|^2_{1,w_2} \leqslant C\bar{M}_0^2 e^{-\theta t}. \tag{5.23}$$

THEOREM 5.3 (Algebraic Rates). *Assume* $f'(u_+) < s \leqslant f'(u_-)$, *the essential assumption of* (4.82), *and let* $a \gg 1$. *Suppose* $\phi_0(x) \in H^2 \cap L^2_\alpha$ *and* $\phi_1(x) \in H^1 \cap L^2_\alpha$ *for a constant* $\alpha > 0$, *and* $d(t)$ *satisfies* (5.7). *Then the solution* $\phi(x, t)$ *of* (5.12), *with the initial boundary conditions* (5.15), (5.16), *and* (5.17), *satisfies the estimate*

$$\sup_{x \in R_+} |(\phi, \phi_x, \phi_t)(x, t)| \leqslant CM_1(1+t)^{-\alpha/2}, \tag{5.24}$$

where $M_1^2 := |(\phi_0, \phi_1)|^2_\alpha + M^2$.

Remark 5.4. (1) Since $d(t) > 0$ and $d(t) + x + x_1 > -\infty$ does not go to $-\infty$ for all $t \geqslant 0$ and $x \geqslant 0$, we have, as in Subsection 4.2, that $U(x + d(t) + x_1)$ does not go to u_-. As a consequence, the properties of the weights $w_1(U)$ and $w_2(U)$ in the case $f'(u_+) < 0 = f'(u_-)$ are the same to those in the case $f'(u_+) < 0 < f'(u_-)$. So, even for the degenerate case $f'(u_+) < s = 0 = f'(u_-)$, the decay rates and the conditions on $d(t)$ are same to the nondegenerate case $f'(u_+) < s = 0 < f'(u_-)$, as we mentioned in the above theorems.

2. To have the exponential decay, in Theorem 5.2 we assume two smallness hypotheses $a^{-1} \ll 1$ and $|u_+ - u_-| \ll 1$. We don't know if these conditions can be dropped. Further contributions are expected in this direction.

5.2. *Proofs of Main Theorems*

Since the local existence for the IBVP (5.12), (5.15), (5.16), and (5.17) is standard, we are going to show only the *a priori* estimates. Let us define the solution spaces as

$$\begin{aligned} Y_1(0, T) = \{&\phi \in C^0([0, T]; H^2), \quad \phi_x \in L^2([0, T]; H^1), \\ &\phi_t \in C^0([0, T]; H^1) \cap L^2([0, T]; H^1)\} \\ Y_2(0, T) = \{&\phi \in C^0([0, T]; H^2_{w_2}), \cap L^2([0, T]; H^1_{w_2}), \\ &\phi_t \in C^0([0, T]; H^1_{w_2}) \cap L^2([0, T]; H^1_{w_2})\} \end{aligned}$$

for any given constant $0 \leqslant T \leqslant +\infty$, and

$$N_1(T) = \sup_{0 \leqslant t \leqslant T} (\|\phi(t)\|_2 + \|\phi_t(t)\|_1),$$

$$N_2(T) = \sup_{0 \leqslant t \leqslant T} (|\phi(t)|_{2,w_2} + \|\phi_t(t)\|_{1,w_2}).$$

Thanks to (5.6) (or (5.7) or (5.8)), by a similar procedure used in Lemma 4.9, we may prove the following estimates for the boundary. We omit the details of the proof.

LEMMA 5.5. *It holds*

$$\int_0^t \left[|aw_i\phi\phi_x| + \frac{a}{2}|w_{i,x}\phi^2| + \frac{1}{2}|w_i f'\phi^2| + |2aw_i\phi_t\phi_x| \right]\Big|_{x=0} d\tau$$
$$\leqslant aCe^{-c_-x_1/2}, \qquad i=1,2, \tag{5.25}$$
$$\int_0^t (1+\tau)^\alpha \left[|aw_1\phi\phi_x| + \frac{a}{2}|w_{1,x}\phi^2| + \frac{1}{2}|w_1 f'\phi^2| + |2aw_1\phi_t\phi_x| \right]\Big|_{x=0} d\tau$$
$$\leqslant aCe^{-c_-x_1/2}, \tag{5.26}$$

where $C>0$ is independent of a and x_1.

The main goal of this subsection will be the proof of the basic estimates. To this end we need before to obtain a technical result.

LEMMA 5.6. *Let $U=U(x+d(t)+x_1)$ be the shifted stationary wave for any $d(t)\geqslant 0$. Then it holds*

$$|w_1'(U)| \sim w_1(U) \sim O(1). \tag{5.27}$$

Proof. By a straightforward calculation, we have

$$|w_1'(U)| = \frac{(U-u_+)(u_- - U)}{h(U)^2} \left| \frac{h(U)}{U-u_+} + \frac{h(U)}{U-u_-} - h'(U) \right|.$$

Due to the Taylor's formula $0=h(u_+)=h(U)+h'(U)(u_+ - U)+O(1)|u_+ - U|^2$, we have

$$\left| \frac{h(U)}{U-u_+} - h'(U) \right| \sim |U-u_+| \sim |h(u)|. \tag{5.28}$$

Since $U(x+d(t)+x_1)$ will remain away from u_- for all $(x,t)\in\mathbb{R}_+\times\mathbb{R}_+$, we get $|h(U)/(U-u_-)| \sim |U-u_+|$. Thus (5.27) is proved.

