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Uniqueness and stability of traveling waves for cellular neural networks with multiple delays

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Abstract

In this paper, we investigate the properties of traveling waves to a class of lattice differential equations for cellular neural networks with multiple delays. Following the previous study [38] on the existence of the traveling waves, here we focus on the uniqueness and the stability of these traveling waves. First of all, by establishing the *a priori* asymptotic behavior of traveling waves and applying Ikehara's theorem, we prove the uniqueness (up to translation) of traveling waves $\phi(n-ct)$ with $c \le c_*$ for the cellular neural networks with multiple delays, where $c_* < 0$ is the critical wave speed. Then, by the weighted energy method together with the squeezing technique, we further show the global stability of all non-critical traveling waves for this model, that is, for all monotone waves with the speed $c < c_*$, the original lattice solutions converge time-exponentially to the corresponding traveling waves, when the initial perturbations around the monotone traveling waves decay exponentially at far fields, but can be arbitrarily large in other locations.

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1. Introduction

Cellular neural networks (CNN) were first proposed by Chua and Yang [8,9] as an achievable alternative to fully-connected neural networks in electric circuit systems. Since then, the study of cellular neural networks has been one of hot research topics due to their many significant applications to a broad scope of problems arising from, for example, image and video signal processing, robotic and biological visions, and higher brain functions [7–9,29]. The infinite system of the ordinary differential equations for the one-dimensional CNN with a neighborhood of radius m but without inputs is of the form

$$x_n'(t) = -x_n(t) + z + \sum_{i=1}^{m} a_i f(x_{n-i}(t)) + \alpha f(x_n(t)) + \sum_{i=1}^{m} \beta_i f(x_{n+i}(t)), \tag{1.1}$$

for $n \in \mathbb{Z}$, $m \in \mathbb{N}$. Here, $x_n(t)$ denotes the state function of cell C_n at time t. The quantity z is called a threshold or bias term and is related to independent voltage sources in electric circuits. The nonnegative constant coefficients a_i, α and β_i of the output function f constitute the so-called space-invariant template that measure the synaptic weights of self-feedback and neighborhood interactions. When the cells are taken account of the instantaneous self-feedback and neighborhood interaction with distributed delays, because of the finite switching speed of signal transmission, the dynamic system can be presented by the following nonlocal lattice differential equation with multi-delays [32,38]

$$\begin{aligned} x_n'(t) = -x_n(t) &+ \sum_{i=1}^{m} a_i \int_0^{\tau} J_i(y) f(x_{n-i}(t-y))dy + \alpha \int_0^{\tau} J_{m+1}(y) f(x_n(t-y))dy \\ &+ \sum_{j=1}^{l} \beta_j \int_0^{\tau} J_{m+1+j}(y) f(x_{n+j}(t-y))dy \end{aligned} \tag{1.2}$$

for $n \in \mathbb{Z}$, $m, l \in \mathbb{N}$, where $J_i : [0, \tau] \to [0, \infty)$ is the density function for delay effect of the neighbors. Particularly, if the kernels are taken as some delta-functions $J_i = \delta(y - \tau_i)$, $i = 1, 2, \cdots, m + l + 1$, where $\tau_i > 0$ are the time-delays, then the equation (1.2) is reduced to the following multiple time-delayed lattice differential equation for the cellular neural networks [12–16,19,30]

$$\begin{aligned} x_n'(t) = -x_n(t) &+ \sum_{i=1}^{m} a_i f(x_{n-i}(t-\tau_i)) + \alpha f(x_n(t-\tau_{m+1})) \\ &+ \sum_{j=1}^{l} \beta_j f(x_{n+j}(t-\tau_{m+1+j})), \end{aligned} \tag{1.3}$$

subjected to the initial data

$$x_n(s) = x_n^0(s), \quad s \in [-r, 0], \quad r = \max_{1 \le i \le m+1+l} \{\tau_i\}. \tag{1.4}$$

This will be the targeted equation concerned in the present paper. Here, the nonlinear output function f is assumed to satisfy:

(H1) Equation (1.3) possesses two constant equilibria: 0 and $K > 0$, such that $f(0) = 0$ and $(a+\alpha+\beta)f(K) = K$, where $a := \sum_{i=1}^{m} a_i \geq 0$, $\alpha \geq 0$ and $\beta := \sum_{j=1}^{l} \beta_j > 0$;

(H2) $f \in C([0, K], [0, K])$ is non-decreasing with restrictions of $(\alpha+\beta)f'(0) > 1$, $(a+\alpha+\beta)f(u) > u$ for $u \in (0, K)$, and $|f(u) - f(v)| \leq f'(0)|u - v|$ for $u, v \in [0, K]$;

(H3) There exists $\sigma \in (0, 1]$ such that

$$\limsup_{u\to 0^+}[f'(0) - f(u)/u]u^{-\sigma} < +\infty. \tag{1.5}$$

(H4) $f''(u) \leq 0$ for $u \in (0, K)$ and $f'(K)(\alpha + a + \beta) < 1$.

Remark 1.1.

i) In (H2), $(\alpha+\beta)f'(0) > 1$ means that $(a+\alpha+\beta)f'(0) > 1$ and 0 is an unstable node for the linearized equation around 0 of the following homogeneous equation

$$x'(t) = -x(t) + (a+\alpha+\beta)f(x(t)). \tag{1.6}$$

On the other hand, $f(0) = 0$ and $|f(u) - f(v)| \leq f'(0)|u - v|$ for $u, v \in [0, K]$ immediately indicates $f'(0)u \geq f(u)$ for $u \in [0, K]$;

ii) In (H3), the condition (1.5) implies that there exist $d > 0$ and $\gamma > 0$ such that $f(u) \geq f'(0)u - du^{\sigma+1}$ for $u \in [0, \gamma]$;

iii) In (H4), the condition $f'(K)(\alpha + a + \beta) < 1$ is equivalent to that the equilibrium K is a stable node for the linearized equation of (1.6) around K.

The typical example of the output function to satisfy the conditions (H1)–(H3) is Nicholson's function

$$f(u) = pue^{-u} \quad \text{for } 1 < (a+\alpha+\beta)p < e.$$

In this case, $K = \ln[(a+\alpha+\beta)p] > 0$ and $\sigma = 1$.

In the study of CNN and delayed CNN (simply denoted by DCNN), traveling waves in the form of $\phi(n - ct)$ play a crucial role in understanding the fundamental structure of the dynamical systems, where c is the wave speed. Let $x_n(t) = \phi(n - ct) =: \phi(\xi)$ be the solution of (1.3) connecting the constant equilibria 0 with K at far fields, then

$$\begin{cases} -c\phi'(\xi) = -\phi(\xi) + \displaystyle\sum_{i=1}^{m} a_i f(\phi(\xi - i + c\tau_i)) + \alpha f(\phi(\xi + c\tau_{m+1})) \\ \qquad\qquad + \displaystyle\sum_{j=1}^{l} \beta_j f(\phi(\xi + j + c\tau_{m+1+j})), \\ \phi(-\infty) = 0, \quad \phi(+\infty) = K. \end{cases} \tag{1.7}$$

The existence of the monotone traveling waves $\phi(n-ct)$ connecting the constant equilibria 0 with K for the equation (1.3) has been proved by the method of upper and lower solutions [12–16,19,30], and by the Schauder's fixed point theorem [38], respectively. Here, we just need the first three conditions (H1)–(H3) for the existence of traveling waves. That is,

Proposition 1.1. *(See [38].) Assume that* (H1)–(H3) *hold. Then there exists* $c_* < 0$ *such that for any* $c \le c_*$, (1.3) *admits a non-decreasing positive traveling wave solution* $\phi(n-ct)$ *satisfying* (1.7).

Remark 1.2. The positivity of the traveling wave solution can be directly obtained by the construction of upper–lower solutions. Indeed, according to the above arguments in [38], $\underline{\phi}(\xi) \le \phi(\xi) \le \overline{\phi}(\xi)$, where $\overline{\phi}(\xi) := \min\{K, e^{\lambda_1 \xi}\}$, $\underline{\phi}(\xi) := \max\{0, e^{\lambda_1 \xi} - q e^{\eta \lambda_1 \xi}\}$. Note that there exists $B < 0$ such that $\underline{\phi}(\xi) > 0$ for $\xi < B$. Thus, $\phi(\xi) \ge \underline{\phi}(\xi) > 0$ for $\xi < B$, and the monotonicity of ϕ implies that ϕ is positive.

As a continuity of the previous study [38], it is very natural and interesting to investigate the uniqueness of the traveling waves (up to translation) and their asymptotic stabilities. This will be the main purpose of the present paper.

