Champlain College – St.-Lambert

MATH 201-203: Calculus II

Review Questions for Test # 1

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Questions

1. Let $f(x)$ be
$$f(x) = \begin{cases} \sqrt{1-x^2}, & \text{if } -1 \le x \le 1, \\ 1-x, & \text{if } 1 < x \le 2. \end{cases}$$
   (a) Sketch the graph of $f(x)$.
   (b) Evaluate $\int_{-1}^{2} f(x)dx$ by interpreting it in terms of area.

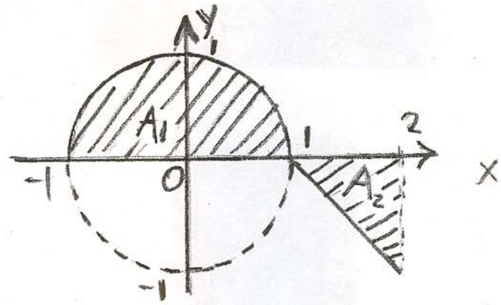
2. Find integrals
   (a) $\int \frac{1}{x\sqrt{\ln x}}\, dx$
   (b) $\int \frac{x^2+x}{x+1}\, dx$
   (c) $\int \frac{x}{e^{x^2+3}}\, dx$
   (d) $\int (2x+1)\sqrt[3]{x^2+x}\, dx$

3. The quantity demanded x (in units of a hundred) of the Mikado miniature cameras/week is related to the unit price p (in dollars) by $p = -0.2x^2 + 80$, and the quantity x (in units of a hundred) that the supplier is willing to make available in the market is related to the unit price p (in dollars) by $p = 0.1x^2 + x + 40$. If the market price is set at the equilibrium price, find the consumers' surplus and the producers' surplus.

4. Let A be a region completely enclosed by two curves $y = x^2$ and $y = 2x - x^2$, and V_1 and V_2 be the solids obtained by rotating the region A about the x-axis and the line $y = 1$, respectively.
   (a) Find the area of A;
   (b) Find the volume of V_1;
   (c) Find the volume of V_2.

Solutions to Review Questions

1(a).



1(b).

$$\int_{-1}^{2} f(x)dx = A_1 - A_2 = \frac{1}{2}\cdot\pi\cdot 1^2 - \frac{1}{2}\cdot 1\cdot 1 = \frac{1}{2}(\pi - 1).$$

2(a). Substitute $u = \ln x$, then $du = \frac{1}{x}dx$. Thus, we have

$$\int \frac{1}{x\sqrt{\ln x}}dx = \int \frac{1}{\sqrt{u}}du = \int u^{-1/2}du = 2u^{1/2} + C = 2\sqrt{\ln x} + C.$$

2(b).

$$\int \frac{x^2+x}{x+1}dx = \int \frac{x(x+1)}{x+1}dx = \int xdx = \frac{x^2}{2} + C.$$

2(c).

$$\begin{aligned}
\int \frac{x}{e^{x^2+3}}dx &= \int \frac{1}{e^u}\frac{du}{2} = \frac{1}{2}\int e^{-u}du & &[\text{ substitute: } u = x^2+3, du = 2dx] \\
&= \frac{1}{2}\int e^v(-dv) & &[\text{ substitute: } v = -u, dv = -du] \\
&= -\frac{1}{2}\int e^v dv = -\frac{1}{2}e^v + C \\
&= -\frac{1}{2}e^{-u} + C = -\frac{1}{2}e^{-x^2-3} + C.
\end{aligned}$$

2(d). Substitute $u = x^2 + x$, then $du = (2x+1)dx$. Thus, we have

$$\int (2x+1)\sqrt[3]{x^2+x}dx = \int \sqrt[3]{u}\, du = \int u^{1/3}\, du = \frac{3u^{4/3}}{4} + C = \frac{3(x^2+x)^{4/3}}{4} + C.$$