LEMMA 5.7 (Basic Energy Estimate). *Assume the solution $\phi\in Y_1(0,T)$. Then the estimate*

$$\|(\phi, \sqrt{a}\phi_x, \phi_t)(t)\|^2 + \int_0^t [a\,\|(\phi_x(\tau)\|^2 + \|\phi_t(\tau)\|^2]\,d\tau$$
$$\leqslant aC\{e^{-c_-x_1/2} + \|(\phi_0, \phi_{0,x}, \phi_1)\|^2\}, \tag{5.29}$$

holds, provided $N_1(T) \ll 1$, *where* $C>0$ *is independent of* a, x_1, *and* (ϕ_0, ϕ_1).

Moreover, if the solution $\phi \in Y_2(0, T)$, *then*

$$|(\phi, \sqrt{a}\phi_x, \phi_t)(t)|^2_{w_2} + \theta_1 \int_0^t \|(\phi, \sqrt{a}\phi_x, \phi_t)(\tau)\|^2_{w_2}]\, d\tau$$

$$\leqslant aC\{e^{-c_-x_1/2} + |(\phi_0, \sqrt{a}\phi_{0,x}, \phi_1)|^2_{w_2}\}, \tag{5.30}$$

holds, for some constant $\theta_1 > 0$, *provided* $N_2(T) \ll 1$, *where* $C>0$ *is independent of* a, x_1, *and* (ϕ_0, ϕ_1).

Proof. Multiplying (5.12) by $2w_i(U)\,\phi$ $(i=1,2)$ and by $2w_i(U)\,\phi_t$, we have respectively

$$2w_i\phi \cdot L(\phi) = -2w_i\phi \cdot (F_1 + F_2), \tag{5.31}$$

and

$$2w_i\phi_t \cdot L(\phi) = -2w_i\phi_t \cdot (F_1 + F_2), \tag{5.32}$$

where

$$2w_i\phi \cdot L(\phi) = \{w_i\phi^2 + 2w_i\phi\phi_t - w_{it}\phi^2\}_t - 2w_i\phi_t^2 + 2aw_i\phi_x^2 - (wh)''\, U_x\phi^2$$
$$+ (w_{itt} - w_{it})\,\phi^2 - \{2aw_i\phi\phi_x - aw_{ix}\phi^2 - 2(w_i f')(U)\,\phi^2\}_x.$$

and

$$2w_i\phi^t \cdot L(\phi) = \{w_i\phi_t^2 + aw_i\phi_x^2\}_t + (2w_i + w_{it})\,\phi_t^2 - aw_{it}\phi_x^2$$
$$+ (2aw_{ix} + 2w_i f'(U))\,\phi_x\phi_t - \{2aw_i\phi_t\phi_x\}_x.$$

Combining $\frac{1}{2}\times(5.31) + (5.32)$, we obtain

$$\{E_7(\phi, \phi_t) + E_8(\phi_x)\}_t + E_9(\phi_x, \phi_t) + E_{10}(\phi) - \{\cdots\}_x$$
$$= -(w_{itt} - w_{it})\,\phi^2 - 2(F_1 + F_2)\,w_i[\tfrac{1}{2}\phi + \phi_t], \tag{5.33}$$

where

$$E_7(\phi, \phi_t) = w_i\left[\phi_t^2 + \phi\phi_t + \frac{1}{2}\left(1 + d'(t)\,\frac{w_{ix}}{w_i}\right)\phi^2\right], \tag{5.34}$$

$$E_8(\phi_x) = aw_i\phi_x^2, \tag{5.35}$$

$$E_9(\phi_x, \phi_t) = w_i \left[\left(1 + d'(t) \frac{w_{ix}}{w_i}\right) \phi_t^2 + 2\left(f' + a \frac{w_{ix}}{w_i}\right) \phi_x \phi_t \right.$$

$$\left. + a\left(1 + d'(t) \frac{w_{ix}}{w_i}\right) \phi_x^2 \right], \tag{5.36}$$

$$E_{10}(\phi) = -\frac{1}{2} (w_i h)'' (U) U_x \phi^2, \tag{5.37}$$

$$\{ \cdots \} = \left\{ a w_i \phi \phi_x - \frac{1}{2} w_{i,x} \phi^2 - \frac{a}{2} w_i f' \phi^2 + 2 a w_i \phi_t \phi_x \right\}. \tag{5.38}$$