First of all, inspired by the approach developed by Carr and Chmaj [1] for integro-differential equation without time-delays, by establishing the *a priori* analysis of the asymptotic behavior of the traveling waves and applying Ikehara's theorem, we will be able to show the uniqueness of all traveling waves for $c \le c_*$. This method (see also the application in [35,36] for the non-monotone delayed systems on Lattices) is different from the traditional ones used in [3–5,10,11,20,23] for the proof of uniqueness of traveling waves to the other types of evolution systems. Our first main result is as follows.

Theorem 1.1 *(Uniqueness). Assume that* (H1)–(H3) *hold. Let* $\phi(n-ct)$ *be a traveling wave of* (1.3) *with the wave speed* $c \le c_*$*, which is given in* Proposition 1.1. *If* $\psi(n-ct)$ *is any positive traveling wave of* (1.3) *with the same wave speed c satisfying* (1.7)*, then* ϕ *is a translation of* ψ*; more precisely, there exists* $\bar{\xi} \in \mathbb{R}$ *such that* $\phi(n-ct) = \psi(n-ct+\bar{\xi})$.

As a direct result of Proposition 1.1 and Theorem 1.1, we have

Corollary 1.1 *(Monotonicity). Assume that* (H1)–(H3) *hold. If* $\psi(n-ct)$ *is any positive traveling wave of* (1.3) *with the same wave speed* $c \le c_*$ *and satisfying* (1.7)*, then* $\psi(\xi)$ *is non-decreasing on* $\xi \in \mathbb{R}$*, where* $\xi = n - ct$.

Remark 1.3. For the uniqueness of the monotone traveling waves (up to translation) presented in Theorem 1.1 and Corollary 1.1, we need that f satisfies the conditions (H1)–(H3) only. But the equation (1.3) may be non-monotone for $u \in [0, K]$ if f is non-monotone, see [39]. Consequently, the equation (1.3) may not hold the comparison principle. This means, we can still have the uniqueness of the traveling waves for the non-monotone equation (1.3) by the technique of *a priori* analysis on the behavior of the traveling waves at far fields with the help of Ikehara's theorem. This is essentially different from the other studies [3–5,10,11,20,23] where the comparison principle plays a key role in the uniqueness proof.

Our next goal is to show the stability of the traveling waves with $c < c_*$. By using the technical weighted-energy method developed recently in [2,18,24–27,17,28,33,34] for the time-delayed reaction–diffusion equations in the continuous forms, we will prove that, for all monotone traveling waves with the wave speed $c < c_*$, the original lattice solutions converge time-exponentially to the corresponding traveling waves, when the initial perturbations around the monotone traveling waves decay exponentially at far fields, but can be arbitrarily large in other locations. The exponential convergence rate will be also derived. Note that, for the discrete reaction–diffusion equations with time-delay, the stability of the traveling waves was also shown in [6,37] for the faster traveling waves with the speed $|c| \gg |c_*|$. As we know, when the weighted functions were chosen as the piecewise continuous functions, just like the original studies [24–26], it always causes us to take $|c| \gg 1$ in order to control some bad terms in establishing the basic energy estimates. For such a reason, this makes the stability open for the case when the wave speed c is close to c_*. To overcome this shortage, inspired by [18], by choosing the weight function as the optimal exponential function, and applying the weighted energy method and the Gronwall inequality plus the squeezing technique, we can further prove the global stability for all non-critical traveling waves. Here the wave speed c can be arbitrarily close to the critical wave speed c_*. For other studies on the stability of traveling waves for delayed lattice reaction–diffusion equations with bistable and monostable nonlinearities, we refer to Ma and Zhao [21] and Ma and Zou [22,23] by the method of upper and lower solutions.

Before stating our second main result, let us introduce the following notation. Throughout this paper, l_v^2 denotes the weighted l^2-space with a weighted function $0 < v(\xi) \in C(\mathbb{R})$, that is,

$$l_v^2 := \left\{ \zeta = \{\zeta_i\}_{i\in\mathbb{Z}}, \zeta_i \in \mathbb{R} \,\middle|\, \sum_i v(\xi_i)\zeta_i^2 < \infty \right\}$$

and its norm is defined by

$$\|\zeta\|_{l_v^2} = \left(\sum_i v(\xi_i)\zeta_i^2 \right)^{\frac{1}{2}}, \quad \text{for } \zeta \in l_v^2.$$

In particular, when $v \equiv 1$, we denote l_v^2 by l^2. We define a weighted function $v(\xi)$ by

$$v(\xi) = e^{-2\lambda(\xi-\xi_0)}, \ \lambda \in (\lambda_1, \lambda_2), \ \xi_0 \gg 1, \tag{1.8}$$

where λ_1, λ_2 are given in Lemma 2.1. Our second main result on the stability of the traveling waves is as follows.

Theorem 1.2 *(Stability). Assume that* (H1)–(H4) *hold and let $\phi(n - ct)$ be a traveling wave with $c < c_*$ and satisfy* (1.7). *If the initial data satisfy*

$$0 \le w_n^0(s) \le K \quad \textit{for } s \in [-r, 0], \ n \in \mathbb{Z}, \tag{1.9}$$

where $r = \max_{1\le i\le m+1+l}\{\tau_i\}$ and the initial perturbation $w_n^0(s) - \phi(n - cs)$ is in $C([-r, 0], l_v^2)$, where v is the weighted function given in (1.8), *then Eq.* (1.3) *with the initial data* (1.9) *admits a* *unique solution* $\{w_n(t)\}_{n\in\mathbb{Z}}$ *such that*

$$0 \le w_n(t) \le K, \quad \text{for } t \in [0, +\infty),\ n \in \mathbb{Z}$$

and

$$\{w_n(t) - \phi(n - ct)\}_{n\in\mathbb{Z}} \in C([0, \infty), l_v^2).$$

In particular, the solution $\{w_n(t)\}_{n\in\mathbb{Z}}$ *converges to the traveling wave* $\phi(n - ct)$ *exponentially in time* t*, that is,*

$$\sup_{n\in\mathbb{Z}} |w_n(t) - \phi(n - ct)| \le Ce^{-\mu t},\ t \ge 0,$$

for some positive constants C *and* μ*.*

Remark 1.4.

i) Unfortunately, the stability for the critical wave with speed $c = c_*$ for the lattice equation (1.3) cannot be solved at the current stage, due to some technical difficulties in the *a priori* estimates. We have to leave it for the future work.
ii) The stability of the traveling waves presented in Theorem 1.2 is global in the weighted space l_v^2, because the initial perturbation around the wavefront can be arbitrarily large. The monotone conditions (H2) and (H4) play the key role in the proof of global stability, because the comparison principle holds for the equation (1.3). If the equation losses its comparison principle, by the same approach (the weighted energy method), we may still obtain the local stability by taking the initial perturbation small enough. This is the advantage for the energy method working out for the non-monotone equations. While, the monotone technique and the upper–lower solution methods adopted in [3,11,20,23] usually requests the monotonicity of the working equations for the proof about the stability of the traveling waves.
iii) When we use the l^2-weighted energy method, as showed in [26], the nonlocal terms (integral terms) will cause us some troubles in the l^2-weighted energy estimates. In order to treat these nonlocal terms properly, we usually need to take $|c| \gg |c_*|$. So, the technique developed in the present paper for treating the local equation (1.3) cannot perfectly work out for the nonlocal equation (1.2), and we need a different strategy. This will be another target in future.

The rest of this paper is organized as follows. In Section 2, we prove the uniqueness of traveling waves for DCNNs. Section 3 is devoted to proving the stability result.

2. Uniqueness of traveling waves

In order to prove the uniqueness of traveling waves for (1.1), we need to investigate asymptotic behavior of any traveling wave.

2.1. Asymptotic behavior of traveling waves

In this subsection, we show the asymptotic behavior of traveling waves of (1.3) for $c \le c_*$ with the help of Ikehara's theorem.

