Journal of Differential Equations **158**, 314–340 (1999)
Article ID jdeq.1999.3638, available online at http://www.idealibrary.com on IDEAL®

L_q-Decay Rates of Solutions for Benjamin–Bona–Mahony–Burgers Equations

Ming Mei[1]

Department of Computational Science, Faculty of Science, Kanazawa University, Kanazawa 920-1192, Japan
E-mail: mei@kappa.s.kanazawa-u.ac.jp

Received September 16, 1998; revised December 16, 1998

This paper studies the asymptotic behavior of solutions for the Benjamin–Bona–Mahony–Burgers equations $u_t - u_{xxt} - \alpha u_{xx} + \beta u_x + u^p u_x = 0$, $x \in R$, $t \geqslant 0$, with the initial data $u|_{t=0} = u_0(x) \to 0$ as $x \to \pm\infty$. Under the restrictions $\int_{-\infty}^{\infty} u_0(x)\, dx = 0$ and $\int_{-\infty}^{x} u_0(y)\, dy \in W^{2p+1,1}$, we obtain more results on the energy decay rates of the solutions in the forms that if $p \geqslant 1$, then $\|\partial_x^j u(t)\|_{L^2} = O(1)\, t^{-(2j+3)/4}$ for $j = 0, 1, ..., 2p-1$, and $\|\partial_x^j u(t)\|_{L^q} = O(1)\, t^{-((j+2)q-1)/(2q)}$ for $2 < q \leqslant \infty$ and $j = 0, 1, ..., 2p-2$; furthermore, if $p \geqslant 2$, then $\|\partial_x^j u(t)\|_{L^q} = O(1)\, t^{-((j+2)q-1)/(2q)}$ for $1 \leqslant q < 2$, $j = 0, 1, ..., 2p-3$, and $\|\partial_x^j u_t(t)\|_{L^q} = O(1)\, t^{-((j+4)q-1)/(2q)}$ for $2 \leqslant q \leqslant \infty$, $j = 0, 1, ..., 2p-3$, which are optimal. The proof is dependent on the Fourier transform method, the energy method and the point wise method of the Green function.

Key Words: BBM-Burgers equations; asymptotic behavior; L^q-decay rates.

1. INTRODUCTION AND MAIN RESULTS

We consider the Cauchy problem of the Benjamin–Bona–Mahony–Burgers (BBM-B) equations in the form

$$u_t - u_{xxt} - \alpha u_{xx} + \beta u_x + u^p u_x = 0, \qquad x \in R^1, \quad t \geqslant 0, \tag{1.1}$$

with the initial data

$$u|_{t=0} = u_0(x) \to 0 \qquad \text{as} \quad x \to \pm\infty, \tag{1.2}$$

where $\alpha > 0$, $\beta \in R^1$ are some given constants, and $p \geqslant 1$ is an integer.

When $\alpha = 0$, $\beta = 1$, $p = 1$, Eq. (1.1) is the alternative regularized long-wave equation proposed by Peregrine [19] and Benjamin *et al.* [2]. This equation features a balance between the nonlinear dispersive effect but takes no account of dissipation. When $\alpha > 0$, Eq. (1.1) is given if the good predicitive power is desired in the physical sense, such as the phenomenon

[1] Current address: Institut für Angewandte und Numerische Mathematik, Technische Universitaet Wien, Wiedner Hauptstraße 8–10, 1040 Wien, Austria.

0022-0396/99 $30.00

of bore propagation and water waves. Since the dispersive effect of (1.1) is the same as the Benjamin–Bona–Mahony equation

$$u_t - u_{xxt} + u_x + u^p u_x = 0,$$

while the dissipative effect is the same as the Burgers equation

$$u_t - \alpha u_{xx} + u_x + u^p u_x = 0,$$

we call (1.1) the Benjamin–Bona–Mahony–Burgers equation (BBM-B), but it is proposed neither by Benjamin, Bona, and Mahony nor by Burgers. To analyze the asymptotic behavior of the solution of (1.1) is an interesting problem both from a mathematical and physical point of view. Such a problem is widely studied by many mathematicians, cf. [1–9, 11–19, 21–24], and the references therein. Subsquent to our previous work [15], in this paper we are further going to show more decay results of the solution for the Cauchy problem (1.1) and (1.2). We are interested in the time-asymptotic decay of solution in L^q spaces for $1 \leqslant q \leqslant \infty$, especially in L^1 and L^∞ spaces, and obtain the optimal decay rates of $\|\partial_x^j u(t)\|_{L^q}$ and $\|\partial_x^j u_t(t)\|_{L^q}$, where $j = 0, 1, ..., k$, k is a positive integer. We discover that the highest orders k of derivatives $\partial_x^k u(x, t)$ and $\partial_x^k u_t(x, t)$ we can have are dependent on the number p in the nonlinear term $u^p u_x$ of Eq. (1.1), if we desire the optimal decay rates in our present method. Precisely saying, we have the following main results.

THEOREM 1.1. *Suppose that* $\int_{-\infty}^{\infty} u_0(x)\, dx = 0$ *and* $v_0(x) := \int_{-\infty}^{x} u_0(y)\, dy \in W^{2p+1,1}$. *Then there exists a positive constant* δ_0 *such that when* $\|v_0\|_{W^{2p+1,1}} \leqslant \delta_0$, *then the Cauchy problem* (1.1) *and* (1.2) *has a unique global solution* $u(x, t)$ *satisfying that:*

(i) *If* $p \geqslant 1$, *then the following estimates hold,*

$$\|\partial_x^j u(t)\|_{L^2} = O(1)(1+t)^{-(2j+3)/4} \tag{1.3}$$

for $j = 0, 1, ..., 2p-1$, *and*

$$\|\partial_x^j u(t)\|_{L^q} = O(1)(1+t)^{-((j+2)q-1)/(2q)} \tag{1.4}$$

for $2 \leqslant q \leqslant \infty$, $j = 0, 1, ..., 2p-2$;

(ii) *Furthermore, if* $p \geqslant 2$, *then* (1.3), (1.4) *and the following estimates hold,*

$$\|\partial_x^j u(t)\|_{L^q} = O(1)\, t^{-((j+2)q-1)/(2q)} \tag{1.5}$$

for $1 \leqslant q < 2$, $j = 0, 1, ..., 2p-3$, *and*

$$\|\partial_x^j u_t(t)\|_{L^q} = O(1)(1+t)^{-((j+4)q-1)/(2q)} \tag{1.6}$$

for $2 \leqslant q \leqslant \infty$, $j = 0, 1, ..., 2p-3$.

The results represented in Theorem 1.1 are new, especially in the sense of L^1. Herewith, we improve and develop the previous works [1, 3–9, 11, 15–18, 21–23] essentially. For the proof of these decay rates, we divide it into two steps. First, as in our previous work [15] to consider a strong form of a zero-mass perturbation, we extend our previous decay results in [15] to the case of higher derivatives $\partial_x^k u$ in L^∞ and L^2-decay by means of the Fourier transform method together with the energy method. Second, we further show the L^q-decay rates of the solution $u(x, t)$ for (1.1) and (1.2) for $1 \leqslant q \leqslant \infty$. In particular, in the sense of L^1 which is a difficult case, we will have to make a bit more effort on it by the point wise method of the Green function. These considerations can be also applied to the generalized BBM-Burger equations. We will remark on it in the last part of this paper.

Notations. We now make some notations for simplicity. C always denotes some positive constants without confusion. $\partial_x^k f := \partial^k f/\partial x^k$. L^p presents the Lebesque integrable space with the norm $\|\cdot\|_{L^p}$. Especially, L^2 is the square integrable space with the norm $\|\cdot\|_{L^2}$, and L^∞ is the essential bounded space with the norm $\|\cdot\|_{L^\infty}$. H^k and $W^{k,p}$ denote the usual Sobolev space with the norms $\|\cdot\|_{H^k}$ and $\|\cdot\|_{W^{k,p}}$, respectively. Suppose that $f(x) \in L^1 \cap L^2(R)$; we define the Fourier transforms of $f(x)$ as

$$F[f](\xi) \equiv \hat{f}(\xi) = \int_R f(x)\, \mathrm{e}^{-\mathrm{i}x\xi}\, dx.$$

Let T and B be a positive constant and a Banach space, respectively. $C^k(0, T; B)$ $(k \geqslant 0)$ denotes the space of B-valued k-times continuously differentiable functions on $[0, T]$, and $L^2(0, T; B)$ denotes the space of B-valued L^2-functions on $[0, T]$. The corresponding spaces of the B-valued function on $[0, \infty)$ are defined similarly.

2. REFORMULATATION OF THE PROBLEM AND PROOF OF MAIN RESULTS

Suppose that

$$\int_{-\infty}^{\infty} u_0(x)\, dx = 0. \tag{2.1}$$

Integrating Eq. (1.1) over $(-\infty, \infty) \times [0, t]$ with respect to x and t, respectively, we then have formally

$$\int_{-\infty}^{\infty} u(x, t)\, dx = \int_{-\infty}^{\infty} u_0(x)\, dx = 0.$$

Thus, let

$$v_0(x) = \int_{-\infty}^{x} u_0(y)\, dy, \qquad v(x, t) = \int_{-\infty}^{x} u(y, t)\, dy, \tag{2.2}$$

that is,

$$v_x(x, t) = u(x, t); \tag{2.3}$$

we reformulate the initial value problem (1.1) and (1.2) as the "integrated" equation

$$v_t - v_{xxt} - \alpha v_{xx} + \beta v_x + F(v_x) = 0, \tag{2.4}$$

with the initial data

$$v|_{t=0} = v_0(x), \tag{2.5}$$

where

$$F(v_x) := (v_x)^{p+1}/(p+1). \tag{2.6}$$

We now state our main theorems as follows, which imply Theorem 1.1.