Denote D_7 and D_9 as the discriminantes of E_7 and E_9, respectively. Since $|w_{ix}/w_i| = O(1) |u_+ - u_-|/a$ (here $s = 0$) and $a > f'(U)^2$ (see (1.4)), using the condition $|d'(t)| \leqslant C$ in (5.6) and (5.7) for $w_1(U)$, and in (5.8) for $w_2(U)$, respectively, we get

$$\begin{aligned}
D_7 &:= 1 - 4 \times \frac{1}{2} \left(1 + d'(t) \frac{w_{ix}}{w_i}\right) \\
&= -\left(1 + 2d'(t) \frac{w_{ix}}{w_i}\right) \\
&\leqslant \left(1 - 2 |d'(t)| \frac{O(1) |u_+ - u_-|}{a}\right) \\
&\leqslant -C < 0
\end{aligned}$$

$$\begin{aligned}
D_9 &:= 4\left(f' + a \frac{w_{ix}}{w_i}\right)^2 - 4a\left(1 + d'(t) \frac{w_{ix}}{w_i}\right)^2 \\
&= -4\left(\sqrt{a} + f' + (\sqrt{a}\, d'(t) + a) \frac{w_{ix}}{w_i}\right)\left(\sqrt{a} - f' + (\sqrt{a}\, d'(t) - a) \frac{w_{ix}}{w_i}\right) \\
&\leqslant -4\left(\sqrt{a} + f' - O(1) |u_+ - u_-| \left(\frac{1}{\sqrt{a}} + 1\right)\right) \\
&\quad \times \left(\sqrt{a} - f' - O(1) |u_+ - u_-| \left(\frac{1}{\sqrt{a}} + 1\right)\right) \\
&\leqslant -C < 0
\end{aligned}$$

for $a \gg 1$, so we have

$$c_1 w_i \phi^2 + c_2 w_i \phi_t^2 \leqslant E_7(\phi, \phi_t) \leqslant c_1' w_i \phi^2 + c_2' w_i \phi_t^2, \tag{5.39}$$

$$c_3 w_i \phi_x^2 + c_4 w_i \phi_t^2 \leqslant E_9(\phi_x, \phi_t) \leqslant c_3' w_i \phi_x^2 + c_4' w_i \phi_t^2, \tag{5.40}$$

for some positive constants c_j, c_j', $j = 1, 2, 3, 4$.

Integrating (5.33) over $[0, +\infty) \times [0, t]$ and using (5.39), (5.40), and the boundary estimate (5.5), we obtain

$$\begin{aligned}
&c_1 |\phi(t)|_{w_i}^2 + c_2 |\phi_t(t)|_{w_i}^2 + a |\phi_x(t)|_{w_i}^2 + \int_0^t \int_0^\infty [-\tfrac{1}{2}(w_i h)'' \, U_x] \, \phi^2 \, dx \, d\tau \\
&\qquad + \int_0^t [c_3 |\phi_x(\tau)|_{w_i}^2 + c_4 |\phi_t(\tau)|_{w_i}^2] \, d\tau \\
&\quad \leqslant aC(|\phi_0|_{1, w_i}^2 + |\phi_1|_{w_i}^2 + e^{-c_+ x_1/2}) \\
&\qquad + \int_0^t \int_0^\infty (\tfrac{1}{2} |w_{itt} - w_{it}| \, \phi^2 + w_i \, |(F_1 + F_2)(\phi + 2\phi_t)|) \, dx \, d\tau.
\end{aligned} \tag{5.41}$$

We now have, by using $|u_+ - U| \sim (a - s^2) \, |U_x| \sim e^{-c_+(x + d(t) + x_1)}$, the following estimates for the nonlinear terms:

$$|F_1| \leqslant C \, |\phi_x|^2, \qquad |F_2| \leqslant C[\, |d''(t)| + d'(t)^2 + |d'(t)| \,] \, e^{-c_+ d(t)} e^{-c_+(x + x_1)}.$$

So, by using the condition (5.6), and $w_1(U) \sim O(1)$, $w_2(U) \sim e^{c_+(x + d(t) + x_1)/2}$, we obtain

$$\begin{aligned}
&\int_0^t \int_0^\infty w_i \, |(F_1 + F_2)(\phi + 2\phi_t)|] \, dx \, d\tau \\
&\quad \leqslant CN_i(T) \int_0^t \int_0^\infty w_i [\phi_x^2 + (|d''(t)| + d'(t)^2 + |d'(t)|) \\
&\qquad \times e^{-c_+ d(t)} e^{-c_+(x + x_1)}] \, dx \, d\tau \\
&\quad \leqslant CN_i(T) \left(\int_0^t |\phi_x(\tau)|_{w_i}^2 \, d\tau + \int_0^t \int_0^\infty [\, |d''(t)| + d'(t)^2 + |d'(t)| \,] \right. \\
&\qquad \left. + e^{-c_+ d(t)} e^{-c_+(x + x_1)/2} \, dx \, d\tau \right) \\
&\quad \leqslant CN_i(T) \left(\int_0^t |\phi_x(\tau)|_{w_i}^2 \, d\tau + e^{-c_+ x_1/2} \right).
\end{aligned} \tag{5.42}$$