Define the characteristic equation

$$\Delta(c,\lambda) = -c\lambda + 1 - f'(0)\left[\sum_{i=1}^{m} a_i e^{\lambda(-i+c\tau_i)} + \alpha e^{\lambda c\tau_{m+1}} + \sum_{j=1}^{l} \beta_j e^{\lambda(j+c\tau_{m+1+j})}\right]. \quad (2.1)$$

According to Lemma 2.1 in [38], we have

Lemma 2.1. *Assume that $\beta > 0$ and $(\alpha+\beta)f'(0) > 1$ hold. Then, there exist a unique pair of $c_* < 0$ and $\lambda_* > 0$ such that the following assertions hold.*

(i) $\Delta(c_*, \lambda_*) = 0$, $\frac{\partial \Delta(c,\lambda)}{\partial \lambda}|_{c=c_*,\lambda=\lambda_*} = 0$;
(ii) *For any $c > c_*$ and $\lambda \in [0, +\infty)$, $\Delta(c,\lambda) < 0$;*
(iii) *For any $c < c_*$, $\Delta(c,\lambda) = 0$ has two positive roots $\lambda_1 \geq \lambda_2 > 0$. Moreover, if $c < c_*$, $\lambda_2 > \lambda_1$ and $\Delta(c,\lambda) > 0$ for any $\lambda \in (\lambda_1, \lambda_2)$; if $c = c_*$, then $\lambda_1 = \lambda_2 = \lambda_*$.*

We recall a version of Ikehara's Theorem.

Lemma 2.2. *(See [1], Proposition 2.3.) Let $F(\lambda) = \int_0^{+\infty} u(x)e^{-\lambda x}dx$, with u being a positive decreasing function. Assume that $F(\lambda)$ has the representation*

$$F(\lambda) = \frac{h(\lambda)}{(\lambda+\alpha)^{k+1}},$$

where $k > -1$ and h is analytic in the strip $-\alpha \leq \operatorname{Re}\lambda < 0$. Then

$$\lim_{x\to+\infty} \frac{u(x)}{x^k e^{-\alpha x}} = \frac{h(-\alpha)}{\Gamma(\alpha+1)} > 0.$$

We now state asymptotic behaviors of positive traveling waves for (1.3).

Lemma 2.3. *Assume that* (H1)–(H3) *hold and let $\psi(n-ct)$ be a positive traveling wave of* (1.3) *with $c \leq c_*$ and satisfy* (1.7). *Then there exists $\rho > 0$ such that $\psi(\xi) = O(e^{-\rho\xi})$ as $\xi \to -\infty$.*

Proof. Since $f'(0)(a+\alpha+\beta) > 1$, there exists $\epsilon_0 > 0$ such that

$$A := (1-\epsilon_0)f'(0)(a+\alpha+\beta) - 1 > 0.$$

For such $\epsilon_0 > 0$, there exists $\delta_1 > 0$ such that $f(u) \geq (1-\epsilon_0)f'(0)u$ for any $u \in [0,\delta_1]$. Since $\psi(-\infty) = 0$, there exists $M > 0$ such that $\psi(\xi) < \delta_1$ for any $\xi \leq -M$. Integrating (1.7) from η to ξ with $\xi \leq -l - M$, it follows that

$$\begin{aligned}
-c[\psi(\xi) - \psi(\eta)] = &-\int_\eta^\xi \psi(y)dy + \sum_{i=1}^m a_i \int_\eta^\xi f(\psi(y-i+c\tau_i))dy \\
&+ \alpha \int_\eta^\xi f(\psi(y+c\tau_{m+1}))dy + \sum_{j=1}^l \beta_j \int_\eta^\xi f(\psi(y+j+c\tau_{m+1+j}))dy
\end{aligned}$$

$$\begin{aligned}
&\geq -\int_\eta^\xi \psi(y)dy + f'(0)(1-\epsilon_0)\Big[\sum_{i=1}^m a_i \int_\eta^\xi \psi(y-i+c\tau_i)dy \\
&\quad + \alpha\int_\eta^\xi \psi(y+c\tau_{m+1})dy + \sum_{j=1}^l \beta_j \int_\eta^\xi \psi(y+j+c\tau_{m+1+j})dy\Big] \\
&= A\int_\eta^\xi \psi(y)dy + f'(0)(1-\epsilon_0)\Big[\sum_{i=1}^m a_i \int_\eta^\xi (\psi(y-i+c\tau_i)-\psi(y))dy \\
&\quad + \alpha\int_\eta^\xi (\psi(y+c\tau_{m+1})-\psi(y))dy \\
&\quad + \sum_{j=1}^l \beta_j \int_\eta^\xi (\psi(y+j+c\tau_{m+1+j})-\psi(y))dy\Big]. \qquad (2.2)
\end{aligned}$$

Since $\psi(\xi)$ is differentiable, we have

$$\begin{aligned}
\int_\eta^\xi (\psi(y-i+c\tau_i)-\psi(y))dy &= \int_\eta^\xi \left(\int_0^{-i+c\tau_i} \psi'(y+x)dx\right)dy \\
&= \int_0^{-i+c\tau_i} (\psi(\xi+x)-\psi(\eta+x))dx.
\end{aligned}$$

Similarly, it follows that

$$\int_\eta^\xi (\psi(y+c\tau_{m+1})-\psi(y))dy = \int_0^{c\tau_{m+1}} (\psi(\xi+x)-\psi(\eta+x))dx$$

and

$$\int_\eta^\xi (\psi(y+j+c\tau_{m+1+j})-\psi(y))dy = \int_0^{j+c\tau_{m+1+j}} (\psi(\xi+x)-\psi(\eta+x))dx.$$

Letting $\eta\to-\infty$ in (2.2), we obtain

$$\begin{aligned}
A\int_{-\infty}^\xi \psi(y)dy \leq -c\psi(\xi) - f'(0)(1-\epsilon_0)\Big[\sum_{i=1}^m a_i \int_0^{-i+c\tau_i} \psi(\xi+x)dx \\
+ \alpha\int_0^{c\tau_{m+1}} \psi(\xi+x)dx + \sum_{j=1}^l \beta_j \int_0^{j+c\tau_{m+1+j}} \psi(\xi+x)dx\Big]. \qquad (2.3)
\end{aligned}$$

It then follows from (2.3) that $\int_{-\infty}^{\xi}\psi(y)dy<+\infty$. Letting $\Phi(\xi)=\int_{-\infty}^{\xi}\psi(y)dy$ and integrating (2.3) from $-\infty$ to ξ, we have

$$
\begin{aligned}
A\int_{-\infty}^{\xi}\Phi(y)dy \le & -c\Phi(\xi)-f'(0)(1-\epsilon_0)\Big[\sum_{i=1}^{m}a_i\int_0^{-i+c\tau_i}\Phi(\xi+x)dx \\
& +\alpha\int_0^{c\tau_{m+1}}\Phi(\xi+x)dx+\sum_{j=1}^{l}\beta_j\int_0^{j+c\tau_{m+1+j}}\Phi(\xi+x)dx\Big] \\
\le & \,\varrho\Phi(\xi+\kappa)
\end{aligned}
\tag{2.4}
$$

for some $\kappa>0$ and $\varrho>0$ according to the monotonicity of $\Phi(\xi)$. Letting $\varpi>0$ such that $\varrho<A\varpi$, and for $\xi\le -l-M$, it then follows that

$$
\Phi(\xi-\varpi)\le\frac{1}{\varpi}\int_{\xi-\varpi}^{\xi}\Phi(y)dy\le\frac{1}{\varpi}\int_{-\infty}^{\xi}\Phi(y)dy\le\frac{\varrho}{A\varpi}\Phi(\xi+\kappa).
\tag{2.5}
$$

Define $h(\xi)=\Phi(\xi)e^{-\rho\xi}$, where $\rho=\frac{1}{\kappa+\varpi}\ln\frac{A\varpi}{\varrho}>0$. Hence,

$$
h(\xi-\varpi)=\Phi(\xi-\varpi)e^{-\rho(\xi-\varpi)}\le\frac{\varrho}{A\varpi}e^{\rho(\kappa+\varpi)}h(\xi+\kappa)=h(\xi+\kappa),
$$

which implies h is bounded. Therefore, $\Phi(\xi)=O(e^{\rho\xi})$ when $\xi\to-\infty$. Integrating (1.7) from $-\infty$ to ξ, it follows from (H2) that

$$
\begin{aligned}
-c\psi(\xi)= & -\Phi(\xi)+\sum_{i=1}^{m}a_i\int_{-\infty}^{\xi}f(\psi(y-i+c\tau_i))dy+\alpha\int_{-\infty}^{\xi}f(\psi(y+c\tau_{m+1}))dy \\
& +\sum_{j=1}^{l}\beta_j\int_{-\infty}^{\xi}f(\psi(y+j+c\tau_{m+1+j}))dy \\
\le & -\Phi(\xi)+f'(0)\sum_{i=1}^{m}a_i\int_{-\infty}^{\xi}\psi(y-i+c\tau_i)dy+\alpha f'(0)\int_{-\infty}^{\xi}\psi(y+c\tau_{m+1})dy \\
& +\sum_{j=1}^{l}\beta_j f'(0)\int_{-\infty}^{\xi}\psi(y+j+c\tau_{m+1+j})dy \\
= & -\Phi(\xi)+f'(0)\sum_{i=1}^{m}a_i\Phi(\xi-i+c\tau_i)+\alpha f'(0)\Phi(\xi+c\tau_{m+1}) \\
& +\sum_{j=1}^{l}\beta_j f'(0)\Phi(\xi+j+c\tau_{m+1+j}).
\end{aligned}
\tag{2.6}
$$

