Math 198-141C: **Final Examination** (2001/2002)

Notice:

1. No calculators allowed.
2. No textbooks, classnotes or integral formulas allowed.
3. Show all your work.

1. (16 pts, 4 pts for each) Evaluate integrals:

a). $\displaystyle\int_0^{\pi/2} \cos^3 \pi x \; dx$; b). $\displaystyle\int x \ln x \; dx$;

c). $\displaystyle\int \frac{x+1}{x^2 - x} \; dx$; d). $\displaystyle\int \frac{x^2}{\sqrt{1-x^2}} \; dx$.

2. (8 pts, 4 pts for each) For each of the following integrals, determine whether it is convergent or divergent. If it is convergent, find its value.

a). $\displaystyle\int_0^{\infty} \sin x \; dx$; b). $\displaystyle\int_0^3 \frac{1}{(x-1)^{4/5}} \; dx$.

3. (8 pts, 4 pts for each) For each of the following sequences, determine whether it is convergent or divergent. If it is convergent, find its value.

a). $\left\{ n \sin \frac{\pi}{n} \right\}$; b). $\{2^n e^{-n}\}$.

4. (8 pts, 4 pts for each) For each of the following series, determine whether it is convergent or divergent, conditionally convergent and/or absolutely convergent.

a). $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^2}{2^n}$; b). $\displaystyle\sum_{n=1}^{\infty} (-1)^n (\sqrt{n+2} - \sqrt{n})$.

5. (10 pts) Find the area that is inside the circle $r = 3\cos\theta$ and outside the curve $r = 2 - \cos\theta$.

6. (10 pts) For the curve given parametrically by $x = 2t^3 + t^2$ and $y = 2 - t^2$, determine
   a). the equation of the tangent line at the point $(x, y) = (3, 1)$;
   b). the value of $\frac{d^2y}{dx^2}$ at the point $(x, y) = (3, 1)$.

7. (40 pts, 10 pts for each sub-question) For the arc of the parabola $y = x^2$ from $A(0,0)$ to $B(1,1)$, use the methods of the calculus to find
   a). the length of the arc; (**Hint**: $\int \sqrt{a^2 + x^2} dx = \frac{1}{2} x \sqrt{a^2 + x^2} + \frac{a^2}{2} \ln(x + \sqrt{a^2 + x^2}) + C$)
   b). the area of the region bounded by the arc, the x-axis and the line $x = 1$;
   c). the volume of the solid obtained by rotating the region specified in (b) about the x-axis;
   d). the surface area of the solid obtained in (c).
   (**Hint**: $\int x^2 \sqrt{a^2 + x^2} dx = \frac{x}{8}(a^2 + 2x^2)\sqrt{a^2 + x^2} - \frac{a^4}{8} \ln(x + \sqrt{a^2 + x^2}) + C$)

Solutions to Final Examination of Math 141C (2001-2002)

1. Solution.

a). Let $u = \sin \pi x$, then $du = \pi \cos \pi x dx$ and $u = 0$ for $x = 0$, $u = \sin \frac{\pi^2}{2}$ for $x = \frac{\pi}{2}$. By the substitution rule, we have

$$\begin{aligned}
&\int_0^{\pi/2} \cos^3 \pi x dx = \int_0^{\pi/2} \cos^2 \pi x \cos \pi x dx = \int_0^{\pi/2} (1 - \sin^2 \pi x) \cos \pi x dx \\
&= \frac{1}{\pi} \int_0^{\sin \frac{\pi^2}{2}} (1 - u^2) du = \frac{1}{\pi} \Big(u - \frac{u^3}{3} \Big) \Big]_0^{\sin \frac{\pi^2}{2}} \\
&= \frac{1}{\pi} \Big(\sin \frac{\pi^2}{2} - \frac{1}{3} \sin^3 \frac{\pi^2}{2} \Big).
\end{aligned}$$

b). We integrat it by parts to have

$$\int x \ln x dx = \frac{1}{2} x^2 \ln x - \int \frac{1}{2} x^2 \cdot \frac{1}{x} dx = \frac{1}{2} x^2 \ln x - \frac{1}{2} \int x dx = \frac{1}{2} x^2 \ln x - \frac{1}{4} x^2 + C.$$

c). Since $x^2 - x = x(x-1)$, then the integrand can be expressed in the form

$$\frac{x+1}{x^2 - x} = \frac{A}{x} + \frac{B}{x-1}$$

for some constants A and B. Multiplying both sides of the above equation by $x(x-1)$ yields