Let us control the term $\int_0^t \int_0^\infty \frac{1}{2} |w_{itt} - w_{it}| \phi^2 \, dx \, d\tau$. Since $s = 0$ and $aU_x = h(U)$, a straightforward calculation yields

$$w_{itt}(U) - w_{it}(U) = \frac{1}{a} (w_i h)'' (U) U_x d'(t)^2 - \frac{1}{a} w_{ix}(U) h'(U) d'(t)^2$$
$$- \frac{1}{a} w_i(U) h''(U) U_x d'(t)^2 + w_{ix}(U)[d''(t) - d'(t)]. \tag{5.43}$$

In particular, let us consider the weight function $w_1(U)$. We have $a |U_x| = |h(U)| \sim e^{-c_+(x + d(t) + x_1)}$, and by Lemma 5.6 $|w_1'(U)| \sim w_1(U) \sim O(1)$, namely, $|w_{1x}(U)| \sim |U_x|$. Moreover, $(w_1 h)'' (U) = 2$, $|h'(U)|$, and $|h''(U)|$ are bounded. So, by the condition (5.6), we have from (5.43)

$$\begin{aligned}
&\int_0^t \int_0^\infty \tfrac{1}{2} |w_{1tt} - w_{1t}| \phi^2 \, dx \, d\tau \\
&\quad \leqslant CN_1^2(T) \int_0^t \int_0^\infty |w_{1tt} - w_{1t}| \, dx \, d\tau \\
&\quad \leqslant CN_1^2(T) \int_0^t \int_0^\infty \left(\frac{d'(\tau)^2}{a} + |d''(\tau) - d'(\tau)| \right) |U_x| \, dx \, d\tau \\
&\quad \leqslant \frac{CN_1^2(T)}{a} \int_0^t \int_0^\infty [a^{-1} d'(\tau)^2 + |d''(\tau) - d'(\tau)|] \\
&\qquad + \exp(-c_+(x + d(\tau) + x_1) \, dx \, d\tau \\
&\quad \leqslant \frac{CN_1^2(T)}{a} e^{-c_+ x_1} \\
&\quad \leqslant \frac{C}{a} e^{-c_+ x_1}
\end{aligned}$$

for $N_1(T) \ll 1$.

Substituting (5.42) and (5.44) into (5.41), by dropping the positive term

$$\int_0^t \int_0^\infty (-\tfrac{1}{2} (w_1 h)'' U_x) \phi^2 \, dx \, d\tau = \int_0^t \int_0^\infty |U_x| \phi^2 \, dx \, d\tau,$$

we proved the basic estimate (5.29).

Let we examine now the weight function $w_2(U)$. We have $|w_{2x}(U)/w_2(U)| = O(1)(|u_+ - u_-|/a)$ and $-(w_2 h)'' U_x = -(w_2 h)'' h(U)/a = O(1) w_2(U)/a$, see (2.18) and (2.19). Moreover, $|w_{2x}| = O(1) |u_+ - u_-| |(w_2 h)'' U_x|$. By using this facts and the boundedness of $|d'(t)|$ and $|d''(t)|$ in (5.8), we also have

$$
\begin{aligned}
&\int_0^t \int_0^\infty \frac{1}{2}\,|w_{2tt}-w_{2t}|\,\phi^2\,dx\,d\tau \\
&\quad \leqslant \int_0^t \int_0^\infty \left(\frac{d'(\tau)^2}{a} + \frac{|u_+ - u_-|}{a}\, d'(\tau)^2 + |U_x|\, d'(\tau)^2 \right.\\
&\qquad \left. + |u_+ - u_-|\,|d''(\tau) - d'(\tau)| \right) |(w_2 h)''\, U_x|\,\phi^2\,dx\,d\tau \\
&\quad \leqslant C\left(\frac{1}{a} + \frac{|u_+ - u_-|}{a} + |u_+ - u_-| \right) \int_0^t \int_0^\infty |(w_2 h)''\, U_x|\,\phi^2\,dx\,d\tau.
\end{aligned}
\tag{5.45}
$$

Substituting (5.42) and (5.45) into (5.41) yields

$$
\begin{aligned}
&c_1\,|\phi(t)|^2_{w_2} + c_2\,|\phi_t(t)|^2_{w_2} + a\,|\phi_x(t)|^2_{w_2} \\
&\quad + \left[\frac{1}{2} - C\left(\frac{1}{a} + \frac{|u_+ - u_-|}{a} + |u_+ - u_-| \right) \right] \int_0^t \int_0^\infty |(w_2 h)''\, U_x|\,\phi^2\,dx\,d\tau \\
&\quad + (c_3 - CN_2(T)) \int_0^t |\phi_x(\tau)|^2_{w_2}\,d\tau + c_4 \int_0^t |\phi_t(\tau)|^2_{w_2}\,d\tau \\
&\leqslant aC(|\phi_0|^2_{1,\,w_2} + |\phi_1|^2_{w_2} + e^{-c_+ x_1/2}).
\end{aligned}
$$

Finally, if we assume $a \gg 1$, $|u_+ - u_-| \ll 1$, and $N_2(T) \ll 1$, we proved (5.30) for some constant $\theta_1 > 0$. ∎

By the same argument used in Subsection 4.1.4, by applying the basic energy estimates in Lemma 5.7, we can similarly prove the estimates for the higher order case.