THEOREM 2.1. *Suppose that* (2.1) *and* $v_0(x) \in W^{2p+1,1}(R)$ *hold. Then there exists a positive constant* δ_1 *such that when* $\|v_0\|_{W^{2p+1,1}} < \delta_1$, *then* (2.4) *and* (2.5) *have a unique global solution* $v(x, t)$ *satisfying*

$$v(x, t) \in C(0, \infty; H^{2p}(R) \cap W^{2p-1,\infty}(R)),$$

and the asymptotic decay rates in $L^2(R)$ *and* $L^\infty(R)$ *as*

$$\|\partial_x^j v(t)\|_{L^2} \leqslant C(1+t)^{-(2j+1)/4}, \qquad \text{for} \quad j = 0, 1, ..., 2p, \tag{2.7}$$

$$\|\partial_x^j v(t)\|_{L^\infty} \leqslant C(1+t)^{-(j+1)/2}, \qquad \text{for} \quad j = 0, 1, ..., 2p-1. \tag{2.8}$$

THEOREM 2.2. *Under the assumptions in Theorem* 2.1, *the unique global solution* $v(x, t)$ *of* (2.4) *and* (2.5) *further satisfies that*

$$\|\partial_x^j v_t(t)\|_{L^2} \leqslant C(1+t)^{-(2j+5)/4}, \qquad \text{for} \quad j = 0, 1, ..., 2p-2, \tag{2.9}$$

$$\|\partial_x^j v_t(t)\|_{L^\infty} \leqslant C(1+t)^{-(j+3)/2}, \qquad \text{for} \quad j = 0, 1, ..., 2p-2; \tag{2.10}$$

and that, when $2<q<\infty$,

$$\|\partial_x^j v(t)\|_{L^q} \leqslant C(1+t)^{-((j+1)q-1)/(2q)}, \qquad \textit{for} \quad j=0, 1, ..., 2p-1, \tag{2.11}$$

$$\|\partial_x^j v_t(t)\|_{L^q} \leqslant C(1+t)^{-((j+3)q-1)/(2q)}, \qquad \textit{for} \quad j=0, 1, ..., 2p-2; \tag{2.12}$$

in particular, when $1 \leqslant q < 2$,

$$\|\partial_x^j v(t)\|_{L^q} \leqslant Ct^{-((j+1)q-1)/(2q)}, \qquad \textit{for} \quad j=0, 1, ..., 2p-2. \tag{2.13}$$

Proof of Theorem 1.1. Once Theorems 2.1 and 2.2 are proved, noting $u(x, t)=v_x(x, t)$, we then prove Theorem 1.1 immediately. ∎

Proving Theorems 2.1 and 2.2 is our main purpose in the rest of this paper. We are first going to prove Theorem 2.1 based on the following local existence (Proposition 2.3) and the *a priori* estimates (Proposition 2.4) by the continuation extension method. The *a priori* estimates will be shown in Section 3. For the proof of Theorem 2.2, we leave it to Section 4. We note also that these considerations can be applied to the generalized BBM-Burger equation, which will be remarked in Section 5.

For a positive constant $0 \leqslant T \leqslant +\infty$, we define the solution space as

$$X(0, T)=\{v \in C(0, T; H^{2p}(R) \cap W^{2p-1, \infty}(R))\}$$

and let

$$\begin{aligned} M(T) = \sup_{0 \leqslant t \leqslant T} \Bigg\{ & \sum_{j=0}^{2p} (1+t)^{(2j+1)/4} \|\partial_x^j v(t)\|_{L^2} \\ & + \sum_{j=0}^{2p-1} (1+t)^{(j+1)/2} \|\partial_x^j v(t)\|_{L^\infty} \Bigg\}. \end{aligned} \tag{2.14}$$

We are going to prove that there exists a unique solution of Eqs. (2.4) and (2.5) in the space $X(0, +\infty)$ for some small initial data.

Proposition 2.3 (Local Existence). *Suppose that* $v_0 \in W^{2p+1, 1}$ *holds. Then there is a positive constant* T_0 *such that the Cauchy problem* (2.4) *and* (2.5) *has a unique solution* $v(x, t) \in X(0, T_0)$ *satisfying* $M(T_0) \leqslant 2M(0)$.

Proposition 2.4 (*A Priori* Estimate). *Let* T *be a positive constant, and* $v(x, t) \in X(0, T)$ *be a solution of the problem* (2.4) *and* (2.5). *Then there exist positive constants* δ_2 *and* C_1 *independent of* T *such that if* $M(T) \leqslant \delta_2$, *then the estimate*

$$\sum_{j=0}^{2p} (1+t)^{(2j+1)/4} \|\partial_x^j v(t)\|_{L^2} + \sum_{j=0}^{2p-1} (1+t)^{j+1/2} \|\partial_x^j v(t)\|_{L^\infty}$$
$$\leqslant C_1 \|v_0\|_{W^{2p+1,1}} \tag{2.15}$$

holds for $t \in [0, T]$.

Since Proposition 2.3 can be proved in the standard way, we omit its proof. Once Proposition 2.4 is proved, using the continuation arguments based on Propositions 2.3 and 2.4, we can show Theorem 2.1 as follows.

Proof of Theorem 2.1. We note that

$$M(0) = \|v_0\|_{H^{2p}} + \|v_0\|_{W^{2p-1,\infty}} \leqslant C_2 \|v_0\|_{W^{2p+1,1}}$$

holds for some positive constant C_2 by Sobolev's embedding inequality. Let $\delta_1 = \min\{\delta_2/(2C_2),\ \delta_2/(2C_1)\}$, and $\|v_0\|_{W^{2p+1,1}} \leqslant \delta_1$. Then Proposition 2.3 ensures a unique solution $v(x, t) \in X(0, T_0)$ satisfying $M(T_0) \leqslant 2M(0) \leqslant 2C_2 \|v_0\|_{W^{2p+1,1}} \leqslant \delta_2$. So, Proposition 2.4 with $T = T_0$ guarantees $M(T_0) \leqslant C_1 \|v_0\|_{W^{2p+1,1}} \leqslant \delta_2/2$. Now considering the Cauchy problem (2.4) and (2.5) with the "initial datum" $v(x, T_0)$ at the "initial time" T_0, thanks to Proposition 2.3 again, we have the unique solution $v(x, t)$ on $[T_0, 2T_0]$, eventually $[0, 2T_0]$, and $M(2T_0) \leqslant 2M(T_0) \leqslant \delta_2$. Thus, applying Proposition 2.4 again with $T = 2T_0$, we obtain $M(t) \leqslant C_1 \|v_0\|_{W^{2p+1,1}} \leqslant \delta_2/2$ for all $t \in [0, 2T_0]$. Therefore, repeating this continuation process, we can obtain a unique global solution $v(x, t) \in X(0, +\infty)$ satisfying (2.15) for all $t \in [0, +\infty)$. Thus, we have completed the proof of Theorem 2.1. ∎

3. A PRIORI ESTIMATES

As in [15], we take the Fourier transform to (2.4) to yield

$$\hat{v}_t - (\mathrm{i}\xi)^2\, \hat{v}_t - \alpha(\mathrm{i}\xi)^2 \hat{v} + \mathrm{i}\beta\xi\hat{v} + \widehat{F(v_x)} = 0,$$

namely,

$$\hat{v}_t + \frac{\alpha\xi^2 + \mathrm{i}\beta\xi}{1+\xi^2}\,\hat{v} + \frac{\widehat{F(v_x)}}{1+\xi^2} = 0, \tag{3.1}$$

which gives us

$$\hat{v}(\xi, t) = \mathrm{e}^{-A(\xi)t}\hat{v}_0(\xi) - \int_0^t \mathrm{e}^{-A(\xi)(t-s)}\, \frac{\widehat{F(v_x)}(\xi, s)}{1+\xi^2}\, ds, \tag{3.2}$$

where

$$A(\xi)=B(\xi)+\mathrm{i}\,\frac{\beta\xi}{1+\xi^2}, \qquad B(\xi)=\frac{\alpha\xi^2}{1+\xi^2}. \tag{3.3}$$

Then taking the inverse Fourier transform to (3.2) yields

$$\begin{aligned} v(x,t)=&\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\xi x}\mathrm{e}^{-A(\xi)t}\hat{v}_0(\xi)\,d\xi \\ &-\frac{1}{2\pi}\int_0^t\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\xi x}\mathrm{e}^{-A(\xi)(t-s)}\frac{\widehat{F(v_x)}(\xi,s)}{1+\xi^2}\,d\xi\,ds. \end{aligned} \tag{3.4}$$

By ∂_x^j (3.4) for any $j\in\mathbf{N}_+$, $\mathbf{N}_+$ we denote the set of non-negative integers, and we have

$$\begin{aligned} \partial_x^j v(x,t)=&\frac{1}{2\pi}\int_{-\infty}^{\infty} (\mathrm{i}\xi)^j\,\mathrm{e}^{\mathrm{i}\xi x}\mathrm{e}^{-A(\xi)t}\hat{v}_0(\xi)\,d\xi \\ &-\frac{1}{2\pi}\int_0^t\int_{-\infty}^{\infty} (\mathrm{i}\xi)^j\,\mathrm{e}^{\mathrm{i}\xi x}\mathrm{e}^{-A(\xi)(t-s)}\frac{\widehat{F(v_x)}(\xi,s)}{1+\xi^2}\,d\xi\,ds. \end{aligned} \tag{3.5}$$

Before starting the proof of the *a priori* estimates, we first give several preparation lemmas as follows.

LEMMA 3.1. *Suppose that* $a>0$ *and* $b>0$, *and* $\max(a,b)>1$, *then*

$$\int_0^t (1+s)^{-a}\,(1+t-s)^{-b}\,ds\leqslant C(1+t)^{-\min(a,b)}. \tag{3.6}$$

The proof of Lemma 3.1 was given in [20], and several applications can be found in [10, 13–15].

LEMMA 3.2. *For any positive constant* c, *the following*

$$\int_{-\infty}^{\infty}\frac{|\xi|^j\,\mathrm{e}^{-cB(\xi)t}}{(1+\xi^2)(1+|\xi|)^j}\,d\xi\leqslant C(1+t)^{-(j+1)/2}, \qquad j\in\mathbf{N}_+ \tag{3.7}$$

holds for all $t\geqslant 0$.