Thus, we have $\psi(\xi)=O(e^{\rho\xi})$ when $\xi\to-\infty$. $\square$

Proposition 2.1. *Assume that* (H1)–(H3) *hold and let* $\psi(n-ct)$ *be a positive traveling wave of* (1.3) *with* $c \le c_*$ *and satisfy* (1.7). *Then*

$$\lim_{\xi\to-\infty}\frac{\psi(\xi)}{e^{\lambda_1\xi}} \text{ exists for } c<c_*, \quad \lim_{\xi\to-\infty}\frac{\psi(\xi)}{|\xi|e^{\lambda_*\xi}} \text{ exists for } c=c_*. \tag{2.7}$$

Proof. According to Lemma 2.3, for any $0<\mathrm{Re}\,\lambda<\rho$, we can define the two-sided Laplace transform of ψ:

$$L(\lambda)\equiv\int_{\mathbb{R}}\psi(y)e^{-\lambda y}dy.$$

We claim that $L(\lambda)$ is analytic for any $\mathrm{Re}\,\lambda\in(0,\lambda_1)$ and has a singularity at $\lambda=\lambda_1$. Indeed, since

$$\begin{aligned}
&-c\psi'(\xi)+\psi(\xi)-f'(0)\sum_{i=1}^m a_i\psi(\xi-i+c\tau_i)-\alpha f'(0)\psi(\xi+c\tau_{m+1})\\
&\quad-\sum_{j=1}^l\beta_j f'(0)\psi(\xi+j+c\tau_{m+1+j})\\
&=\sum_{i=1}^m a_i[f(\psi(\xi-i+c\tau_i))-f'(0)\psi(\xi-i+c\tau_i)]\\
&\quad+\alpha[f(\psi(\xi+c\tau_{m+1}))-f'(0)\psi(\xi+c\tau_{m+1})]\\
&\quad+\sum_{j=1}^l\beta_j[f(\psi(\xi+j+c\tau_{m+1+j}))-f'(0)\psi(\xi+j+c\tau_{m+1+j})]\\
&=:R(\psi)(\xi),
\end{aligned}$$

we obtain

$$\Delta(c,\lambda)L(\lambda)=\int_{-\infty}^{\infty}e^{-\lambda y}R(\psi)(y)dy. \tag{2.8}$$

It is easily seen that the left hand side of (2.8) is analytic for $\mathrm{Re}\,\lambda\in(0,\rho)$. According to Remark 1.1, for any $\bar{u}>0$, there exist $\bar{d}>0$ such that $f(u)\ge f'(0)u-\bar{d}u^{\sigma+1}$, $\forall u\in[0,\bar{u}]$, where $\bar{d}:=\max\left\{d,\gamma^{-(\sigma+1)}\max_{u\in[\gamma,\bar{u}]}\{f'(0)u-f(u)\}\right\}$. Thus,

$$\begin{aligned}
&-\bar{d}\left[\sum_{i=1}^m a_i\psi^{\sigma+1}(\xi-i+c\tau_i)+\alpha\psi^{\sigma+1}(\xi+c\tau_{m+1})+\sum_{j=1}^l\beta_j\psi^{\sigma+1}(\xi+j+c\tau_{m+1+j})\right]\\
&\quad\le R(\psi)(\xi)\le 0.
\end{aligned}$$

Choose $\nu > 0$ such that $\frac{\nu}{\sigma} < \rho$. Then for any $\operatorname{Re}\lambda \in (0, \rho + \nu)$, we have

$$\begin{aligned}
&\left| \int_{-\infty}^{\infty} e^{-\lambda y} R(\psi)(y) dy \right| \\
&\le \bar{d} \int_{-\infty}^{\infty} e^{-\lambda y} \Big[\sum_{i=1}^{m} a_i \psi^{\sigma+1}(y - i + c\tau_i) + \alpha \psi^{\sigma+1}(y + c\tau_{m+1}) \\
&\quad + \sum_{j=1}^{l} \beta_j \psi^{\sigma+1}(y + j + c\tau_{m+1+j}) \Big] dy \\
&= \bar{d} \left[\sum_{i=1}^{m} a_i e^{\lambda(-i + c\tau_i)} + \alpha e^{\lambda c\tau_{m+1}} + \sum_{j=1}^{l} \beta_j e^{\lambda(j + c\tau_{m+1+j})} \right] \int_{-\infty}^{\infty} e^{-\lambda y} \psi^{\sigma+1}(y) dy \\
&\le \bar{d} \left[\sum_{i=1}^{m} a_i e^{\lambda(-i + c\tau_i)} + \alpha e^{\lambda c\tau_{m+1}} + \sum_{j=1}^{l} \beta_j e^{\lambda(j + c\tau_{m+1+j})} \right] L(\lambda - \nu) \left(\sup_{\xi \in \mathbb{R}} \psi(\xi) e^{-\frac{\nu\xi}{\sigma}} \right)^{\sigma} \\
&< +\infty. \qquad (2.9)
\end{aligned}$$

We now use the property of Laplace transform (p. 58, [31]). Since $\psi > 0$, there exists a real number B such that $L(\lambda)$ is analytic for $0 < \operatorname{Re}\lambda < B$ and $L(\lambda)$ has a singularity at $\lambda = B$. Hence for $c \le c_*$, $L(\lambda)$ is analytic for $\operatorname{Re}\lambda \in (0, \lambda_1)$ and $L(\lambda)$ has a singularity at $\lambda = \lambda_1$.

We rewrite (2.8) as

$$\int_{-\infty}^{0} \psi(\theta) e^{-\lambda\theta} d\theta = \frac{\int_{\mathbb{R}} e^{-\lambda\theta} R(\psi)(\theta) d\theta}{\Delta(c, \lambda)} - \int_{0}^{\infty} \psi(\theta) e^{-\lambda\theta} d\theta.$$

Note that $\int_0^\infty \psi(\theta) e^{-\lambda\theta} d\theta$ is analytic for $\operatorname{Re}\lambda > 0$. Also, $\Delta(c, \lambda) = 0$ does not have any zero with $\operatorname{Re}\lambda = \lambda_1$ other than $\lambda = \lambda_1$. Indeed, let $\lambda = \lambda_1 + i\tilde{\lambda}$, then

$$\begin{aligned}
0 = &- c\lambda_1 + 1 - f'(0) \Big[\sum_{k=1}^{m} a_k e^{\lambda_1(-k + c\tau_k)} \cos(-k + c\tau_k)\tilde{\lambda} \\
&+ \alpha e^{\lambda_1 c\tau_{m+1}} \cos(c\tau_{m+1}\tilde{\lambda}) + \sum_{j=1}^{l} \beta_j e^{\lambda_1(j + c\tau_{m+1+j})} \cos(j + c\tau_{m+1+j})\tilde{\lambda} \Big] \qquad (2.10)
\end{aligned}$$

and

$$\begin{aligned}
0 = &- c\tilde{\lambda} - f'(0) \Big[\sum_{k=1}^{m} a_k e^{\lambda_1(-k + c\tau_k)} \sin(-k + c\tau_k)\tilde{\lambda} \\
&+ \alpha e^{\lambda_1 c\tau_{m+1}} \sin(c\tau_{m+1}\tilde{\lambda}) + \sum_{j=1}^{l} \beta_j e^{\lambda_1(j + c\tau_{m+1+j})} \sin(j + c\tau_{m+1+j})\tilde{\lambda} \Big]. \qquad (2.11)
\end{aligned}$$

Thus, according to (2.10) and $\Delta(c,\lambda_1)=0$, we have

$$0=\sum_{k=1}^{m} a_i e^{\lambda_1(-k+c\tau_k)}[1-\cos(-k+c\tau_k)\tilde{\lambda}]$$
$$+\alpha e^{\lambda_1 c\tau_{m+1}}[1-\cos(c\tau_{m+1}\tilde{\lambda})]+\sum_{j=1}^{l}\beta_j e^{\lambda_1(j+c\tau_{m+1+j})}[1-\cos(j+c\tau_{m+1+j})\tilde{\lambda}], \tag{2.12}$$

which implies that

$$\cos(-k+c\tau_k)\tilde{\lambda}=\cos(c\tau_{m+1}\tilde{\lambda})=\cos(j+c\tau_{m+1+j})\tilde{\lambda}=1. \tag{2.13}$$

Combining (2.11) and (2.13), we obtain $\tilde{\lambda}=0$.