3. The demanded function is $p = -0.2x^2 + 80 =: D(x)$, and the supply function $p = 0.1x^2 + x + 40 =: S(x)$. The equilibrium price $\bar{p}$ and the equilibrium quantity $\bar{x}$ satisfy the following system of equations

$$\begin{cases} p = -0.2x^2 + 80, \\ p = 0.1x^2 + x + 40, \end{cases}$$

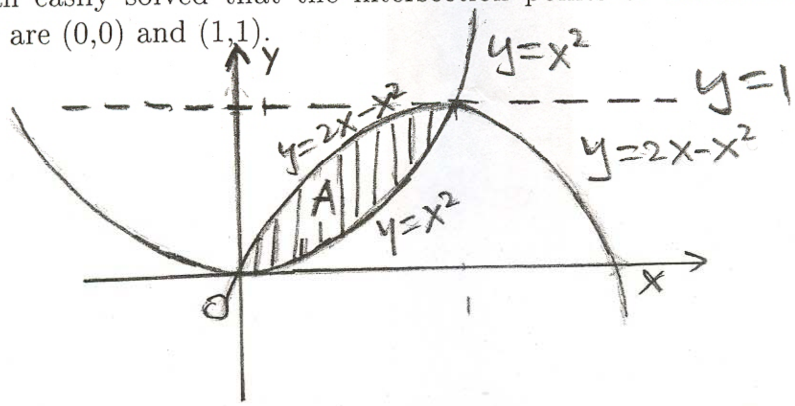
which can solved as $(\bar{x}, \bar{p}) = (10, 60)$. So, the consumers' surplus is

$$\begin{aligned} CS &= \int_0^{\bar{x}} [D(x) - \bar{p}]dx = \int_0^{10} [(-0.2x^2 + 80) - 60]dx \\ &= \int_0^{10} [-0.2x^2 + 20]dx = [-0.2\frac{x^3}{3} + 20x]\Big|_0^{10} = \frac{400}{3} \text{ dollars,} \end{aligned}$$

and the producers' surplus is

$$\begin{aligned} PS &= \int_0^{\bar{x}} [\bar{p} - S(x)]dx = \int_0^{10} [60 - (0.1x^2 + x + 40)]dx \\ &= \int_0^{10} [20 - 0.1x^2 - x]dx = [20x - 0.1\frac{x^3}{3} - \frac{x^2}{2}]\Big|_0^{10} = \frac{350}{3} \text{ dollars.} \end{aligned}$$

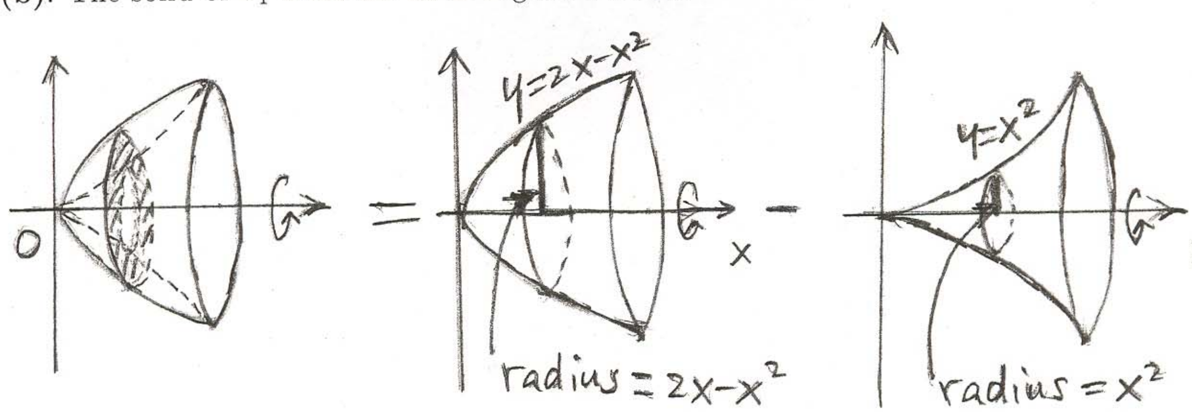
4(a). It can easily solved that the intersection points of the two curves $y = x^2$ and $y = 2x - x^2$ are (0,0) and (1,1).