Proof. Inequallity (3.7) can be similarly proved as in [14, 15, 21, 22]. We first note that

$$\int_{-\infty}^{\infty}\frac{|\xi|^j\,\mathrm{e}^{-cB(\xi)t}}{(1+\xi^2)(1+|\xi|)^j}\,d\xi=2\left(\int_0^1+\int_1^{\infty}\right)\frac{\xi^j\mathrm{e}^{-cB(\xi)t}}{(1+\xi^2)(1+\xi)^j}\,d\xi. \tag{3.8}$$

When $\xi \in [0, 1]$, it is clear that $\frac{1}{2} \leqslant (1+\xi^2)^{-1} \leqslant 1$ and $2^{-(j+1)} \leqslant (1+\xi^2)^{-1}(1+\xi)^{-j} \leqslant 1$, thus,

$$\begin{aligned}
&\int_0^1 \frac{\xi^j e^{-cB(\xi)t}}{(1+\xi^2)(1+\xi)^j}\,d\xi \\
&\quad\leqslant \int_0^1 \xi^j e^{-(c\alpha\xi^2/2)t}\,d\xi \\
&\quad= e^{c\alpha/2} \int_0^1 \xi^j e^{-(c\alpha\xi^2/2)(1+t)}\,d\xi \\
&\quad= \frac{e^{c\alpha/2}}{2}(1+t)^{-(j+1)/2} \int_0^1 [\xi^2(1+t)]^{(j-1)/2}\, e^{-(c\alpha/2)\,\xi^2(1+t)}\, d[\xi^2(1+t)] \\
&\quad\leqslant C(1+t)^{-(j+1)/2},
\end{aligned} \tag{3.9}$$

where we used the fact that, letting $\eta = \xi^2(1+t)$, then

$$\begin{aligned}
&\int_0^1 [\xi^2(1+t)]^{(j-1)/2}\, e^{-(c\alpha/2)\,\xi^2(1+t)}\, d[\xi^2(1+t)] \\
&\quad= \int_0^{1+t} \eta^{(j-1)/2} e^{-(c\alpha/2)\eta}\,d\eta \\
&\quad\leqslant \int_0^{\infty} \eta^{(j-1)/2} e^{-(c\alpha/2)\eta}\,d\eta \leqslant C.
\end{aligned}$$

When $\xi \in [1, +\infty]$, we have that $\frac{1}{2} \leqslant \xi^2/(1+\xi^2) \leqslant 1$ and $\xi^j/(1+\xi)^j \leqslant 1$, which imply

$$\int_1^{\infty} \frac{\xi^j e^{-cB(\xi)t}}{(1+\xi^2)(1+\xi)^j}\,d\xi \leqslant \int_1^{\infty} \frac{e^{-(c\alpha/2)t}}{1+\xi^2}\,d\xi \leqslant Ce^{-(c\alpha/2)t}. \tag{3.10}$$

Thus, applying (3.9) and (3.10) into (3.8), we prove (3.7). ∎

LEMMA 3.3. *If $v(x, t) \in X(0, T)$, then*

$$\sup_{\xi \in R} \left((1+|\xi|)^j \,|\widehat{F(v_x)}(\xi, t)|\right) \leqslant CM(T)^{p+1}(1+t)^{-(2p+1)/2} \tag{3.11}$$

for $j = 0, 1, ..., 2p-1$.

Proof. Since $v \in X(0, T)$, by the definition of $M(t)$, we first have the estimates

$$\int_{-\infty}^{\infty} |v_x^{p+1}(x,t)|\,dx \leqslant \sup_{x\in R} |v_x|^{p-1} \int_{-\infty}^{\infty} |v_x|^2\,dx = \|v_x(t)\|_{L^\infty}^{p-1}\,\|v_x(t)\|_{L^2}^2$$
$$\leqslant M(T)^{p+1}\,(1+t)^{-(p-1)}\,(1+t)^{-3/2}$$
$$= M(T)^{p+1}\,(1+t)^{-(2p+1)/2}, \tag{3.12}$$

$$\int_{-\infty}^{\infty} |\partial_x(v_x^{p+1})(x,t)|\,dx = (p+1)\int_{-\infty}^{\infty} |v_x^p\,v_{xx}|\,dx$$
$$\leqslant (p+1)\,\|v_x(t)\|_{L^\infty}^{p-1}\,\|v_x(t)\|_{L^2}\,\|v_{xx}(t)\|_{L^2}$$
$$\leqslant (p+1)\,M(T)^{p+1}\,(1+t)^{-(p-1)}$$
$$\times (1+t)^{-3/4}\,(1+t)^{-5/4}$$
$$= (p+1)\,M(T)^{p+1}\,(1+t)^{-(2p+2)/2}. \tag{3.13}$$

Generally, we can prove the following in the same way

$$\int_{-\infty}^{\infty} |\partial_x^j(v_x^{p+1})(x,t)|\,dx \leqslant CM(T)^{p+1}\,(1+t)^{-(2p+1+j)/2} \tag{3.14}$$

for $j=0, 1, ..., 2p-1$.

Thus, making use of (3.14), (2.6), and the property of the Fourier transform

$$|(i\xi)^j \hat{f}| = |\widehat{\partial_x^j f}| \leqslant \int_{-\infty}^{\infty} |\partial_x^j f|\,dx,$$

we prove (3.11) as

$$\sup_{\xi\in R} ((1+|\xi|)^j\,|\widehat{F(v_x)}(\xi,t)|)$$
$$\leqslant \frac{C}{p+1} \sup_{\xi\in R} ((1+|\xi|^j)\,|\widehat{v_x^{p+1}}(\xi,t)|)$$
$$\leqslant \frac{C}{p+1}\int_{-\infty}^{\infty} |v_x^{p+1}(x,t)|\,dx + \frac{C}{p+1}\int_{-\infty}^{\infty} |\partial_x^j(v_x^{p+1})(x,t)|\,dx$$
$$\leqslant \frac{C}{p+1}\,M(T)^{p+1}\,(1+t)^{-(2p+1)/2} + \frac{C}{p+1}\,M(T)^{p+1}\,(1+t)^{-(2p+1+j)/2}$$
$$\leqslant \frac{2C}{p+1}\,M(T)^{p+1}\,(1+t)^{-(2p+1)/2},$$

for $j=0, 1, ..., 2p-1$. The proof of this lemma is complete. ∎

LEMMA 3.4. *If* $v_0(x) \in W^{2p+1,1}(R)$, *then*

$$\left\| \frac{1}{2\pi} \int_{-\infty}^{\infty} (i\xi)^j e^{i\xi x} e^{-A(\xi)t} \hat{v}_0(\xi)\, d\xi \right\|_{L^2} \leqslant C \|v_0\|_{W^{j+1,1}} (1+t)^{-(2j+1)/4} \tag{3.15}$$

for $j = 0, 1, ..., 2p$, *and*

$$\left\| \frac{1}{2\pi} \int_{-\infty}^{\infty} (i\xi)^j e^{i\xi x} e^{-A(\xi)t} \hat{v}_0(\xi)\, d\xi \right\|_{L^\infty} \leqslant C \|v_0\|_{W^{j+2,1}} (1+t)^{-(j+1)/2} \tag{3.16}$$

for $j = 0, 1, ..., 2p-1$.

Proof. For the proof of (3.15), noting the fact

$$\begin{aligned} \sup_{\xi \in R} ((1+|\xi|)^{j+1} |\hat{v}_0|) &\leqslant C \sup_{\xi \in R} ((1+|\xi|^{j+1}) |\hat{v}_0|) \\ &\leqslant C \int_{-\infty}^{\infty} (|v_0(x)| + |\partial_x^{j+1} v_0(x)|)\, dx \leqslant C \|v_0\|_{W^{j+1,1}} \end{aligned} \tag{3.17}$$

for $j = 0, 1, ..., 2p$, and by Parseval's equality and Lemma 3.2, we have

$$\begin{aligned} &\left\| \frac{1}{2\pi} \int_{-\infty}^{\infty} (i\xi)^j e^{i\xi x} e^{-A(\xi)t} \hat{v}_0(\xi)\, d\xi \right\|_{L^2} \\ &\quad = \|(i\xi)^j e^{-A(\xi)t} \hat{v}_0(\xi)\|_{L^2} = \left(\int_{-\infty}^{\infty} |\xi|^{2j} e^{-2B(\xi)t} |\hat{v}_0(\xi)|^2\, d\xi \right)^{1/2} \\ &\quad = \left(\int_{-\infty}^{\infty} \frac{|\xi|^{2j} e^{-2B(\xi)t}}{(1+\xi^2)(1+|\xi|)^{2j}} (1+\xi^2)(1+|\xi|)^{2j} |\hat{v}_0(\xi)|^2\, d\xi \right)^{1/2} \\ &\quad \leqslant \sup_{\xi \in R} ((1+|\xi|)^{j+1} |\hat{v}_0(\xi)|) \left(\int_{-\infty}^{\infty} \frac{|\xi|^{2j} e^{-2B(\xi)t}}{(1+\xi^2)(1+|\xi|)^{2j}}\, d\xi \right)^{1/2} \\ &\quad \leqslant C \|v_0\|_{W^{j+1,1}} (1+t)^{-(2j+1)/4} \end{aligned}$$

for $j = 0, 1, ..., 2p$.