Assume that $\psi(\xi)$ is increasing for large $-\xi>0$. Then we can choose a translation of ψ such that it is increasing for $\xi<0$. Letting $u(\xi)=\psi(-\xi)$ and $T(u)(\xi):=R(\psi)(-\xi)$, it is clear that $u(\xi)$ is decreasing $\xi>0$ and

$$\int_0^{\infty} u(\theta)e^{\lambda\theta}d\theta=\frac{\int_{\mathbb{R}} e^{\lambda\theta}T(u)(\theta)d\theta}{\Delta(c,\lambda)}-\int_{-\infty}^{0} u(\theta)e^{\lambda\theta}d\theta:=\frac{h(\lambda)}{(\lambda-\lambda_1)^{k+1}},$$

where $k=0$ for $c<c_*$, and $k=1$ for $c=c_*$, and

$$h(\lambda)=\frac{(\lambda-\lambda_1)^{k+1}\int_{\mathbb{R}} e^{\lambda\theta}T(u)(\theta)d\theta}{\Delta(c,\lambda)}-(\lambda-\lambda_1)^{k+1}\int_{-\infty}^{0} u(\theta)e^{\lambda\theta}d\theta.$$

By Lemma 2.1, $\lim_{\lambda\to\lambda_1} h(\lambda)$ exists. Therefore, $h(\lambda)$ is analytic for all $0<\operatorname{Re}\lambda\le\lambda_1$. Then Lemma 2.2 implies that

$$\lim_{\xi\to+\infty}\frac{u(\xi)}{\xi^k e^{-\lambda_1\xi}} \text{ exists i.e., } \lim_{\xi\to-\infty}\frac{\psi(\xi)}{|\xi|^k e^{\lambda_1\xi}} \text{ exists,}$$

that is,

$$\lim_{\xi\to-\infty}\frac{\psi(\xi)}{e^{\lambda_1\xi}} \text{ exists for } c<c_*,\quad \lim_{\xi\to-\infty}\frac{\psi(\xi)}{|\xi|e^{\lambda_*\xi}} \text{ exists for } c=c_*.$$

Now we assume that $\psi(\xi)$ is not monotone for large $-\xi>0$. Letting $p=-c>0$ and $\hat{\psi}(\xi)=\psi(\xi)e^{p\xi}>0$, it follows that

$$\hat{\psi}'(\xi)$$
$$=\left[\sum_{i=1}^{m} a_i f(\psi(\xi-i+c\tau_i))+\alpha f(\psi(\xi+c\tau_{m+1}))+\sum_{j=1}^{l}\beta_j f(\psi(\xi+j+c\tau_{m+1+j}))\right]e^{p\xi},$$

which implies that $\hat{\psi}'(\xi) > 0$ for any $\xi \in \mathbb{R}$. Let $\hat{u}(\xi) = \hat{\psi}(-\xi)$. Obviously, $\hat{u}(\xi)$ is decreasing on $\xi > 0$. Let $\widehat{L}(\lambda) = \int_{-\infty}^{\infty} e^{-\lambda\xi}\hat{\psi}(\xi)d\xi$. Noting that $\widehat{L}(\lambda) = L(\lambda - p)$ and repeating the above argument, we have

$$\lim_{\xi\to+\infty} \frac{\hat{u}(\xi)}{\xi^k e^{-(p+\lambda_1)\xi}} = \lim_{\xi\to-\infty} \frac{\psi(\xi)}{|\xi|^k e^{\lambda_1\xi}} \text{ exists.}$$

Thus, it follows that

$$\lim_{\xi\to-\infty} \frac{\psi(\xi)}{e^{\lambda_1\xi}} \text{ exists for } c < c_*, \quad \lim_{\xi\to-\infty} \frac{\psi(\xi)}{|\xi|e^{\lambda_*\xi}} \text{ exists for } c = c_*.$$

This completes the proof. $\square$

2.2. *Proof of Theorem 1.1*

From Proposition 2.1, there exist some positive numbers θ_1 and θ_2 such that

$$\lim_{\xi\to-\infty} \frac{\phi(\xi)}{|\xi|^k e^{\lambda_1\xi}} = \omega_1^k \text{ and } \lim_{\xi\to-\infty} \frac{\psi(\xi)}{|\xi|^k e^{\lambda_1\xi}} = \omega_2^k,$$

where $k = 0$ for $c < c_*$, and $k = 1$ for $c = c_*$. For $\epsilon > 0$, let us define

$$w(\xi) := \frac{\phi(\xi) - \psi(\xi + \bar{\xi})}{e^{\lambda_1\xi}} \text{ for } c < c_*, \text{ and } w_\epsilon(\xi) := \frac{\phi(\xi) - \psi(\xi + \bar{\xi})}{(\epsilon|\xi| + 1)e^{\lambda_*\xi}} \text{ for } c = c_*, \tag{2.14}$$

where $\bar{\xi} = \frac{1}{\lambda_1} \ln \frac{\omega_1^k}{\omega_2^k}$. Then $w(\pm\infty) = 0$ and $w_\varepsilon(\pm\infty) = 0$.

First, we consider $c < c_*$. Since $w(\pm\infty) = 0$, $\sup_{\xi\in\mathbb{R}}\{w(\xi)\}$ and $\inf_{\xi\in\mathbb{R}}\{w(\xi)\}$ are finite. Without loss of generality, we assume $\sup_{\xi\in\mathbb{R}}\{w(\xi)\} \geq |\inf_{\xi\in\mathbb{R}}\{w(\xi)\}|$ (otherwise, we may take $w(\xi) := \frac{\psi(\xi+\bar{\xi})-\phi(\xi)}{e^{\lambda_1\xi}}$). If $w(\xi) \not\equiv 0$, there exists ξ_0 such that

$$w(\xi_0) = \max_{\xi\in\mathbb{R}}\{w(\xi)\} = \sup_{\xi\in\mathbb{R}}\{w(\xi)\} > 0 \text{ and } w'(\xi_0) = 0.$$

We claim that

$$w(\xi_0 - i + c\tau_i) = w(\xi_0 + c\tau_{m+1}) = w(\xi_0 + j + c\tau_{m+1+j}) = w(\xi_0)$$

for all i and j. Suppose for the contrary that one of three inequalities $w(\xi_0 - i_0 + c\tau_{i_0}) < w(\xi_0)$, $w(\xi_0 + c\tau_{m+1}) < w(\xi_0)$ and $w(\xi_0 + j_0 + c\tau_{m+1+j_0}) < w(\xi_0)$ for some i_0 and j_0 must hold. According to (1.7) and (H2), we have

$$
\begin{aligned}
0 =& cw'(\xi_0)\\
=& -c\lambda_1 w(\xi_0) + w(\xi_0) - e^{-\lambda_1\xi}\sum_{i=1}^{m} a_i\Big[f(\phi(\xi - i + c\tau_i)) - f(\psi(\xi - i + c\tau_i))\Big]\\
&- e^{-\lambda_1\xi}\Big[f(\phi(\xi - i + c\tau_i)) - f(\psi(\xi - i + c\tau_i))\Big]\\
&- e^{-\lambda_1\xi}\sum_{j=1}^{l}\beta_j\Big[f(\phi(\xi + j + c\tau_{m+1+j})) - f(\psi(\xi + j + c\tau_{m+1+j}))\Big]\\
<& -c\lambda_1 w(\xi_0) + w(\xi_0)\\
&- f'(0)w(\xi_0)\left[\sum_{i=1}^{m} a_i e^{\lambda_1(-i+c\tau_i)} + \alpha f'(0)e^{\lambda_1 c\tau_{m+1}} + \sum_{j=1}^{l}\beta_j e^{\lambda_1(j+c\tau_{m+1+j})}\right]\\
=& -w(\xi_0)\Delta(c,\lambda_1) = 0,
\end{aligned}
\tag{2.15}
$$

which is a contradiction. Again by bootstrapping, $w(\xi_0 + kc\tau_{m+1}) = w(\xi_0)$ for all $k \in \mathbb{Z}$ and $w(+\infty) = 0$, we get $\phi(\xi) \equiv \psi(\xi + \bar{\xi})$ for $\xi \in \mathbb{R}$, which contradicts $w(\xi) \not\equiv 0$.