LEMMA 5.8. *Let us assume the solution $\phi \in Y_i(0, T)$, $i = 1, 2$. Then there exists a constant $\theta_2 > 0$ such that the estimate*

$$
\begin{aligned}
&|(\phi_x, \phi_{xt})(t)|^2_{w_i} + a\,|\phi_{xx}(t)|^2_{w_i} \\
&\quad + \theta_2 \int_0^t [a\,|(\phi_{xx}(\tau)|^2_{w_2} + |(\phi_x, \phi_{xt})(\tau)|^2_{w_2}]\,d\tau \leqslant aCM^2
\end{aligned}
\tag{5.46}
$$

holds, provided $N_i(T) \ll 1$, $i = 1, 2$, where $M = M_0$ or $\bar{M}_0$ when $i = 1$ or 2 respectively.

Combining Lemma 5.7 and Lemma 5.8, we prove Theorem 5.1 for the convergence and Theorem 5.2 for the exponential decay rate.

To conclude the result of Theorem 5.3 on the algebraic decay rates, it will be useful at this point to use an argument similar to the one used in

Subsection 4.1.5 and in particular on the Lemma 4.13. However, the weight function $w_1(U)$ cannot be useful to this purpose, since we cannot have a $z^* = x^* + d(t^*) + x_1$ such that $U(z^*) = (u_+ + u_-)/2$. However, we will see that the result can be obtained by using the new weight function

$$w_3(U) = \frac{u_+ - U}{h(U)} > 0, \qquad \text{for} \quad U \in [u_+, U(d_0 + x_0)]. \tag{5.47}$$

Here $x + d(t) + x_1 \geqslant d_0 + x_1 > 0$. It is clear that this weight function has some properties as follows:

$$w_3(U) \sim |w_3'(U)| \sim w_1(U) \sim O(1) \qquad \text{for} \quad U \in [u_+, U(d_0 + x_0)], \tag{5.48}$$

and

$$(w_3 h)'(U) = -1, \qquad (w_3 h)''(U) = 0, \quad |w_{3x}(U)| \sim O(1)\, |h(U)|/a. \tag{5.49}$$

Now, denoting

$$\bar{K}_3(x, t) := (1+t)^\gamma (1+x)^\beta, \qquad \text{and} \qquad K_3(x, t) := \bar{K}_3(x, t)\, w_3(U),$$

we get

$$\int_0^t \int_0^\infty K_3(\phi + 2\phi_t) \cdot L(\phi)\, dx\, d\tau = -\int_0^t \int_0^\infty K_3(\phi + 2\phi_t) \cdot (F_1 + F_2)\, dx\, d\tau. \tag{5.50}$$

To complete the analogy with Subsection 4.1.5, we define a function P_β as

$$\begin{aligned} P_\beta &:= -\bar{K}_{3x}(w_3 h)' - K_3 (w_3 h)'' = \bar{K}_{3x}(x, t) \\ &= \beta(1+t)^\gamma (1+x)^{\beta - 1}. \end{aligned} \tag{5.51}$$

Thus, without any difficulty, repeating a procedure similar to Subsection 4.1.5, applying the basic estimates in Lemma 5.7, Lemma 5.8 and the boundary estimate (5.26), we may prove the following lemma. The details are omitted.

Lemma 5.9. *It holds for any $\varepsilon > 0$*

$$\begin{aligned} &(1+t)^\alpha \|(\phi, \phi_x, \phi_t)(t)\|^2 + (1+t)^{-\varepsilon} \int_0^t (1+\tau)^{\alpha+\varepsilon} \|(\phi_x, \phi_t)(\tau)\|^2\, d\tau \\ &\quad \leqslant C(|(\phi, \phi_x, \phi_t)(0)|_\alpha^2 + e^{-c_- x_1/2}). \end{aligned} \tag{5.52}$$

Furthermore, we have the estimate for the higher order case.

LEMMA 5.10. *For any $\varepsilon > 0$, the following estimate holds,*

$$(1+t)^{\alpha} \|\partial_x(\phi, \phi_x, \phi_t)(t)\|^2 + (1+t)^{-\varepsilon} \int_0^t (1+\tau)^{\alpha+\varepsilon} \|\partial_x(\phi_x, \phi_t)(\tau)\|^2 \, d\tau$$
$$\leqslant C(\|(\phi, \phi_x, \phi_t)(0)\|_2^2 + |(\phi, \phi_x, \phi_t)(0)|_\alpha^2 + e^{-c_- x_1/2}). \quad (5.53)$$

Combining Lemmas 5.9 and 5.10 yields the algebraic rate of Theorem 5.3.