For the proof of (3.16), using (3.17) and Lemma 3.2, we show

$$\begin{aligned} &\left\| \frac{1}{2\pi} \int_{-\infty}^{\infty} (i\xi)^j e^{i\xi x} e^{-A(\xi)t} \hat{v}_0(\xi)\, d\xi \right\|_{L^\infty} \\ &\quad = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{|\xi|^j e^{-B(\xi)t}}{(1+\xi^2)(1+|\xi|)^j} (1+\xi^2)(1+|\xi|)^j |\hat{v}_0(\xi)|\, d\xi \end{aligned}$$

$$\leqslant \frac{1}{2\pi} \sup_{\xi \in R} ((1+|\xi|)^{j+2} |\hat{v}_0|) \int_{-\infty}^{\infty} \frac{|\xi|^j \, e^{-B(\xi) t}}{(1+\xi^2)(1+|\xi|)^j} \, d\xi$$

$$\leqslant C \, \|v_0\|_{W^{j+2,1}} (1+t)^{-(j+1)/2}$$

for $j = 0, 1, ..., 2p-1$. ∎

Lemma 3.5. *Suppose that $v(x, t) \in X(0, T)$. Then*

$$\left\| \frac{1}{2\pi} \int_{-\infty}^{\infty} (\mathrm{i}\xi)^j \, e^{i\xi x} \frac{e^{-A(\xi)(t-s)}}{1+\xi^2} \widehat{F(v_x)}(\xi, s) \, d\xi \right\|_{L^2} \leqslant CM(T)^{p+1} (1+t-s)^{-(2j+1)/4} (1+s)^{-(2p+1)/2} \tag{3.18}$$

for $j = 0, 1, ..., 2p$, and

$$\left\| \frac{1}{2\pi} \int_{-\infty}^{\infty} (\mathrm{i}\xi)^j \, e^{i\xi x} \frac{e^{-A(\xi)(t-s)}}{1+\xi^2} \widehat{F(v_x)}(\xi, s) \, d\xi \right\|_{L^\infty} \leqslant CM(T)^{p+1} (1+t-s)^{-(j+1)/2} (1+s)^{-(2p+1)/2} \tag{3.19}$$

for $j = 0, 1, ..., 2p-1$.

Proof. When $j = 0$, we can prove (3.18) by Parseval's equality and Lemmas 3.2 and 3.3 as

$$\begin{aligned}
&\left\| \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\xi x} \frac{e^{-A(\xi)(t-s)}}{1+\xi^2} \widehat{F(v_x)}(\xi, s) \, d\xi \right\|_{L^2} \\
&\quad = \left\| \frac{e^{-A(\xi)(t-s)}}{1+\xi^2} \widehat{F(v_x)}(\xi, s) \right\|_{L^2} \\
&\quad = \left(\int_{-\infty}^{\infty} \frac{e^{-2B(\xi)(t-s)}}{(1+\xi^2)^2} |\widehat{F(v_x)}(\xi, s)|^2 \, d\xi \right)^{1/2} \\
&\quad \leqslant \sup_{\xi \in R} |\widehat{F(v_x)}(\xi, s)| \left(\int_{-\infty}^{\infty} \frac{e^{-2B(\xi)(t-s)}}{(1+\xi^2)^2} \, d\xi \right)^{1/2} \\
&\quad \leqslant \sup_{\xi \in R} |\widehat{F(v_x)}(\xi, s)| \left(\int_{-\infty}^{\infty} \frac{e^{-2B(\xi)(t-s)}}{1+\xi^2} \, d\xi \right)^{1/2} \\
&\quad \leqslant CM(T)^{p+1} (1+s)^{-(2p+1)/2} (1+t-s)^{-1/4}.
\end{aligned} \tag{3.20}$$

When $1 \leqslant j \leqslant 2p$, noting $(1+|\xi|)^2/(1+\xi^2) \leqslant 2$ for all $\xi \in R$, we have similarly

$$\left\| \frac{1}{2\pi} \int_{-\infty}^{\infty} (\mathrm{i}\xi)^j \, \mathrm{e}^{i\xi x} \frac{\mathrm{e}^{-A(\xi)(t-s)}}{1+\xi^2} \widehat{F(v_x)}(\xi, s) \, d\xi \right\|_{L^2}$$

$$= \left\| (\mathrm{i}\xi)^j \frac{\mathrm{e}^{-A(\xi)(t-s)}}{1+\xi^2} \widehat{F(v_x)}(\xi, s) \right\|_{L^2}$$

$$= \left(\int_{-\infty}^{\infty} \frac{|\xi|^{2j} \, \mathrm{e}^{-2B(\xi)(t-s)}}{(1+\xi^2)^2} |\widehat{F(v_x)}(\xi, s)|^2 \, d\xi \right)^{1/2}$$

$$= \left(\int_{-\infty}^{\infty} \frac{|\xi|^{2j} \, \mathrm{e}^{-2B(\xi)(t-s)}}{(1+\xi^2)(1+|\xi|)^{2j}} \cdot \frac{(1+|\xi|)^2}{1+\xi^2} \times (1+|\xi|)^{2j-2} \, |\widehat{F(v_x)}(\xi, s)|^2 \, d\xi \right)^{1/2}$$

$$\leqslant \sqrt{2} \sup_{\xi \in R} \left((1+|\xi|)^{j-1} \, |\widehat{F(v_x)}(\xi, s)| \right) \left(\int_{-\infty}^{\infty} \frac{|\xi|^{2j} \, \mathrm{e}^{-2B(\xi)(t-s)}}{(1+\xi^2)(1+|\xi|)^{2j}} \, d\xi \right)^{1/2}$$

$$\leqslant CM(T)^{p+1} \, (1+s)^{-(2p+1)/2} \, (1+t-s)^{-(2j+1)/4}. \tag{3.21}$$

Thus, (3.20) and (3.21) give us (3.18).

To prove (3.19), by means of Lemmas 3.2 and 3.3, we obtain that for $j = 0, 1, ..., 2p-1$

$$\left\| \frac{1}{2\pi} \int_{-\infty}^{\infty} (\mathrm{i}\xi)^j \, \mathrm{e}^{i\xi x} \frac{\mathrm{e}^{-A(\xi)(t-s)}}{1+\xi^2} \widehat{F(v_x)}(\xi, s) \, d\xi \right\|_{L^\infty}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{|\xi|^j \, \mathrm{e}^{-B(\xi)(t-s)}}{1+\xi^2} |\widehat{F(v_x)}(\xi, s)| \, d\xi$$

$$\leqslant \frac{1}{2\pi} \sup_{\xi \in R} \left((1+|\xi|)^j \, |\widehat{F(v_x)}(\xi, s)| \right) \int_{-\infty}^{\infty} \frac{|\xi|^j \, \mathrm{e}^{-B(\xi)(t-s)}}{(1+\xi^2)(1+|\xi|)^j} \, d\xi$$

$$\leqslant CM(T)^{p+1} \, (1+s)^{-(2p+1)/2} \, (1+t-s)^{-(j+1)/2}. \tag{3.22}$$

This completes the proof of Lemma 3.5. ∎

Proof of Proposition 2.4 (*A Priori Estimates*). Let $v(x, t) \in X(0, T)$. From (3.5), thanks to Lemmas 3.1, 3.4, and 3.5, and the fact $(2p+1)/2 > 1$ due to $p \geqslant 1$, $(2p+1)/2 > (2j+1)/4$ for $j = 0, 1, ..., 2p$, we obtain

$$
\begin{aligned}
\|\partial_x^j v(t)\|_{L^2} \leqslant & \left\| \frac{1}{2\pi} \int_{-\infty}^{\infty} (\mathrm{i}\xi)^j \, \mathrm{e}^{i\xi x} \mathrm{e}^{-A(\xi)t} \, \hat{v}_0(\xi) \, d\xi \right\|_{L^2} \\
& + \int_0^t \left\| \frac{1}{2\pi} \int_{-\infty}^{\infty} (\mathrm{i}\xi)^j \, \mathrm{e}^{i\xi x} \, \frac{\mathrm{e}^{-A(\xi)(t-s)}}{1+\xi^2} \, \widehat{F(v_x)}(\xi, s) \, d\xi \right\|_{L^2} ds \\
\leqslant & \; C \, \|v_0\|_{W^{j+1,1}} \, (1+t)^{-(2j+1)/4} \\
& + C M(t)^{p+1} \int_0^t (1+t-s)^{-(2j+1)/4} \, (1+s)^{-(2p+1)/2} \, ds \\
\leqslant & \; \bar{C}_1 (\|v_0\|_{W^{2p+1,1}} + M(t)^{p+1})(1+t)^{-(2j+1)/4}
\end{aligned}
\tag{3.23}
$$

for $j = 0, 1, ..., 2p$. Here $\bar{C}_1$ is some positive constant independent of T.

Similarly, noting $(2p+1)/2 > (j+1)/2$ for $j = 0, 1, ..., 2p-1$, by using Lemmas 3.1, 3.4, and 3.5, we have also

$$
\begin{aligned}
\|\partial_x^j v(t)\|_{L^\infty} \leqslant & \left\| \frac{1}{2\pi} \int_{-\infty}^{\infty} (\mathrm{i}\xi)^j \, \mathrm{e}^{i\xi x} \mathrm{e}^{-A(\xi)t} \, \hat{v}_0(\xi) \, d\xi \right\|_{L^\infty} \\
& + \int_0^t \left\| \frac{1}{2\pi} \int_{-\infty}^{\infty} (\mathrm{i}\xi)^j \, \mathrm{e}^{i\xi x} \, \frac{\mathrm{e}^{-A(\xi)(t-s)}}{1+\xi^2} \, \widehat{F(v_x)}(\xi, s) \, d\xi \right\|_{L^\infty} ds \\
\leqslant & \; C \, \|v_0\|_{W^{j+2,1}} \, (1+t)^{-(j+1)/2} \\
& + C M(t)^{p+1} \int_0^t (1+t-s)^{-(j+1)/2} \, (1+s)^{-(2p+1)/2} \, ds \\
\leqslant & \; \bar{C}_2 (\|v_0\|_{W^{2p+1,1}} + M(t)^{p+1})(1+t)^{-(j+1)/2}
\end{aligned}
\tag{3.24}
$$

for $j = 0, 1, ..., 2p$. Here $\bar{C}_2$ denotes also some positive constant independent of T.