Next, we consider $c = c_*$. Similar to the above argument, assume $\sup_{\xi\in\mathbb{R}}\{w_\epsilon(\xi)\} \geq |\inf_{\xi\in\mathbb{R}}\{w_\epsilon(\xi)\}|$. If $w_\epsilon(\xi) \not\equiv 0$, there exists ξ_0^ϵ such that

$$
w_\epsilon(\xi_0^\epsilon) = \max_{\xi\in\mathbb{R}}\{w_\epsilon(\xi)\} = \sup_{\xi\in\mathbb{R}}\{w_\epsilon(\xi)\} > 0 \text{ and } w_\epsilon'(\xi_0) = 0.
$$

We first suppose that $\xi_0^\epsilon \to \infty$ as $\epsilon \to 0$. Choose $\epsilon > 0$ sufficiently small such that

$$
\xi_0^\epsilon > \max_{1\leq i\leq m}\{-i + c\tau_i\}.
$$

Note that

$$
\phi'(\xi_0^\epsilon) - \psi'(\xi_0^\epsilon + \bar{\xi}) = w_\epsilon'(\xi_0^\epsilon)(\epsilon\xi_0^\epsilon + 1)e^{\lambda_*\xi_0^\epsilon} + w_\epsilon(\xi_0^\epsilon)\epsilon e^{\lambda_*\xi_0^\epsilon} + w_\epsilon(\xi_0^\epsilon)(\epsilon\xi_0^\epsilon + 1)\lambda_* e^{\lambda_*\xi_0^\epsilon}.
$$

Thus, we get

$$
\begin{aligned}
&-c_* w_\epsilon(\xi_0^\epsilon)\epsilon - c_*\lambda_* w_\epsilon(\xi_0^\epsilon)(\epsilon\xi_0^\epsilon + 1)\\
&\leq -(\epsilon\xi_0^\epsilon + 1)w_\epsilon(\xi_0^\epsilon) + f'(0)\sum_{i=1}^{m} a_i[(\epsilon\xi_0^\epsilon - i + c\tau_i) + 1]e^{\lambda_*(-i+c\tau_i)}w_\epsilon(\xi_0^\epsilon - i + c\tau_i)\\
&\quad + \alpha f'(0)[(\epsilon\xi_0^\epsilon + c\tau_{m+1}) + 1]e^{\lambda_* c\tau_{m+1}}w_\epsilon(\xi_0^\epsilon + c\tau_{m+1})\\
&\quad + f'(0)\sum_{j=1}^{l}\beta_j[(\epsilon\xi_0^\epsilon + j + c\tau_{m+1+j}) + 1]e^{\lambda_*(j+c\tau_{m+1+j})}w_\epsilon(\xi_0^\epsilon + j + c\tau_{m+1+j}).
\end{aligned}
\tag{2.16}
$$

It follows from (2.16) and Lemma 2.1 (i) that

$$
w_\epsilon(\xi_0^\epsilon) = w_\epsilon(\xi_0^\epsilon - i + c\tau_i) = w_\epsilon(\xi_0^\epsilon + c\tau_{m+1}) = w_\epsilon(\xi_0^\epsilon + j + c\tau_{m+1+j})
$$

for all i and j. Indeed, as showed before, we may assume one of three inequalities $w_\epsilon(\xi_0^\epsilon - i + c\tau_i) < w_\epsilon(\xi_0^\epsilon)$, $w_\epsilon(\xi_0^\epsilon + c\tau_{m+1}) < w_\epsilon(\xi_0^\epsilon)$ and $w_\epsilon(\xi_0^\epsilon + j + c\tau_{m+1+j}) < w_\epsilon(\xi_0^\epsilon)$ for some i_0 and j_0 holds, then

$$\begin{aligned}
&-c_* w_\epsilon(\xi_0^\epsilon)\epsilon - c_*\lambda_* w_\epsilon(\xi_0^\epsilon)(\epsilon\xi_0^\epsilon + 1) \\
&\quad < -(\epsilon\xi_0^\epsilon + 1)w_\epsilon(\xi_0^\epsilon) + f'(0)\sum_{i=1}^{m} a_i[(\epsilon\xi_0^\epsilon - i + c\tau_i) + 1]e^{\lambda_*(-i+c\tau_i)} w_\epsilon(\xi_0^\epsilon) \\
&\qquad + \alpha f'(0)[(\epsilon\xi_0^\epsilon + c\tau_{m+1}) + 1]e^{\lambda_* c\tau_{m+1}} w_\epsilon(\xi_0^\epsilon) \\
&\qquad + f'(0)\sum_{j=1}^{l} \beta_j[(\epsilon\xi_0^\epsilon + j + c\tau_{m+1+j}) + 1]e^{\lambda_*(j+c\tau_{m+1+j})} w_\epsilon(\xi_0^\epsilon),
\end{aligned} \tag{2.17}$$

which implies that

$$\begin{aligned}
&-c_* w_\epsilon(\xi_0^\epsilon)\epsilon - w_\epsilon(\xi_0^\epsilon)(\epsilon\xi_0^\epsilon + 1)\Delta(c_*, \lambda_*) \\
&\quad < f'(0)\sum_{i=1}^{m} a_i\epsilon(-i + c\tau_i)e^{\lambda_*(-i+c\tau_i)} w_\epsilon(\xi_0^\epsilon) + \alpha f'(0)c\tau_{m+1}e^{\lambda_* c\tau_{m+1}} w_\epsilon(\xi_0^\epsilon) \\
&\qquad + f'(0)\sum_{j=1}^{l} \beta_j(j + c\tau_{m+1+j})e^{\lambda_*(j+c\tau_{m+1+j})} w_\epsilon(\xi_0^\epsilon).
\end{aligned} \tag{2.18}$$

This is a contradiction with $\frac{\partial\Delta(c,\lambda)}{\partial\lambda}|_{c=c_*,\lambda=\lambda_*} = 0$. Repeating the above arguments, we have $w_\epsilon(\xi_0^\epsilon) = w_\epsilon(\xi_0^\epsilon + kc\tau_{m+1})$ for $k \in \mathbb{Z}$ which implies w_ϵ is a constant. Since $w_\epsilon(+\infty) = 0$, we get $\phi(\xi) \equiv \psi(\xi + \bar{\xi})$ for $\xi \in \mathbb{R}$, which contradicts $w_\epsilon(\xi) \not\equiv 0$.

Next we assume that $\xi_0^\epsilon \to -\infty$ as $\epsilon \to 0$, then $w_\epsilon(\xi_0^\epsilon) \to 0$, as $\epsilon \to 0$. Since

$$\lim_{\epsilon\to 0} w_\epsilon(\xi) = w_0(\xi) := \frac{\phi(\xi) - \psi(\xi + \bar{\xi})}{e^{\lambda_*\xi}} \quad \text{for all } \xi \in \mathbb{R}, \tag{2.19}$$

and $w_\epsilon(x) \le w_\epsilon(\xi_0^\epsilon)$, we have $w_0(\xi) \le 0$ for all $\xi \in \mathbb{R}$. Note that $w_\epsilon(\xi_0^\epsilon) > 0$ implies $\phi(\xi_0^\epsilon) - \psi(\xi_0^\epsilon + \bar{\xi}) > 0$ and hence $w_0(\xi_0^\epsilon) > 0$, which gives a contradiction.

Last we assume $\{\xi_0^\epsilon\}$ is bounded, then we can take a subsequence $\xi_0^\epsilon \to \xi_1$ as $\epsilon \to 0$, for some finite ξ_1. From uniform convergence of w_ϵ to w on compact sets, $w_\epsilon(\xi_0^\epsilon) \to w(\xi_1)$ as $\epsilon \to 0$, where $w_0(\xi)$ is defined by (2.19). Thus, $w_0(\xi) = \lim_{\epsilon\to 0} w_\epsilon(\xi) \le \lim_{\epsilon\to 0} w_\epsilon(\xi_0^\epsilon) = w_0(\xi_1)$ for all $\xi \in \mathbb{R}$ and $w_0(\xi_1) \ge 0$. Similar to the argument in the case $c > c_*$ and we can also get $w_0(\xi) \equiv 0$, that is, we have $\phi(\xi) \equiv \psi(\xi + \bar{\xi})$ for $\xi \in \mathbb{R}$. This completes the proof. □

Remark 2.1. Based on the above arguments, the asymptotic behavior and uniqueness of traveling waves are still obtained without assuming that the equation be monotone.

3. Stability of traveling waves

In this section, we will give the proof of stability of traveling waves, i.e. Theorem 1.2. First of all, we state the following boundedness and the comparison principle which are given in Lemma 2.4 of [32].