The enclosed region A is shown in the above Figure 1, and its area can be calculated as

$$A = \int_a^b [Y_{top} - Y_{bottom}]dx = \int_0^1 [(2x - x^2) - x^2]dx = \int_0^1 [2x - 2x^2]dx = \left[x^2 - \frac{2x^3}{3}\right]\Big|_0^1 = \frac{1}{3}.$$

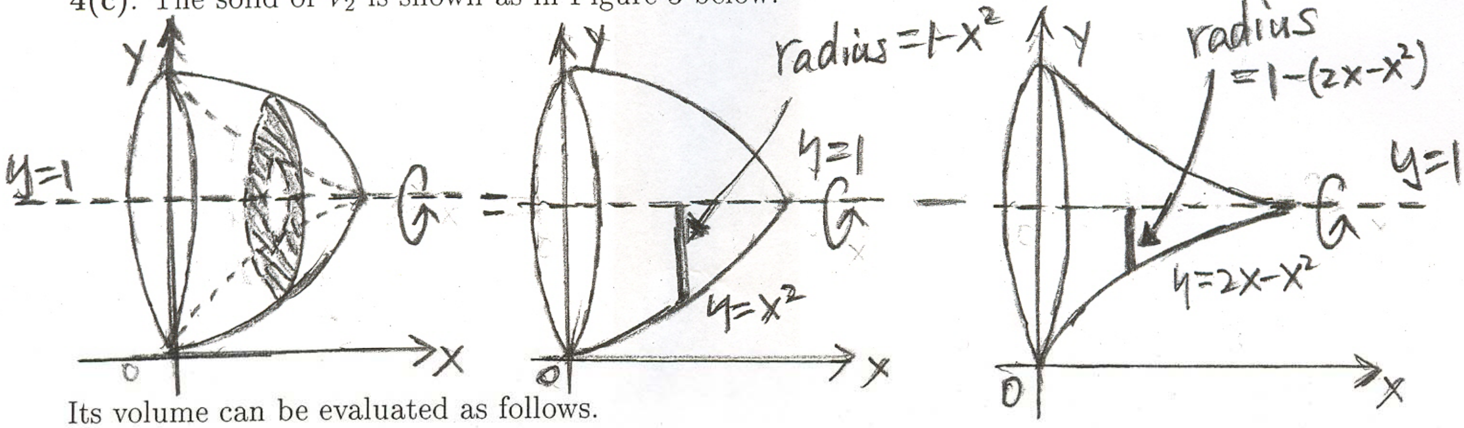
4(b). The solid of V_1 is shown as in Figure 2 below.



Its volume can be evaluated as follows.

$$\begin{aligned} V_1 &= \pi\int_0^1 (2x-x^2)^2dx - \pi\int_0^1 (x^2)^2dx \\ &= \pi\int_0^1 (4x^2-4x^3+x^4)dx - \pi\int_0^1 x^4dx \\ &= \pi\int_0^1 (4x^2-4x^3)dx = \pi\Big(\frac{4x^3}{3}-x^4\Big)\Big|_0^1 = \frac{\pi}{3}. \end{aligned}$$

4(c). The solid of V_2 is shown as in Figure 3 below.



Its volume can be evaluated as follows.

$$\begin{aligned} V_2 &= \pi\int_0^1 (1-x^2)^2dx - \pi\int_0^1 [1-(2x-x^2)]^2dx \\ &= \pi\int_0^1 (1-2x^2+x^4)dx - \pi\int_0^1 [1-4x+6x^2-4x^3+x^4]dx \\ &= \pi\int_0^1 (4x-8x^2+4x^3)dx = \pi\Big(2x^2-\frac{8x^3}{3}+x^4\Big)\Big|_0^1 = \frac{\pi}{3}. \end{aligned}$$