6. RESULTS FOR THE GENERAL CASE

In this section, we briefly discuss the convergence to the traveling waves for a general boundary condition $g(t)$. More precisely, we will prove the solutions of (1.2) and (1.3) converge to the corresponding traveling waves $(U, V)(x-st)$ time-asymptotically, with some decay rates, under some restrictions on $g(t)$, say $g(t)$ small perturbation of the wave $V(-st)$, with initial perturbations also sufficiently small.

For the traveling waves $(U, V)(x-st)$, suppose that $g(t) - V(-st) \in L^1(R_+)$ and satisfies

$$\int_0^\infty [u_0(x) - U(x)] \, dx + \int_0^\infty [g(t) - V(-st)] \, dt = 0, \quad (6.1)$$

Then we have from the first equation of (1.2), $(u-U)_t = -(v-V)_x$, and

$$\int_0^\infty [u(x,t) - U(x-st)] \, dx = \int_0^\infty [u_0(x) - U(x)] \, dx + \int_0^t [g(\tau) - V(-s\tau)] \, d\tau = -\int_t^\infty [g(\tau) - V(-s\tau)] \, d\tau \to 0, \quad \text{as} \quad t \to +\infty. \quad (6.2)$$

Define

$$\phi(x,t) := -\int_x^\infty [u(y,t) - U(y-st)] \, dy, \qquad \psi(x,t) := v(x,t) - V(x-st), \quad (6.3)$$

which implies, by using (6.2), that

$$\phi|_{x=0} = \int_t^\infty [g(\tau) - V(-s\tau)]\, d\tau =: G(t), \tag{6.4}$$

we can rewrite the original IBVP (1.2) and (1.3) as

$$\begin{cases} \phi_t + \psi = 0, \\ \psi_t + a\phi_{xx} - f'(U)\,\phi_x + \psi = F, \qquad x > 0, \quad t > 0, \\ (\phi, \psi)|_{t=0} = \left(-\int_x^\infty [u_0(y) - U(y)]\, dy, v_0(x) - V(x) \right) =: (\phi_0, \psi_0)(x), \\ \phi|_{x=0} = G(t), \end{cases} \tag{6.5}$$

where $F = f(U + \phi_x) - f(U) - f'(U)\,\phi_x$.

If the boundary perturbation $G(t) \in W^{3,1}(R_+)$ is small and initial perturbations $(\phi_0, \psi_0)(x)$ are also small, we can prove the following theorem.

THEOREM 6.1 (Algebraic Rates). *Under the assumption* (6.1), *let a be a suitably large but fixed constant.*

(i) *Case* $f'(u_+) < s < f'(u_-)$: *Suppose that* $G(t) \in W^{3,1}(R_+)$ *and* $(\phi_0, \psi_0)(x) \in L^2_\alpha \cap H^2$ *for some* $\alpha > 0$. *Then there exists a constant* $\delta_4 > 0$ *such that if* $a(\|G\|^{1/2}_{W^{3,1}} + |(\phi_0, \psi_0)|_\alpha + \|(\phi_0, \psi_0)\|_2) < \delta_4$, *then the system* (1.2) *and* (1.3) *has a unique global solution* $(u, v)(x, t)$ *satisfying*

$$\sup_{x \in R_+} |(u, v)(x, t) - (U, V)(x - st)| \leqslant C(1+t)^{-\alpha/2} (\|G\|^{1/2}_{W^{3,1}} + |(\phi_0, \psi_0)|_\alpha + \|(\phi_0, \psi_0)\|_2).$$

(ii) *Case* $f'(u_+) = s < f'(u_-)$: *Suppose that* $G(t) \in W^{3,1}(R_+)$ *and* $(\phi_0, \psi_0)(x) \in L^2_{\alpha\langle x\rangle_+} \cap H^2$ *for some* $0 < \alpha < 2/n$. *Then there exists a constant* $\delta_5 > 0$ *such that if* $a(\|G\|^{1/2}_{W^{3,1}} + |(\phi_0, \psi_0)|_{\alpha\langle x\rangle_+} + \|(\phi_0, \psi_0)\|_2 < \delta_5$, *then the system* (1.2) *and* (1.3) *has a unique global solution* $(u, v)(x, t)$ *satisfying*

$$\sup_{x \in \mathbb{R}_+} |(u, v)(x, t) - (U, V)(x - st)| \leqslant C(1+t)^{-\alpha/4} (\|G\|^{1/2}_{W^{3,1}} + |(\phi_0, \psi_0)|_{\alpha\langle x\rangle_+} + \|(\phi_0, \psi_0)\|_2).$$

If the boundary perturbation $G(t) \in W^{3,1}_{w_2}(R_+)$, namely, $w_2(-st)\, G^{(k)}(t) \in L^1(R_+)$ $(k = 0, 1, 2, 3)$, are small and the initial perturbations $(w_2(x))^{1/2} (\phi_0, \psi_0)(x)$ are small too, we have the exponential decay rate as follows.