Lemma 3.1 *(Boundedness [32]). Assume that* (H1)–(H2) *hold. Then* (1.3) *with the initial data* (1.4) *admits a unique solution* $\{w_n(t)\}_{n\in\mathbb{Z}}$ *satisfying*

$$0 \le w_n(t) \le K, \quad \text{for } t \in [0,+\infty),\ n \in \mathbb{Z}.$$

Lemma 3.2 *(Comparison principle [32]). Assume that* (H1)–(H2) *hold. Let* $\{\overline{W}_n(t)\}_{n\in\mathbb{Z}}$ *and* $\{\underline{W}_n(t)\}_{n\in\mathbb{Z}}$ *be the solutions of* (1.3) *with the initial data* $\{\overline{W}_n^0(t)\}_{n\in\mathbb{Z}}$ *and* $\{\underline{W}_n^0(t)\}_{n\in\mathbb{Z}}$*, respectively. If*

$$0 \le \underline{W}_n^0(s) \le \overline{W}_n^0(s) \le K \quad \text{for } s \in [-r,0],\ n \in \mathbb{Z}$$

then

$$0 \le \underline{W}_n(t) \le \overline{W}_n(t) \le K \quad \text{for } t \in [0,+\infty),\ n \in \mathbb{Z}.$$

For any given traveling wave $\phi(n-ct)$ with the wave speed $c < c_*$ satisfying (1.7), it follows from Corollary 1.1 that $\phi(\xi)$ is non-decreasing on $\xi \in \mathbb{R}$.

Let the initial data $w_n^0(s)$ satisfy

$$0 \le w_n^0(s) \le K \qquad \text{for } s \in [-r,0],\ n \in \mathbb{Z},$$

and define

$$\begin{cases} \overline{W}_n^0(s) = \max\{w_n^0(s), \phi(n-cs)\}, \\ \underline{W}_n^0(s) = \min\{w_n^0(s), \phi(n-cs)\}, \end{cases} \quad \text{for } s \in [-r,0],\ n \in \mathbb{Z}. \tag{3.1}$$

It is obvious that

$$\begin{cases} 0 \le \underline{W}_n^0(s) \le w_n^0(s) \le \overline{W}_n^0(s) \le K, \\ 0 \le \underline{W}_n^0(s) \le \phi(n-cs) \le \overline{W}_n^0(s) \le K, \end{cases} \quad \text{for } s \in [-r,0],\ n \in \mathbb{Z}. \tag{3.2}$$

Let $\overline{W}_n(t)$ and $\underline{W}_n(t)$ be the corresponding solutions of (1.3) with the initial data $\overline{W}_n^0(s)$ and $\underline{W}_n^0(s)$, respectively. According to Lemmas 3.1 and 3.2, we easily obtain

$$\begin{cases} 0 \le \underline{W}_n(t) \le w_n(t) \le \overline{W}_n(t) \le K, \\ 0 \le \underline{W}_n(t) \le \phi(n-ct) \le \overline{W}_n(t) \le K \end{cases} \quad \text{for } t \in [0,+\infty),\ n \in \mathbb{Z}. \tag{3.3}$$

Proof of Theorem 1.2. We divide the proof into three steps.

Step 1. $\overline{W}_n(t)$ converges to $\phi(n-ct)$ for $n\in\mathbb{Z}$. For the sake of convenience, we always take $\xi=\xi(t,n):=n-ct$. Let

$$u_n(t)=\overline{W}_n(t)-\phi(n-ct),\ t\in[0,+\infty),\ n\in\mathbb{Z}$$

and

$$u_n^0(s)=\overline{W}_n^0(s)-\phi(n-cs),\ s\in[-r,0],\ n\in\mathbb{Z}.$$

Therefore, it follows from (3.2) and (3.3) that

$$u_n(t)\geq 0 \text{ and } u_n(s)\geq 0. \tag{3.4}$$

From (1.3) and (1.7), $u_n(t)$ satisfies

$$\begin{aligned}\frac{du_n(t)}{dt}=&-u_n(t)+\sum_{i=1}^m a_i[f(\overline{W}_{n-i}(t-\tau_i))-f(\phi(n-ct-i+c\tau_i))]\\&+\alpha[f(\overline{W}_n(t-\tau_{m+1}))-f(\phi(n-ct+c\tau_{m+1}))]\\&+\sum_{j=1}^l\beta_j[f(\overline{W}_{n+j}(t-\tau_{m+1+j}))-f(\phi(n-ct+j+c\tau_{m+1+j}))]\\=&-u_n(t)+\sum_{i=1}^m a_i f'(\phi(n-ct-i+c\tau_i))u_{n-i}(t-\tau_i)\\&+\alpha f'(\phi(n-ct+c\tau_{m+1}))u_n(t-\tau_{m+1})\\&+\sum_{j=1}^l\beta_j f'(\phi(n-ct+j+c\tau_{m+1+j}))u_{n+j}(t-\tau_{m+1+j})\\&+Q_n(t)\end{aligned}\tag{3.5}$$

where

$$\begin{aligned}Q_n(t)=&\sum_{i=1}^m a_i[f(\overline{W}_{n-i}(t-\tau_i))-f(\phi(n-ct-i+c\tau_i))]\\&+\alpha[f(\overline{W}_n(t-\tau_{m+1}))-f(\phi(n-ct+c\tau_{m+1}))]\\&+\sum_{j=1}^l\beta_j[f(\overline{W}_{n+j}(t-\tau_{m+1+j}))-f(\phi(n-ct+j+c\tau_{m+1+j}))]\\&-\Big[\sum_{i=1}^m a_i f'(\phi(n-ct-i+c\tau_i))u_{n-i}(t-\tau_i)\\&+\alpha f'(\phi(n-ct+c\tau_{m+1}))u_n(t-\tau_{m+1})\\&+\sum_{j=1}^l\beta_j f'(\phi(n-ct+j+c\tau_{m+1+j}))u_{n+j}(t-\tau_{m+1+j})\Big].\end{aligned}\tag{3.6}$$

According to Taylor's formula and assumption (H4), we have $Q_n(t) \le 0$. Let $v(\xi) > 0$ be the weight function defined in (1.8). Multiplying (3.5) by $e^{2\mu t}u_n(t)v(\xi(t,n))$, where $\mu > 0$ will be given later in Lemma 3.4, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}[e^{2\mu t}u_n^2(t)v(\xi(t,n))] - (\mu-1)e^{2\mu t}u_n^2(t)v(\xi(t,n)) + \frac{c}{2}e^{2\mu t}u_n^2(t)v'_\xi \\
&\quad - \Big[\sum_{i=1}^m a_i f'(\phi(n-ct-i+c\tau_i))u_{n-i}(t-\tau_i)u_n(t)v(\xi(t,n))e^{2\mu t} \\
&\quad + \alpha f'(\phi(n-ct+c\tau_{m+1}))u_n(t-\tau_{m+1})u_n(t)v(\xi(t,n))e^{2\mu t} \\
&\quad + \sum_{i=1}^l \beta_j f'(\phi(n-ct+j+c\tau_{m+1+j}))u_{n+j}(t-\tau_{m+1+j})u_n(t)v(\xi(t,n))e^{2\mu t}\Big] \\
&= Q_n(t)e^{2\mu t}u_n(t)v(\xi(t,n)) \le 0.
\end{aligned} \tag{3.7}$$

By the Cauchy–Schwarz inequality $|xy| \le \frac{\kappa_i}{2}x^2 + \frac{1}{2\kappa_i}y^2$ for any $\kappa_i > 0, i = 1, \cdots, m+l+1$ and summing them for all $n \in \mathbb{Z}$ and integrating the resultant inequality over $[0,t]$ for (3.7), then we have

$$\begin{aligned}
&e^{2\mu t}\|u(t)\|_{l_v^2}^2 - \|u^0(0)\|_{l_v^2}^2 + \int_0^t \sum_n e^{2\mu s}u_n^2(s)v(\xi(s,n))\Big[-2(\mu-1) + c\frac{v'_\xi(\xi(s,n))}{v(\xi(s,n))}\Big]ds \\
&\le \int_0^t \sum_n \Big[\sum_{i=1}^m a_i f'(\phi(n-cs-i+c\tau_i))\Big(\kappa_i u_{n-i}^2(s-\tau_i) + \frac{1}{\kappa_i}u_n^2(s)\Big)e^{2\mu s}v(\xi(s,n))\Big]ds \\
&\quad + \alpha\int_0^t \sum_n f'(\phi(n-cs+c\tau_{m+1}))\Big(\kappa_{m+1}u_n^2(s-\tau_{m+1}) + \frac{1}{\kappa_{m+1}}u_n^2(s)\Big)e^{2\mu s}v(\xi(s,n))ds \\
&\quad + \int_0^t \sum_n \Bigg[\sum_{j=1}^l \beta_j f'(\phi(n-cs+j+c\tau_{m+1+j})) \\
&\quad \times \Big(\kappa_{m+1+j}u_{n+j}^2(s-\tau_{m+1+j}) + \frac{1}{\kappa_{m+1+j}}u_n^2(s)\Big)e^{2\mu s}v(\xi(s,n))\Bigg]ds.
\end{aligned} \tag{3.8}$$