Adding $(3.23) \times (1+t)^{(2j+1)/4}$ and $(3.24) \times (1+t)^{(j+1)/2}$ gives us

$$
\begin{aligned}
M(T) = & \sup_{0 \leqslant t \leqslant T} \sum_{j=0}^{2p} (1+t)^{(2j+1)/4} \, \|\partial_x^j v(t)\|_{L^2} + \sum_{j=0}^{2p-1} (1+t)^{(j+1)/2} \, \|\partial_x^j v(t)\|_{L^\infty} \\
\leqslant & \; C_p (\|v_0\|_{W^{2p+1,1}} + M(T)^{p+1}),
\end{aligned}
$$

where $C_p = 2p\bar{C}_1 + (2p-1)\,\bar{C}_2$. Namely,

$$
M(T)\{1 - C_p M(T)^p\} \leqslant C_p \, \|v_0\|_{W^{2p+1,1}}.
$$

Now we choose our positive constant δ_2 in Proposition 2.4 as

$$\delta_2 = 1/(2C_p)^{1/p},$$

when $M(T) \leqslant \delta_2$, and we obtain

$$M(T) \leqslant \frac{C_p \|v_0\|_{W^{2p+1,1}}}{1 - C_p M(T)^p} \leqslant 2C_p \|v_0\|_{W^{2p+1,1}}.$$

That is,

$$\sum_{j=0}^{2p} (1+t)^{(2j+1)/4} \|\partial_x^j v(t)\|_{L^2} + \sum_{j=0}^{2p-1} (1+t)^{(j+1)/2} \|\partial_x^j v(t)\|_{L^\infty}$$
$$\leqslant 2C_p \|v_0\|_{W^{2p+1,1}}$$

for $M(T) \leqslant \delta_2 = 1/(2C_p)^{1/p}$ and $t \in [0, T]$. Thus, the proof of Proposition 2.4 is complete. ∎

4. PROOF OF THEOREM 2.2

We are going to prove Theorem 2.2 based on Theorem 2.1. In the same way as mentioned above, we can prove (2.9) and (2.10). For the L^q-decay rates (2.11) and (2.12) with $2 < q < \infty$, it can be easily proved by means of the Sobolev inequalty and Theorem 2.1. When $1 \leqslant q < 2$, the proof of the L^q decay estimate (2.13) is not easy, and we must make a bit of effort on it, in particular, for the case $q = 1$. The approach we will adopt is the point wise method of the Green function.

Proof of (2.9) *and* (2.10). Taking $\partial_x^j \partial_t$ to (3.4) yields

$$\begin{aligned}
\partial_x^j v_t(x,t) = & -\frac{1}{2\pi} \int_{-\infty}^{\infty} (\mathrm{i}\xi)^j A(\xi)\, e^{\mathrm{i}\xi x} e^{-A(\xi)t} \hat{v}_0(\xi)\, d\xi \\
& + \frac{1}{2\pi} \int_0^t \int_{-\infty}^{\infty} (\mathrm{i}\xi)^j A(\xi)\, e^{\mathrm{i}\xi x} e^{-A(\xi)(t-s)} \frac{\widehat{F(v_x)}(\xi, s)}{1+\xi^2}\, d\xi\, ds \\
& - \frac{1}{2\pi} \int_{-\infty}^{\infty} (\mathrm{i}\xi)^j e^{\mathrm{i}\xi x} \frac{\widehat{F(v_x)}(\xi, t)}{1+\xi^2}\, d\xi
\end{aligned} \tag{4.1}$$

for $j = 0, 1, ..., 2p-2$.

We first prove (2.9) in the same way as (3.23). From (4.1) and using Parseval's equality, we have

$$
\begin{aligned}
&\|\partial_x^j v_t(t)\|_{L^2}\\
&\quad\leqslant \left\| \frac{1}{2\pi}\int_{-\infty}^{\infty} (\mathrm{i}\xi)^j\, A(\xi)\, \mathrm{e}^{\mathrm{i}\xi x}\mathrm{e}^{-A(\xi)t}\hat{v}_0(\xi)\, d\xi \right\|_{L^2}\\
&\qquad+\int_0^t \left\| \frac{1}{2\pi}\int_{-\infty}^{\infty} (\mathrm{i}\xi)^j\, A(\xi)\, \mathrm{e}^{\mathrm{i}\xi x}\mathrm{e}^{-A(\xi)(t-s)}\, \frac{\widehat{F(v_x)}(\xi, s)}{1+\xi^2}\, d\xi \right\|_{L^2} ds\\
&\qquad+\left\| \frac{1}{2\pi}\int_{-\infty}^{\infty} (\mathrm{i}\xi)^j\, \mathrm{e}^{\mathrm{i}\xi x}\, \frac{\widehat{F(v_x)}(\xi, t)}{1+\xi^2}\, d\xi \right\|_{L^2}\\
&\quad= \|(\mathrm{i}\xi)^j\, A(\xi)\, \mathrm{e}^{-A(\xi)t}\hat{v}_0(\xi)\|_{L^2}\\
&\qquad+\int_0^t \left\| \frac{1}{2\pi}\int_{-\infty}^{\infty} (\mathrm{i}\xi)^j\, A(\xi)\, \mathrm{e}^{-A(\xi)(t-s)}\, \frac{\widehat{F(v_x)}(\xi, s)}{1+\xi^2} \right\|_{L^2} ds\\
&\qquad+\left\| (\mathrm{i}\xi)^j\, \frac{\widehat{F(v_x)}(\xi, t)}{1+\xi^2} \right\|_{L^2}
\end{aligned}
\tag{4.2}
$$

for $j=0, 1, ..., 2p-2$.

For $j=0$, we have proved in [15]

$$
\|v_t(t)\|_{L^2} \leqslant C(1+t)^{-5/4}. \tag{4.3}
$$

Now we consider the cases $j=1, ..., 2p-2$. Making use of Lemma 3.2 and (3.17), we have

$$
\begin{aligned}
&\|(\mathrm{i}\xi)^j\, A(\xi)\, \mathrm{e}^{-A(\xi)t}\hat{v}_0(\xi)\|_{L^2}\\
&\quad= \left(\int_{-\infty}^{\infty} \frac{|\xi|^{2j+4}\, \mathrm{e}^{-2B(\xi)t}}{(1+\xi^2)^2}\, |\hat{v}_0|^2\, d\xi \right)^{1/2}\\
&\quad= \left(\int_{-\infty}^{\infty} \frac{|\xi|^{2j+4}\, \mathrm{e}^{-2B(\xi)t}}{(1+\xi^2)(1+|\xi|)^{2j+4}} \cdot \frac{(1+|\xi|)^2}{1+\xi^2} \cdot (1+|\xi|)^{2j+2}\, |\hat{v}_0|^2\, d\xi \right)^{1/2}\\
&\quad\leqslant \sqrt{2}\, \sup_{\xi\in R}\, ((1+|\xi|)^{j+1}\, |\hat{v}_0(\xi)|) \left(\int_{-\infty}^{\infty} \frac{|\xi|^{2j+4}\, \mathrm{e}^{-2B(\xi)t}}{(1+\xi^2)(1+|\xi|)^{2j+4}}\, d\xi \right)^{1/2}\\
&\quad\leqslant C\, \|v_0\|_{W^{j+1,1}}\, (1+t)^{-(2j+5)/4}
\end{aligned}
\tag{4.4}
$$

for $j=1, ..., 2p-2$.

Thanks to Lemma 3.2 and Lemma 3.3, that is,

$$
\sup_{\xi\in R}\, ((1+|\xi|)^{j-1}\, |\widehat{F(v_x)}(\xi, t)|) \leqslant C\, \|v_0\|^{p+1}_{W^{2p+1,1}}\, (1+t)^{-(2p+1)/2}
$$

for $j=1, ..., 2p-2$, where we used $M(t) \leqslant C \|v_0\|_{W^{2p+1,1}}$ for all $t \in [0, +\infty)$ due to Theorem 2.1, then we have

$$\begin{aligned}
&\left\| (\mathrm{i}\xi)^j A(\xi)\, \mathrm{e}^{-A(\xi)(t-s)} \frac{\widehat{F(v_x)}(\xi, s)}{1+\xi^2} \right\|_{L^2} \\
&\quad = \left(\int_{-\infty}^{\infty} \frac{|\xi|^{2j+4}\, \mathrm{e}^{-2B(\xi)(t-s)}}{(1+\xi^2)^4} |\widehat{F(v_x)}(\xi, s)|^2\, d\xi \right)^{1/2} \\
&\quad = \left(\int_{-\infty}^{\infty} \frac{|\xi|^{2j+4}\, \mathrm{e}^{-2B(\xi)(t-s)}}{(1+\xi^2)(1+|\xi|)^{2j+4}} \cdot \frac{(1+|\xi|)^6}{(1+\xi^2)^3} \right. \\
&\qquad \left. \times (1+|\xi|)^{2j-2} |\widehat{F(v_x)}(\xi, s)|^2\, d\xi \right)^{1/2} \\
&\quad \leqslant 2\sqrt{2} \sup_{\xi \in R} ((1+|\xi|)^{j-1} |\widehat{F(v_x)}(\xi, s)|) \\
&\qquad \times \left(\int_{-\infty}^{\infty} \frac{|\xi|^{2j+4}\, \mathrm{e}^{-2B(\xi)t}}{(1+\xi^2)(1+|\xi|)^{2j+4}}\, d\xi \right)^{1/2} \\
&\quad \leqslant C \|v_0\|_{W^{2p+1,1}}^{p+1} (1+s)^{-(2p+1)/2} (1+t-s)^{-(2j+5)/4} \qquad (4.5)
\end{aligned}$$

for $j=1, ..., 2p-2$.

Similarly, we prove

$$\begin{aligned}
\left\| (\mathrm{i}\xi)^j \frac{\widehat{F(v_x)}(\xi, t)}{1+\xi^2} \right\|_{L^2} &= \left(\int_{-\infty}^{\infty} \frac{|\xi|^{2j} |\widehat{F(v_x)}(\xi, t)|^2}{(1+\xi^2)^2}\, d\xi \right)^{1/2} \\
&\leqslant \sup_{\xi \in R} (|\xi|^j |\widehat{F(v_x)}(\xi, t)|) \left(\int_{-\infty}^{\infty} \frac{1}{(1+\xi^2)^2}\, d\xi \right)^{1/2} \\
&\leqslant C \|v_0\|_{W^{2p+1,1}}^{p+1} (1+t)^{-(2p+1+j)/2} \\
&\leqslant C \|v_0\|_{W^{2p+1,1}}^{p+1} (1+t)^{-(2j+5)/4} \qquad (4.6)
\end{aligned}$$

for $j=1, ..., 2p-2$. Here we used $p \geqslant 1$, i.e., $(2p+1+j)/2 > (2j+5)/4$.