$$x + 1 = (A + B)x - A.$$

Comparing the coefficients gives

$$A + B = 0, \ -A = 1,$$

which solves $A = -1$ and $B = 2$. Thus

$$\int \frac{x+1}{x^2 - x} dx = \int \Big(\frac{-1}{x} + \frac{2}{x-1} \Big) dx = -\ln |x| + 2 \ln |x - 1| + C.$$

d). Let $x = \sin \theta$ for $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, which gives $\theta = \arcsin x$ and $dx = \cos \theta d\theta$. Then we have

$$\begin{aligned}
\int \frac{x^2}{\sqrt{1 - x^2}} dx &= \int \frac{\sin^2 \theta}{\sqrt{1 - \sin^2 \theta}} \cos \theta d\theta = \int \sin^2 \theta d\theta \\
&= \frac{1}{2} \int (1 - \cos 2\theta) d\theta = \frac{1}{2} \theta - \frac{1}{4} \sin 2\theta + C \\
&= \frac{1}{2} \arcsin x - \frac{1}{2} x \sqrt{1 - x^2} + C.
\end{aligned}$$

2. Solution.

a). Since $\cos t$ is oscillative and $\lim_{t \to \infty} \cos t$ doesn't exist, we see that

$$\int_0^{\infty} \sin x dx = \lim_{t \to \infty} \int_0^t \sin x dx = \lim_{t \to \infty} (1 - \cos t)$$

doesn't exist, too. Hence the improper integral is divergent.

b). Since 1 is the singular point of $\frac{1}{(x-1)^{4/5}}$, we then have

$$\begin{aligned}
\int_0^3 \frac{1}{(x-1)^{4/5}}dx &= \lim_{t\to 1^+}\int_t^3 \frac{1}{(x-1)^{4/5}}dx + \lim_{t\to 1^-}\int_0^t \frac{1}{(x-1)^{4/5}}dx \\
&= \lim_{t\to 1^+}\int_t^3 \frac{1}{(x-1)^{4/5}}dx + \lim_{t\to 1^-}\int_0^t \frac{1}{(1-x)^{4/5}}dx \\
&= \lim_{t\to 1^+} 5(x-1)^{1/5}]_t^3 - \lim_{t\to 1^-} 5(1-x)^{1/5}]_0^t \\
&= 5\lim_{t\to 1^+}[2^{1/5}-(t-1)^{1/5}] - 5\lim_{t\to 1^-}[(1-t)^{1/5}-1] \\
&= 5(2^{1/5}+1).
\end{aligned}$$

So, this improper integral is convergent to $5(2^{1/5}+1)$.

3. Solution.
a). Since

$$\lim_{n\to\infty} n\sin\frac{\pi}{n} = \lim_{n\to\infty}\frac{\sin\frac{\pi}{n}}{\frac{\pi}{n}}\pi = \pi\cdot 1 = \pi,$$

$\{n\sin\frac{\pi}{n}\}$ is convergent to π.
b). Since $\frac{2}{e} < 1$, we have

$$\lim_{n\to\infty} 2^n e^{-n} = \lim_{n\to\infty}\left(\frac{2}{e}\right)^n = 0.$$

So, the sequence $\{2^n e^{-n}\}$ is convergent to 0.

4. Solution. [The solutions to a) and b) are not unique. We present here only one solution to each sub-question.]
a). Since

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}\frac{(n+1)^2}{2^{n+1}}}{(-1)^n\frac{n^2}{2^n}}\right| = \lim_{n\to\infty}\frac{(n+1)^2}{2n^2} = \frac{1}{2}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^2 = \frac{1}{2},$$

by the Ratio Test, we prove that it is absolutely convergent.
b). Since

$$a_n := \sqrt{n+2}-\sqrt{n} = \frac{(\sqrt{n+2}-\sqrt{n})(\sqrt{n+2}+\sqrt{n})}{\sqrt{n+2}+\sqrt{n}} = \frac{2}{\sqrt{n+2}+\sqrt{n}} > 0,$$

the series $\sum_{n=1}^{\infty}(-1)^n(\sqrt{n+2}-\sqrt{n})$ is the alternating seires. Furthermore, noticing that

$$a_{n+1} = \frac{2}{\sqrt{n+3}+\sqrt{n+1}} < \frac{2}{\sqrt{n+2}+\sqrt{n}} = a_n$$

and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2}{\sqrt{n+2}+\sqrt{n}} = 0,$$

we apply the Alternating Series Test to obtain the convergence of $\sum_{n=1}^{\infty}(-1)^n(\sqrt{n+2}-\sqrt{n})$.