THEOREM 6.2 (Exponential Rates). *Let a be a suitably large but fixed constant. Assume the hypothesis* (6.1), $f'(u_+) < s < f'(u_-)$, $G(t) \in W^{3,1}_{w_2}(R_+)$, $\phi_0 \in H^3_{w_2(U)}$, $\psi_0 \in H^2_{w_2(U)}$. *Then, there exist positive constants* δ_6 *and* $\theta = \theta(|u_+ - u_-|, a)$ *such that if* $a(\|G\|^{1/2}_{W^{3,1}_{w_2}} + |(\phi_0, \psi_0)|_{2,w_2}) \leqslant \delta_6$, *the IBVP* (1.2) *and* (1.3) *has a unique global solution* $(u, v)(x, t)$ *satisfying*

$$\sup_{x \in R_+} |(u, v)(x, t) - (U, V)(x - st)| \leqslant Ce^{-\theta t/2} (\|G\|^{1/2}_{W^{3,1}_{w_2}} + |(\phi_0, \psi_0)|_{2,w_2}). \tag{6.6}$$

Since the proofs of the above theorems can be similarly treated as in Subsection 4.1, we just state them here without the details of the proof.

6.1. *Concluding Remarks*

Even most of the important situation, also in the degenerate shock case are solved in this paper for the problem (1.2)–(1.3), there are some unsolved cases that we are at this moment not able to solve. We list them below and we expect more contributions to them.

Problem 1. When $g(t) = v_-$ with $f'(u_-) = s > 0$, the convergence of the solutions $(u, v)(x, t)$ to the corresponding front traveling waves is unknown. In fact we cannot determine a shift by our analysis.

Problem 2. When $g(t) = v_+$ with $f'(u_+) = s < 0$, the convergence of the solutions $(u, v)(x, t)$ to the corresponding back traveling waves is unknown. In fact we cannot control the boundary integration in this case.

Problem 3. When $g(t) = v_\mp$ with $f'(u_+) = s = 0$, is there a nonconstant shift $d(t)$, such that $(u, v)(x, t) \to (U(x + d(t)), v_+)$ as $t \to +\infty$? Here we failed to have a result since we cannot look for a suitable shift function $d(t)$ satisfying the conditions (5.6).

For other situations, when $g(t) = v_-$ with $s < 0$, it should be no convergence to the back waves, since the boundary perturbation is really big

$$|(v - V)|_{x=0}| = |v_- - V(-st)| \geqslant |v_- - V(0)| > 0.$$

Similarly, when $g(t) = v_+$ with $s > 0$, the boundary perturbation is also big

$$|(v - V)|_{x=0}| = |v_+ - V(-st)| \geqslant |v_+ - V(0)| > 0.$$

But this does not mean necessarily instability, since, for example, we are also not sure that there is a convergence to the front wave.

Finally, it could be interesting to discuss the case with boundary condition $u_{|x=0} = g(t)$ and the differences with the present case.

ACKNOWLEDGMENTS

This work was completed when the first author visited the Dipartimento di Matematica Pura ed Applicata of the Universitá degli Studi di L'Aquila. He expresses his thanks for their warm hospitality. Both of the authors express their gratitude to Professor Pierangelo Marcati for his helpful discussion. The research by the first author was supported in part by the Ministry of Education of Japan Grant-in-Aid for JSPS under Contract P-96169 and, during his stay in L'Aquila, in part by the Italian *progetto strategico CNR* "*Metodi e Modelli per la Matematica e l'Ingegneria*." The research by the second author was supported in part by the European *TMR project* "*Hyperbolic Systems of Conservation Laws*," Contract ERB FMRX-CT96-0033 and in part by the Italian *progetto strategico CNR* "*Metodi e Modelli per la Matematica e l'Ingegneria*."