Thus, applying (4.4)–(4.6) into (4.2) and using Lemma 3.1, and noting $(2p+1)/2 > 1$, $(2p+1)/2 > (2j+5)/4$ for $j=1, ..., 2p-2$, we obtain

$$\begin{aligned}
\|\partial_x^j v_t(t)\|_{L^2} &\leqslant C \|v_0\|_{W^{j+1,1}} (1+t)^{-(2j+5)/4} \\
&\quad + C \|v_0\|_{W^{2p+1,1}}^{p+1} \int_0^t (1+s)^{-(2p+1)/2} (1+t-s)^{-(2j+5)/4}\, ds \\
&\quad + C \|v_0\|_{W^{2p+1,1}}^{p+1} (1+t)^{-(2j+5)/4} \\
&\leqslant C(\|v_0\|_{W^{j+1,1}} + \|v_0\|_{W^{2p+1,1}}^{p+1})(1+t)^{-(2j+5)/4} \qquad (4.7)
\end{aligned}$$

for $j=0, 1, ..., 2p-2$. Therefore, (4.3) and (4.7) imply (2.9).

For the proof of (2.10), without any difficulty, we can similarly prove that, for $j=0, ..., 2p-2$,

$$\int_{-\infty}^{\infty} \frac{|\xi|^{j+2} e^{-B(\xi)t}}{1+\xi^2} |\hat{v}_0(\xi)| \, d\xi \leqslant \sup_{\xi\in R} ((1+|\xi|)^{j+2} |\hat{v}_0(\xi)|) \int_{-\infty}^{\infty} \frac{|\xi|^{j+2} e^{-B(\xi)t}}{(1+\xi^2)(1+|\xi|)^{j+2}} \, d\xi \leqslant C \|v_0\|_{W^{j+2,1}} (1+t)^{-(j+3)/2}, \tag{4.8}$$

and

$$\int_{-\infty}^{\infty} \frac{|\xi|^{j+2} e^{-B(\xi)(t-s)}}{(1+\xi^2)^2} |\widehat{F(v_x)}(\xi, s)| \, d\xi \leqslant \sup_{\xi\in R} ((1+|\xi|)^j |\widehat{F(v_x)}(\xi, s)|) \int_{-\infty}^{\infty} \frac{|\xi|^{j+2} e^{-B(\xi)(t-s)}}{(1+\xi^2)(1+|\xi|)^{j+2}} \, d\xi \leqslant C \|v_0\|^{p+1}_{W^{2p+1,1}} (1+s)^{-(2p+1)/2} (1+t-s)^{-(j+3)/2}, \tag{4.9}$$

and

$$\int_{-\infty}^{\infty} \frac{|\xi|^j |\widehat{F(v_x)}(\xi, t)|}{1+\xi^2} \, d\xi \leqslant \sup_{\xi\in R} (|\xi|^j |\widehat{F(v_x)}(\xi, t)|) \int_{-\infty}^{\infty} \frac{1}{1+\xi^2} \, d\xi \leqslant C \|v_0\|^{p+1}_{W^{2p+1,1}} (1+t)^{-(2p+j+1)/2} \leqslant C \|v_0\|^{p+1}_{W^{2p+1,1}} (1+t)^{-(j+3)/2}. \tag{4.10}$$

These estimates imply (2.10) as

$$\|\partial_x^j v_t(t)\|_{L^\infty} \leqslant \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{|\xi|^{j+2} e^{-B(\xi)t}}{1+\xi^2} |\hat{v}_0(\xi)| \, d\xi + \frac{1}{2\pi} \int_0^t \int_{-\infty}^{\infty} \frac{|\xi|^{j+2} e^{-B(\xi)(t-s)}}{(1+\xi^2)^2} \frac{\widehat{F(v_x)}(\xi, s)}{1+\xi^2} \, d\xi \, ds + \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{|\xi|^j |\widehat{F(v_x)}(\xi, t)|}{1+\xi^2} \, d\xi$$

$$\leqslant \frac{C}{2\pi} \|v_0\|_{W^{j+2,1}} (1+t)^{-(j+3)/2}$$

$$+\frac{C}{2\pi} \|v_0\|^{p+1}_{W^{2p+1,1}} \int_0^t (1+s)^{-(2p+1)/2} (1+t-s)^{-(j+3)/2}\, ds$$

$$+\frac{C}{2\pi} \|v_0\|^{p+1}_{W^{2p+1,1}} (1+t)^{-(j+3)/2}$$

$$\leqslant \frac{C}{2\pi} (\|v_0\|_{W^{j+1,1}} + \|v_0\|^{p+1}_{W^{2p+1,1}})(1+t)^{-(j+3)/2}$$

for $j=0, 1, ..., 2p-2$. ∎

Proof of (2.11) *and* (2.12). It can be easily proved that, for $j=0, 1, ..., 2p-1$ and $2<q<\infty$,

$$\begin{aligned} \|\partial_x^j v(t)\|_{L^q} &\leqslant \|\partial_x^j v(t)\|_{L^\infty}^{(q-2)/q} \|\partial_x^j v(t)\|_{L^2}^{2/q} \\ &\leqslant C(1+t)^{-(j+1)(q-2)/2q} (1+t)^{-(2j+1)/2q} \\ &= C(1+t)^{-((j+1)q-1)/2q}, \end{aligned}$$

and, for $j=0, 1, ..., 2p-2$ and $2<q<\infty$,

$$\begin{aligned} \|\partial_x^j v_t(t)\|_{L^q} &\leqslant \|\partial_x^j v_t(t)\|_{L^\infty}^{(q-2)/q} \|\partial_x^j v_t(t)\|_{L^2}^{2/q} \\ &\leqslant C(1+t)^{-(j+3)(q-2)/2q} (1+t)^{-(2j+5)/2q} \\ &= C(1+t)^{-((j+3)q-1)/2q}. \end{aligned}$$

Thus, we have proved (2.11) and (2.12). ∎

Proof of (2.13). Finally, we are going to prove (2.13). When $1 \leqslant q<2$, we note that the above method is unavailable for this case, since we have only the following inequality

$$\|f\|_{L^q} \leqslant \|\hat{f}\|_{L^m}, \qquad \text{for} \quad \frac{1}{q}+\frac{1}{m}=1, \quad 1 \leqslant m \leqslant 2,$$

and

$$\|f\|_{L^q} = \|\hat{f}\|_{L^m}, \qquad \text{only for} \quad q=m=2.$$

Namely, we only obtain our desired estimates for the case $q \in [2, \infty]$. So, in the case $1 \leqslant q<2$, we must find another recipe. To end it, the key step is to estimate $\|\partial_x^j v(t)\|_{L^1}$, $j=0, ..., 2p-2$. We are going to show it by the so-called pointwise method of the Green function.

Observing Eq. (2.4), since we expect that, in general, v_{xxt} decays time-asymptotically faster than v_t, v_x, and v_{xx} behave, we rewrite Eq. (2.4) as

$$v_t - \alpha v_{xx} + \beta v_x = v_{xxt} - F(v_x). \tag{4.11}$$

Formally, the expression of solution of Eq. (4.11) is

$$\begin{aligned} v(x, t) = &\int_{-\infty}^{\infty} G(x-y-\beta t, t)\, v_0(y)\, dy \\ &+ \int_0^t \int_{-\infty}^{\infty} G(x-y-\beta(t-s), t-s)\, v_{yys}(y, s)\, dy\, ds \\ &- \int_0^t \int_{-\infty}^{\infty} G(x-y-\beta(t-s), t-s)\, F(v_y)(y, s)\, dy\, ds, \end{aligned} \tag{4.12}$$

where

$$G(x-\beta t, t) = (4\alpha\pi t)^{-1/2} \exp\left(-\frac{(x-\beta t)^2}{4\alpha t}\right)$$

is the Green function of the following parabolic equation in the whole space $x \in R^1$

$$v_t - \alpha v_{xx} + \beta v_x = 0.$$

Differentiating (4.12) j-times with respect to x, we have

$$\begin{aligned} \partial_x^j v(x, t) = &\int_{-\infty}^{\infty} \partial_x^j G(x-y-\beta t, t)\, v_0(y)\, dy \\ &+ \int_0^t \int_{-\infty}^{\infty} \partial_x^j G(x-y-\beta(t-s), t-s)\, v_{yys}(y, s)\, dy\, ds \\ &- \int_0^t \int_{-\infty}^{\infty} \partial_x^j G(x-y-\beta(t-s), t-s)\, F(v_y)(y, s)\, dy\, ds. \end{aligned} \tag{4.13}$$

Since

$$\partial_x^j G(x-y-\beta t, t) = (-1)^j\, \partial_y^j G(x-y-\beta t, t),$$

and applying the integration by parts, (4.13) is reduced to

$$\begin{aligned}
\partial_x^j v(x,t) &= \int_{-\infty}^{\infty} (-1)^j \partial_y^j G(x-y-\beta t, t)\, v_0(y)\, dy \\
&\quad + \left(\int_0^{t/2} + \int_{t/2}^{t}\right) \int_{-\infty}^{\infty} (-1)^j \\
&\quad \times \partial_y^j G(x-y-\beta(t-s), t-s)\, v_{yys}(y,s)\, dy\, ds \\
&\quad - \left(\int_0^{t/2} + \int_{t/2}^{t}\right) \int_{-\infty}^{\infty} (-1)^j \\
&\quad \times \partial_y^j G(x-y-\beta(t-s), t-s)\, F(v_y)(y,s)\, dy\, ds \\
&= \int_{-\infty}^{\infty} (-1)^j \partial_y^j G(x-y-\beta t, t)\, v_0(y)\, dy \\
&\quad + \int_0^{t/2} \int_{-\infty}^{\infty} (-1)^{j+2} \\
&\quad \times \partial_y^{j+2} G(x-y-\beta(t-s), t-s)\, v_s(y,s)\, dy\, ds \\
&\quad + \int_{t/2}^{t} \int_{-\infty}^{\infty} G_{yy}(x-y-\beta(t-s), t-s)\, \partial_y^j v_s(y,s)\, dy\, ds \\
&\quad - \int_0^{t/2} \int_{-\infty}^{\infty} (-1)^j \partial_y^j G(x-y-\beta(t-s), t-s)\, F(v_y)(y,s)\, dy\, ds \\
&\quad - \int_{t/2}^{t} \int_{-\infty}^{\infty} G(x-y-\beta(t-s), t-s)\, \partial_y^j F(v_y)(y,s)\, dy\, ds \\
&=: I_{j1}(x,t) + I_{j2}(x,t) + I_{j3}(x,t) + I_{j4}(x,t) + I_{j5}(x,t). \qquad (4.14)
\end{aligned}$$

Lemma 4.1. *The following*

$$\int_{-\infty}^{\infty} |I_{j1}(x,t)|\, dx \leqslant C t^{-j/2}, \tag{4.15}$$

$$\int_{-\infty}^{\infty} |I_{j2}(x,t)|\, dx \leqslant C t^{-(j+1)/2}, \tag{4.16}$$

$$\int_{-\infty}^{\infty} |I_{j3}(x,t)|\, dx \leqslant C t^{-(j+2)/2}, \tag{4.17}$$

$$\int_{-\infty}^{\infty} |I_{j4}(x,t)|\,dx \leqslant Ct^{-j/2}, \tag{4.18}$$

$$\int_{-\infty}^{\infty} |I_{j5}(x,t)|\,dx \leqslant Ct^{-(2p-1+j)/2}, \tag{4.19}$$

hold for $j=0, ..., 2p-2$.