On the other hand, let $b_n = \frac{1}{\sqrt{n}}$, since

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2}{\sqrt{n+2}+\sqrt{n}} \Big/ \frac{1}{\sqrt{n}} = \lim_{n\to\infty} \frac{2}{\sqrt{1+\frac{2}{n}}+1} = 1$$

and $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is divergent (p-series with $p = \frac{1}{2}$), by the Comparison Limit Test, we prove that $\sum_{n=1}^{\infty} |(-1)^n(\sqrt{n+2}-\sqrt{n})| = \sum_{n=1}^{\infty} \frac{2}{\sqrt{n+2}+\sqrt{n}}$ is divergent. Hence, $\sum_{n=1}^{\infty}(-1)^n(\sqrt{n+2}-\sqrt{n})$ is conditionally convergent.

5. Solution. Notice that

$$3\cos\theta = 2 - \cos\theta$$

gives $\theta = \pm\frac{\pi}{3}$. So, the inserction points are $(\frac{3}{2}, \frac{\pi}{3})$ and $(\frac{3}{2}, -\frac{\pi}{3})$. The area of the region inside of $r = 3\cos\theta$ and outside $r = 2 - \cos\theta$ is

$$\begin{aligned} A &= \int_{-\pi/3}^{\pi/3} \frac{1}{2}[(3\cos\theta)^2 - (2-\cos\theta)^2]d\theta \\ &= 2\int_{-\pi/3}^{\pi/3} [2\cos^2\theta - 1 + \cos\theta]d\theta \\ &= 2\int_{-\pi/3}^{\pi/3} [\cos 2\theta + \cos\theta]d\theta \\ &= 2\Big(\frac{\sin 2\theta}{2} + \sin\theta\Big)\Big|_{-\pi/3}^{\pi/3} = 3\sqrt{3}. \end{aligned}$$

6. Solution.

a). Since

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{(2-t^2)'}{(2t^3+t^2)'} = -\frac{1}{3t+1},$$

so, the tangent at the point (3,1), i.e., $t = 1$, is

$$\frac{dy}{dx}\Big|_{(3,1)} = -\frac{1}{3t+1}\Big|_{t=1} = -\frac{1}{4}.$$

The tangent line is

$$\frac{y-1}{x-3} = -\frac{1}{4}, \quad i.e., \quad y = -\frac{1}{4}x + \frac{7}{4}.$$

b). Since

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}(\frac{dy}{dt})}{\frac{dx}{dt}} = \frac{(-\frac{1}{3t+1})'}{(2t^3+t^2)'} = \frac{3}{(3t+1)^2(6t^2+2t)},$$

we have

$$\left.\frac{d^2y}{dx^2}\right|_{(3,1)} = \left.\frac{3}{(3t+1)^2(6t^2+2t)}\right|_{t=1} = \frac{3}{128}.$$

7. Solution.

a). Since $y = x^2$, i.e., $\frac{dy}{dx} = 2x$, we adopt the given integral formula to have

$$\begin{aligned} L &= \int_0^1 \sqrt{1+\left(\frac{dy}{dx}\right)^2}dx = \int_0^1 \sqrt{1+4x^2}dx \\ &= \frac{1}{2}\int_0^2 \sqrt{1+u^2}du \qquad \text{(substitute } u = 2x) \\ &= \frac{1}{4}(u\sqrt{1+u^2} + \ln|u+\sqrt{1+u^2}|)]_0^2 \\ &= \frac{\sqrt{5}}{2} + \frac{1}{4}\ln(2+\sqrt{5}) \end{aligned}$$

b).

$$A = \int_0^1 x^2dx = \frac{1}{3}.$$

c).

$$V = \int_0^1 \pi(x^2)^2dx = \pi\int_0^1 x^4dx = \frac{\pi}{5}.$$

d).

$$\begin{aligned} S &= \int_0^1 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx = 2\pi\int_0^1 x^2\sqrt{1+4x^2}dx \\ &= \frac{\pi}{4}\int_0^2 u^2\sqrt{1+u^2}du \qquad \text{(substitute } u = 2x) \\ &= \frac{\pi}{4}\left[\frac{u}{8}(1+2u^2)\sqrt{1+u^2} - \frac{1}{8}\ln(u+\sqrt{1+u^2})\right]_0^2 \\ &= \frac{18\sqrt{5}-\ln(2+\sqrt{5})}{32}\pi. \end{aligned}$$