REFERENCES

1. D. Aregba-Driollet and R. Natalini, Convergence of relaxation schemes for conservation laws, *Appl. Anal.* **61** (1996), 163–190.
2. G.-Q. Chen, C. D. Levermore, and T.-P. Liu, Hyperbolic conservation laws with stiff relaxation terms and entropy, *Comm. Pure Appl. Math.* **47** (1994), 787–830.
3. G.-Q. Chen and T.-P. Liu, Zero relaxation and dissipation limits for hyperbolic conservation laws, *Comm. Pure Appl. Math.* **46** (1993), 755–781.
4. I.-L. Chern, Long-time effect of relaxation for hyperbolic conservation laws, *Comm. Math. Phys.* **172** (1995), 39–55.
5. J. F. Clarke, Gas dynamics with relaxation effects, *Rep. Progr. Phys.* **41** (1978), 807–863.
6. S. Jin, A convex entropy for a hyperbolic system with stiff relaxation, *J. Differential Equations* **127** (1996), 97–109.
7. S. Jin and C. D. Levermore, Numerical schemes for hyperbolic conservation laws with stiff relaxation terms, *J. Comput. Phys.* **126** (1996), 449–467.
8. S. Jin and H. Liu, Diffusion limit of a hyperbolic system with relaxation, *Methods Appl. Anal.* **5** (1998), 317–334.
9. S. Jin and Z. Xin, The relaxing schemes for systems of conservation laws in arbitrary space dimensions, *Comm. Pure Appl. Math.* **48** (1995), 555–563.
10. S. Kawashima and A. Matsumura, Asymptotic stability of traveling wave solutions of systems for one-dimensional gas motion, *Comm. Math. Phys.* **101** (1985), 97–127.
11. S. Kawashima and A. Matsumura, Stability of shock profiles in viscoelasticity with non-convex costitutive relations, *Comm. Pure Appl. Math.* **47** (1994), 1547–1569.
12. H. L. Liu, C. W. Woo, and T. Yang, Decay rate for travelling waves of a relaxation model, *J. Differential Equations* **134** (1997), 343–367.
13. H. L. Liu, J. H. Wang, and T. Yang, Stability in a relaxation model with a nonconvex flux, *SIAM J. Math. Anal.* **29** (1998), 18–29.
14. J.-G. Liu and Z. Xin, Boundary-layer behavior in the fluid-dynamics limit for a nonlinear model Boltzmann equation, *Arch. Rational Mech. Anal.* **135** (1996), 61–105.
15. J.-G. Liu and Z. Xin, Kinetic and viscous boundary layers for Broadwell equations, *Trans. Theory Statist. Phys.* **25** (1996), 447–461.
16. T.-P. Liu, Hyperbolic conservation laws with relaxation, *Comm. Math. Phys.* **108** (1987), 153–175.
17. T.-P. Liu and K. Nishihara, Asymptotic behavior for scalar viscous conservation laws with boundary effect, *J. Differential Equations* **133** (1997), 296–320.

18. T.-P. Liu and S.-H. Yu, Propagation of a stationary shock layer in the presence of a boundary, *Arch. Rational Mech. Anal.* **139** (1997), 57–82.
19. T. Luo, Asymptotic stability of planar rarefaction waves for the relaxation approximation of conservation laws in several dimensions, *J. Differential Equations* **133** (1997), 255–279.
20. T. Luo and Z. Xin, Nonlinear stability of shock fronts for a relaxation system in several space dimensions, *J. Differential Equations* **139** (1997), 365–408.
21. P. Marcati and B. Rubino, Hyperbolic to parabolic relaxation theory for quasilinear first order systems, *J. Differential Equations*, in press.
22. C. Mascia and R. Natalini, L^1 nonlinear stability of travelling waves for a hyperbolic system with relaxation, *J. Differential Equations* **132** (1996), 275–292.
23. C. Mascia and R. Natalini, "Convergence to Rarefaction Waves for Some Relaxation Approximations to Conservation Laws," Technical Report, IAC-CNR, Rome, 1997.
24. A. Matsumura and M. Mei, "Asymptotic Toward Viscous Shock Profile for Solution of the Viscous *p*-System with Boundary Effect," Technical Report, 1997.
25. A. Matsumura and K. Nishihara, Asymptotic stability of traveling waves for scalar viscous conservation laws with non-convex nonlinearity, *Comm. Math. Phys.* **165** (1994), 83–96.
26. M. Mei, Stabitity of shock profiles for nonconvex scalar viscous conservation laws, *Math. Models Methods Appl. Sci.* **5** (1995), 279–296.
27. M. Mei and T. Yang, Convergence rates to travelling waves for a nonconvex relaxation model, *Proc. Roy. Soc. Edinburgh*, in press.
28. R. Natalini, Convergence to equilibrium for the relaxation approximations of conservation laws, *Comm. Pure Appl. Math.* **49** (1996), 795–823.
29. S. Nishibata, The initial boundary value problems for hyperbolic conservation laws with relaxation, *J. Differential Equations* **130** (1996), 100–126.
30. M. Nishikawa, Convergence rate to traveling waves for viscous conservation laws, *Funkcial. Ekvac.* **41** (1998), 107–132.
31. W.-C. Wang and Z. Xin, "Asymptotic Limit of the Initial Boundary Value Problems for Conservation Laws with Relaxational Extensions," Technical Report, 1997.
32. G. Whitham, "Linear and Nonlinear Waves," Wiley–Interscience, New York, 1974.
33. W. Yong, "Boundary Conditions for Hyperbolic Systems with Stiff Source Terms," Technical Report, University Heidelberg, 1997.
34. P. Zingano, Nonlinear stability with decay rate for traveling wave solutions of a hyperbolic system with relaxation, *J. Differential Equations* **130** (1996), 36–58.