Proof. We denote $z=(x-y-\beta t)/\sqrt{4\alpha t}$ here and after here. Before starting the proofs of (4.15)–(4.19), we first show the estimates

$$\begin{aligned}
&\int_{-\infty}^{\infty} |G_y(x-y-\beta t, t)|\,dx \\
&\quad = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi\alpha t}} \exp\left(-\frac{(x-y-\beta t)^2}{4\alpha t}\right)\cdot\left|\frac{2(x-y-\beta t)}{4\alpha t}\right| dx \\
&\quad = O(1)\,t^{-1/2}\int_{-\infty}^{\infty} \mathrm{e}^{-z^2}\,|z|\,dz = O(1)\,t^{-1/2},
\end{aligned} \tag{4.20}$$

and

$$\begin{aligned}
&\int_{-\infty}^{\infty} |G_{yy}(x-y-\beta t, t)|\,dx \\
&\quad = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi\alpha t}} \exp\left(-\frac{(x-y-\beta t)^2}{4\alpha t}\right)\cdot\left|\left(\frac{2(x-y-\beta t)}{4\alpha t}\right)^2-\frac{2}{4\alpha t}\right| dx \\
&\quad = O(1)\,t^{-1}\int_{-\infty}^{\infty} \mathrm{e}^{-z^2}\,|4z^2-2|\,dz = O(1)\,t^{-1}.
\end{aligned} \tag{4.21}$$

In general, without any difficulty, the same calculation gives

$$\int_{-\infty}^{\infty} |\partial_y^j G(x-y-\beta t, t)|\,dx \leqslant Ct^{-j/2}, \qquad \text{for} \quad j\in \mathbf{N}_+. \tag{4.22}$$

Moreover, we have also

$$\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |G_y(x-y-\beta t, t)|\,dy\,dx \\
&\quad = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi\alpha t}} \exp\left(-\frac{(x-y-\beta t)^2}{4\alpha t}\right)\cdot\left|\frac{2(x-y-\beta t)}{4\alpha t}\right| dy\,dx
\end{aligned}$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi\alpha t}} \exp\left(-\frac{(x-y-\beta t)^2}{8\alpha t}\right) dx$$
$$\times \int_{-\infty}^{\infty} \exp\left(-\frac{(x-y-\beta t)^2}{8\alpha t}\right) \cdot \left|\frac{2(x-y-\beta t)}{4\alpha t}\right| dy$$
$$= O(1) \int_{-\infty}^{\infty} \mathrm{e}^{-z^2/2}\, dz \int_{-\infty}^{\infty} \mathrm{e}^{-z^2/2}\, |z|\, dz = O(1), \tag{4.23}$$

and

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |G_{yy}(x-y-\beta t, t)|\, dy\, dx$$
$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi\alpha t}} \exp\left(-\frac{(x-y-\beta t)^2}{4\alpha t}\right) \cdot \left|\left(\frac{2(x-y-\beta t)}{4\alpha t}\right)^2 - \frac{2}{4\alpha t}\right| dy\, dx$$
$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi\alpha t}} \exp\left(-\frac{(x-y-\beta t)^2}{8\alpha t}\right) dx$$
$$\times \int_{-\infty}^{\infty} \exp\left(-\frac{(x-y-\beta t)^2}{8\alpha t}\right) \cdot \left|\left(\frac{2(x-y-\beta t)}{4\alpha t}\right)^2 - \frac{2}{4\alpha t}\right| dy$$
$$= O(1) \int_{-\infty}^{\infty} \mathrm{e}^{-z^2}\, dz \int_{-\infty}^{\infty} t^{-1/2} \mathrm{e}^{-z^2}\, |4z^2 - 2|\, dz = O(1)\, t^{-1/2}. \tag{4.24}$$

Similarly, a straightforward computation yields

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\partial_y^j G(x-y-\beta t, t)|\, dy\, dx \leqslant C t^{-(j-1)/2}, \qquad \text{for} \quad j = 1, 2, ... \tag{4.25}$$

Now we are going to prove (4.15)–(4.19). By the definition of $I_{j1}(x, t)$ and (4.22), we have

$$\int_{-\infty}^{\infty} |I_{j1}(x, t)|\, dx \leqslant \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\partial_y^j G(x-y-\beta t, t)| \cdot |v_0(y)|\, dy\, dx$$
$$= \int_{-\infty}^{\infty} |\partial_y^j G(x-y-\beta t, t)|\, dx \int_{-\infty}^{\infty} |v_0(y)|\, dy$$
$$\leqslant C\, \|v_0\|_{L^1}\, t^{-j/2} \tag{4.26}$$

for $j = 0, ..., 2p-2$. This proves (4.15).

Thanks to (2.10) and (4.25), one can have, for $j=0, ..., 2p-2$,

$$\begin{aligned}
&\int_{-\infty}^{\infty} |I_{j2}(x,t)|\,dx \\
&\quad \leqslant \int_{-\infty}^{\infty} dx \int_0^{t/2} ds \int_{-\infty}^{\infty} |\partial_y^{j+2} G(x-y-\beta(t-s), t-s)| \cdot |v_s(y,s)|\,dy \\
&\quad \leqslant \int_0^{t/2} \left[\|v_s(s)\|_{L^\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\partial_y^{j+2} G(x-y-\beta(t-s), t-s)|\,dy\,dx \right] ds \\
&\quad \leqslant C \int_0^{t/2} (1+s)^{-3/2}\,(t-s)^{-(j+1)/2}\,ds \\
&\quad \leqslant C(t/2)^{-(j+1)/2} \int_0^{t/2} (1+s)^{-3/2}\,ds \\
&\quad \leqslant C(t/2)^{-(j+1)/2} \int_0^{\infty} (1+s)^{-3/2}\,ds \\
&\quad \leqslant Ct^{-(j+1)/2}.
\end{aligned} \tag{4.27}$$

This proves (4.16).

By (2.10) and (4.25), we can also prove (4.17) as

$$\begin{aligned}
&\int_{-\infty}^{\infty} |I_{j3}(x,t)|\,dx \\
&\quad \leqslant \int_{-\infty}^{\infty} dx \int_{t/2}^{t} ds \int_{-\infty}^{\infty} |G_{yy}(x-y-\beta(t-s), t-s)| \cdot |\partial_y^j v_s(y,s)|\,dy \\
&\quad \leqslant \int_{t/2}^{t} \left[\|\partial_y^j v_s(s)\|_{L^\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |G_{yy}(x-y-\beta(t-s), t-s)|\,dy\,dx \right] ds \\
&\quad \leqslant C \int_{t/2}^{t} (1+s)^{-(j+3)/2}\,(t-s)^{-1/2}\,ds \\
&\quad \leqslant C(1+(t/2))^{-(j+3)/2} \int_{t/2}^{t} (t-s)^{-1/2}\,ds \\
&\quad = 2C(1+(t/2))^{-(j+3)/2}\,(t/2)^{1/2} \\
&\quad \leqslant Ct^{-(j+2)/2}
\end{aligned} \tag{4.28}$$

for $j=0, ..., 2p-2$.

Making use of (2.11) and (4.25), we obtain

$$
\begin{aligned}
&\int_{-\infty}^{\infty} |I_{j4}(x, t)|\, dx \\
&\quad \leqslant \int_{-\infty}^{\infty} dx \int_0^{t/2} ds \int_{-\infty}^{\infty} |\partial_y^j G(x-y-\beta(t-s), t-s)| \cdot |F(v_y)(y, s)|\, dy \\
&\quad = \int_0^{t/2} \left[\int_{-\infty}^{\infty} |\partial_y^j G(x-y-\beta(t-s), t-s)|\, dx \int_{-\infty}^{\infty} |F(v_y)(y, s)|\, dy \right] ds \\
&\quad \leqslant C \int_0^{t/2} (t-s)^{-j/2}\, \|v_y(s)\|_{L^{p+1}}^{p+1}\, ds \\
&\quad \leqslant C \int_0^{t/2} (t-s)^{-j/2}\, (1+s)^{-(2p+1)/2}\, ds \\
&\quad \leqslant C(t/2)^{-j/2} \int_{t/2}^{t} (1+s)^{-(2p+1)/2}\, ds \\
&\quad \leqslant C(t/2)^{-j/2} \int_0^{\infty} (1+s)^{-(2p+1)/2}\, ds \\
&\quad \leqslant Ct^{-j/2} \qquad (4.29)
\end{aligned}
$$

for $j=0, ..., 2p-2$. Thus, we proved (4.18).

Finally, by (3.14), i.e.,

$$\int_{-\infty}^{\infty} |\partial_y^j F(v_y)(y, s)|\, dy \leqslant C(1+s)^{-(2p+1+j)/2},$$

and noting

$$\int_{-\infty}^{\infty} G(x-y-\beta t, t)\, dx = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{-z^2}\, dz = 1,$$

where $z=(x-y-\beta t)/\sqrt{4\alpha t}$, we prove (4.19) as

$$
\begin{aligned}
&\int_{-\infty}^{\infty} |I_{j5}(x, t)|\, dx \\
&\quad \leqslant \int_{-\infty}^{\infty} dx \int_{t/2}^{t} ds \int_{-\infty}^{\infty} G(x-y-\beta(t-s), t-s)\, |\partial_y^j F(v_y)(y, s)|\, dy \\
&\quad = \int_{t/2}^{t} \left[\int_{-\infty}^{\infty} G(x-y-\beta(t-s), t-s)\, dx \int_{-\infty}^{\infty} |\partial_y^j F(v_y)(y, s)|\, dy \right] ds
\end{aligned}
$$

$$\leqslant C\int_{t/2}^{t}(1+s)^{-(2p+1+j)/2}\,ds$$

$$\leqslant C(1+(t/2))^{-(2p+1+j)/2}\int_{t/2}^{t}ds$$

$$\leqslant Ct^{-(2p-1+j)/2} \tag{4.30}$$

for $j=0, ..., 2p-2$. Therefore, the proof of Lemma 4.1 is complete. ∎

LEMMA 4.2. *The following*

$$\|\partial_x^j v(t)\|_{L^1}\leqslant Ct^{-j/2} \tag{4.31}$$

holds for $j=0, ..., 2p-2$.

Proof. From (4.14), thanks to Lemma 4.1 and noting $2p-1+j>j$ due to $p\geqslant 1$, we obtain

$$\begin{aligned}\|\partial_x^j v(t)\|_{L^1}&\leqslant\sum_{k=1}^{5}\int_{-\infty}^{\infty}|I_{jk}(x,t)|\,dx\\&\leqslant C(t^{-j/2}+t^{-(j+1)/2}+t^{-(j+2)/2}+t^{-j/2}+t^{-(2p-1+j)/2})\\&\leqslant Ct^{-j/2}\end{aligned}$$

for $j=0, ..., 2p-2$. Thus, the proof of Lemma 4.2 is complete. ∎

Under the above analysis, now we are going to show (2.13). When $1\leqslant q<2$, Lemma 4.2 and (2.8) implies (2.13) as

$$\begin{aligned}\|\partial_x^j v(t)\|_{L^q}&=\left(\int_{-\infty}^{\infty}|\partial_x^j v(x,t)|^q\,dx\right)^{1/q}\\&\leqslant\sup_{x\in R}|\partial_x^j v(x,t)|^{(q-1)/q}\,\|\partial_x^j v(t)\|_{L^1}^{1/q}\\&\leqslant C(1+t)^{-(j+1)(q-1)/(2q)}\,t^{-j/(2q)}\\&\leqslant Ct^{-((j+1)q-1)/(2q)}\end{aligned}$$

for $j=0, ..., 2p-2$. ∎

5. REMARK

Consider the generalized BBM-Burgers equations

$$u_t-u_{xxt}-\alpha u_{xx}+\beta u_x+\phi(u)_x=0,\qquad x\in R^1,\quad t\geqslant 0, \tag{5.1}$$

with the initial data (1.2), where $\phi(u)$ is the nonlinear flux function. If $\phi(u) \in C^{p+1}$ and satisfies

$$\phi(0) = \phi'(0) = \cdots = \phi^{(p)}(0) = 0 \qquad \text{and} \qquad \phi^{(p+1)}(0) \neq 0 \tag{5.2}$$

for some positive integer $p \geqslant 1$, by Taylor's formula, we have

$$|\phi(u)| \leqslant O(1)\, |u|^{p+1}. \tag{5.3}$$

Without any difficulty, we can prove the following asymptotic behavior of the solution for (5.1) and (1.2). We state it as follows but without proof, since it can be similarly proved as Theorem 1.1.

THEOREM 5.1. *Suppose that $\phi(u)$ satisfies* (5.2), *and the initial data $u_0(x)$ satisfies $\int_{-\infty}^{\infty} u_0(x)\, dx = 0$ and $v_0(x) := \int_{-\infty}^{x} u_0(y)\, dy \in W^{2p+1,1}$. Then there exists a positive constant δ_3 such that, when $\|v_0\|_{W^{2p+1,1}} \leqslant \delta_3$, then the Cauchy problem* (5.1) *and* (1.2) *has a unique global solution $u(x, t)$ satisfying the asymptotic behaviors as follows*:

(i) *If $p \geqslant 1$, then*

$$\|\partial_x^j u(t)\|_{L^2} = O(1)(1+t)^{-(2j+3)/4} \qquad \text{for} \quad j = 0, 1, ..., 2p-1, \tag{5.4}$$

and

$$\|\partial_x^j u(t)\|_{L^q} = O(1)(1+t)^{-((j+2)q-1)/(2q)} \tag{5.5}$$

for $2 < q \leqslant \infty$ and $j = 0, 1, ..., 2p-2$;

(ii) *Furthermore, if $p \geqslant 2$, then*

$$\|\partial_x^j u(t)\|_{L^q} = O(1)\, t^{-((j+2)\, q-1)/(2q)} \tag{5.6}$$

for $1 \leqslant q < 2$, $j = 0, 1, ..., 2p-3$, and

$$\|\partial_x^j u_t(t)\|_{L^q} = O(1)(1+t)^{-((j+4)\, q-1)/(2q)} \tag{5.7}$$

for $2 \leqslant q \leqslant \infty$ and $j = 0, 1, ..., 2p-3$.

REFERENCES

1. C. J. Amick, J. L. Bona, and M. E. Schonbek, Decay of solutions of some nonlinear wave equations, *J. Differential Equations* **81** (1989), 1–49.
2. T. B. Benjamin, J. L. Bona, and J. J. Mahony, Model equations for long waves in nonlinear dispersive systems, *Philos. Trans. Roy. Soc. London Ser. A* **272** (1972), 47–78.
3. E. Bisognin, V. Bisognin, and G. Perla Menzala, Uniform stabilization and space-periodic solutions of nonlinear dispersive system, research preprint, LNCC of Brasil, No. 16, 1998.

4. V. Bisognin, On the asymptotic behaviour of the solutions of a nonlinear dispersive system of Benjamin–Bona–Mahony's type, *Boll. Un. Mat. Ital. B (7)* **10** (1996), 99–128.
5. V. Bisognin and G. Perla Menzala, Decay rates of the solutions of nonlinear dispersive equations, *Proc. Roy. Soc. Edinburgh Sect. A* **124** (1994), 1231–1246.
6. V. Bisognin and G. Perla Menzala, Asymptotic behavior in time of KdV type equations with time dependent coefficients, *Appl. Math. Lett.* **7** (1994), 85–89.
7. V. Bisognin and G. Perla Menzala, Asymptotic behaviour of nonlinear dispersive models with variable coefficients, *Ann. Mat. Pura Appl. (4)* **168** (1995), 219–235.
8. J. L. Bona and L. Luo, Decay of solutions to nonlinear, dispersive wave equations, *Differential Integral Equations* **6** (1993), 961–980.
9. J. L. Bona and L. Luo, More results on the decay of solutions to nonlinear, dispersive wave equations, *Discrete Continuous Dynam. Systems* **1** (1995), 151–193.
10. I.-L. Chern and T.-P. Liu, Convergence to diffusion waves of solutions for viscous conservation laws, *Comm. Math. Phys.* **110** (1987), 503–517.
11. D. B. Dix, The dissipation of nonlinear dispersive waves: The case of asymptotically weak nonlinearity, *Comm. Partial Differential Equations* **17** (1992), 1665–1693.
12. B. L. Guo and C. X. Miao, On inhomogenous GBBM equations, *J. Partial Differential Equations* **8** (1995), 193–204.
13. A. Matsumura, On the asymptotic behavior of solutions of semi-linear wave equation, *Publ. Res. Inst. Math. Sci.* **12** (1976), 169–189.
14. M. Mei, Long-time behavior of solution for Rosenau–Burgers equation (I), *Appl. Anal.* **63** (1996), 515–330; (II), **68** (1998), 333–356.
15. M. Mei, Large-time behavior of solution for generalized Benjamin–Bona–Mahony–Burgers equations, *Nonlinear Anal.* **33** (1998), 699–714.
16. P. I. Naumkin, Large-time asymptotic behaviour of a step for the Benjamin–Bona–Mahony–Burgers equation, *Proc. Roy. Soc. Edinburgh Sect. A* **126** (1996), 1–18.
17. P. I. Naumkin, Evolution of a step for the Benjamin–Bona–Mahony–Burgers equation, *Dokl. Akad. Nauk* **352** (1997), 742–745. [In Russian]
18. P. I. Naumkin and I. A. Shishmarv, "Nonlinear Nonlocal Equations in the Theory of Waves," Transl. Math. Monogr., Vol. 133, Amer. Math. Soc., Providence, 1994.
19. D. H. Peregrine, Calculations of the development of an undular bore, *J. Fluid Mech.* **25** (1966), 321–330.
20. I. E. Segal, Quantization and dispersion for nonlinear relativistic equations, *in* "Mathematical Theory of Elementary Particles," pp. 79–108, MIT Press, Cambridge, MA, 1966.
21. L. Zhang, Decay of solutions of generalied Benjamin–Bona–Mahony–Burgers equations, *Acta Math. Sinica (N.S.)* **10** (1994), 428–438.
22. L. Zhang, Decay of solutions of generalied Benjamin–Bona–Mahony–Burgers equations in *n*-space dimensions, *Nonlinear Anal.* **25** (1995), 1343–1369.
23. H. Zhao and B. Xuan, Existence and convergence of solutions for the generalized BBM-Burgers equations with dissipative term, *Nonlinear Anal.* **28** (1997), 1835–1849.
24. Y. Zhou and B. Guo, The large time behavior of Cauchy problems for a dissipative Benjamin–Ono type equation, *in* "Nonlinear Evolutionary Partial Differential Equations, Beijing, 1993" (X. Ding and T.-P. Liu, Eds.), pp. 91–105, AMS/IP Stud. Adv. Math., Vol. 3, Amer. Math. Soc., Providence, RI, 1997.