Note that there exists $C_1 > 0$ such that

$$f'(0)\sum_{i=1}^m a_i\kappa_i e^{2\mu\tau_i}\frac{v(\xi(s+\tau_i,n+i))}{v(\xi(s,n))} \le C_1 \text{ for all } n \in \mathbb{Z},\ s \in [-r,0].$$

In view of (H2) and (H4), we have $0 \le f'(u) \le f'(0)$ for any $u \in [0,K]$, and

$$\begin{aligned}
&\int_0^t \sum_n \Big(\sum_{i=1}^m a_i\kappa_i f'(\phi(n-cs-i+c\tau_i))u_{n-i}^2(s-\tau_i)e^{2\mu s}v(\xi(s,n))\Big)ds \\
&= \int_0^t \sum_{i=1}^m a_i\kappa_i \sum_k f'(\phi(k-cs+c\tau_i))u_k^2(s-\tau_i)e^{2\mu s}v(\xi(s,k+i))\Big)ds \\
&= \int_0^t \sum_{i=1}^m a_i\kappa_i \sum_n f'(\phi(n-cs+c\tau_i))u_n^2(s-\tau_i)e^{2\mu s}v(\xi(s,n+i))\Big)ds \\
&= \sum_{i=1}^m a_i\kappa_i \sum_n \Big(\int_{-\tau_i}^{t-\tau_i} f'(\phi(n-cs))u_n^2(s)e^{2\mu(s+\tau_i)}v(\xi(s+\tau_i,n+i))ds\Big) \\
&\le \sum_{i=1}^m a_i\kappa_i \sum_n \Bigg[\Big(\int_{-r}^0 + \int_0^t\Big) f'(\phi(n-cs))u_n^2(s)e^{2\mu(s+\tau_i)}v(\xi(s+\tau_i,n+i))ds\Bigg] \\
&\le C_1\int_{-r}^0 \|u^0(s)\|_{l_v^2}^2 ds \\
&\quad + \int_0^t \sum_n u_n^2(s)e^{2\mu s}v(\xi(s,n))f'(\phi(n-cs))\Big(\sum_{i=1}^m a_i\kappa_i e^{2\mu\tau_i}\frac{v(\xi(s+\tau_i,n+i))}{v(\xi(s,n))}\Big)ds. \quad (3.9)
\end{aligned}$$

Similarly, there exist $C_2>0$ and $C_3>0$ such that

$$\begin{aligned}
&\alpha\kappa_{m+1}\int_0^t \sum_n f'(\phi(n-cs+c\tau_{m+1}))u_n^2(s-\tau_{m+1})e^{2\mu s}v(\xi(s,n))ds \\
&\quad \le C_2\int_{-r}^0 \|u^0(s)\|_{l_v^2}^2 ds \\
&\qquad + \int_0^t \sum_n u_n^2(s)e^{2\mu s}v(\xi(s,n))f'(\phi(n-cs))\Big(\alpha\kappa_{m+1}e^{2\mu\tau_{m+1}}\frac{v(\xi(s+\tau_{m+1},n))}{v(\xi(s,n))}\Big)ds
\end{aligned} \quad (3.10)$$

and

$$\begin{aligned}
&\int_0^t \sum_n \Big(\sum_{j=1}^l \beta_j\kappa_{m+1+j} f'(\phi(n-cs+j+c\tau_{m+1+j}))u_{n+j}^2(s-\tau_{m+1+j})e^{2\mu s}v(\xi(s,n))\Big)ds \\
&\le C_3\int_{-r}^0 \|u^0(s)\|_{l_v^2}^2 ds
\end{aligned}$$

$$+\int_0^t \sum_n u_n^2(s)e^{2\mu s}v(\xi(s,n))f'(\phi(n-cs))$$

$$\times\Big(\sum_{j=1}^{l}\beta_j\kappa_{m+1+j}e^{2\mu\tau_{m+1+j}}\frac{v(\xi(s+\tau_{m+1+j},n-j))}{v(\xi(s,n))}\Big)ds. \tag{3.11}$$

Thus, it follows from (3.8)–(3.11) that

$$e^{2\mu t}\|u(t)\|^2_{l^2_v}+\int_0^t\sum_n u_n^2(s)v(\xi(s,n))B_{\mu,v}(s,n)ds$$

$$\le \|u^0(0)\|^2_{l^2_v}+C_4\int_{-r}^0\|u^0(s)\|^2_{l^2_v}ds, \tag{3.12}$$

where

$$C_4=C_1+C_2+C_3,$$

$$B_{\mu,v}(t,n)=A_{\mu,v}(t,n)-2\mu-f'(\phi(n-ct))\Bigg[\sum_{i=1}^m a_i\kappa_i\Big(e^{2\mu\tau_i}-1\Big)\frac{v(\xi(t+\tau_i,n+i))}{v(\xi(t,n))}$$

$$+\alpha\kappa_{m+1}\Big(e^{2\mu\tau_{m+1}}-1\Big)\frac{v(\xi(t+\tau_{m+1},n))}{v(\xi(t,n))}$$

$$+\sum_{j=1}^l\beta_j\kappa_{m+1+j}\Big(e^{2\mu\tau_{m+1+j}}-1\Big)\frac{v(\xi(t+\tau_{m+1+j},n-j))}{v(\xi(t,n))}\Bigg] \tag{3.13}$$

and

$$A_{\mu,v}(t,n)=c\frac{v'_\xi(\xi(t,n))}{v(\xi(t,n))}+2-f'(\phi(n-ct))\Bigg[\sum_{i=1}^m a_i\kappa_i\frac{v(\xi(t+\tau_i,n+i))}{v(\xi(t,n))}$$

$$+\alpha\kappa_{m+1}\frac{v(\xi(t+\tau_{m+1},n))}{v(\xi(t,n))}+\sum_{j=1}^l\beta_j\kappa_{m+1+j}\frac{v(\xi(t+\tau_{m+1+j},n-j))}{v(\xi(t,n))}\Bigg]$$

$$-\Bigg[\sum_{i=1}^m\frac{a_i}{\kappa_i}f'(\phi(n-ct-i+c\tau_i))+\frac{\alpha}{\kappa_{m+1}}f'(\phi(n-ct+c\tau_{m+1}))$$

$$+\sum_{j=1}^l\frac{\beta_j}{\kappa_{m+1+j}}f'(\phi(n-ct+j+c\tau_{m+1+j}))\Bigg]. \tag{3.14}$$

The most important step now is to prove $B_{\mu,v}(t,n) > 0$ for $t \in (-\infty, +\infty)$ and $n \in \mathbb{Z}$. In order to obtain this, the following lemma plays a key role in this paper.

Lemma 3.3. *Assume that* (H1)–(H4) *hold and let* $\kappa_i := e^{\lambda(i-c\tau_i)}$ $(i = 1, \cdots, m)$, $\kappa_{m+1} := e^{-\lambda c\tau_{m+1}}$ *and* $\kappa_{m+1+j} := e^{-\lambda(j+c\tau_{m+1+j})}$ $(j = 1, \cdots, l)$. *Then, for any* $c < c_*$, $A_{\mu,v}(t,n)$ *defined in* (3.14) *satisfies* $A_{\lambda,v}(t,n) \geq C_5 > 0$ *for some positive constant* C_5, *which is independent on* t, n *and* μ.

Proof. Notice that $0 \leq f'(w) \leq f'(0)$ for any $w \in [0, K]$ and

$$\frac{v'_\xi(\xi(t,n))}{v(\xi(t,n))} = -2\lambda, \quad \frac{v(\xi(t+\tau_i, n+i))}{v(\xi(t,n))} = e^{-2\lambda(i-c\tau_i)},$$
$$\frac{v(\xi(t+\tau_{m+1}, n))}{v(\xi(t,n))} = e^{2\lambda c\tau_{m+1}}, \quad \frac{v(\xi(t+\tau_{m+1+j}, n-j))}{v(\xi(t,n))} = e^{2\lambda(j+c\tau_{m+1+j})}.$$

According to Lemma 2.1, it follows from (1.8)–(1.9) and (3.14) that

$$\begin{aligned}
A_{\mu,v}(t,n) \geq & -2\lambda c + 2 - f'(0)\left[\sum_{i=1}^m a_i\kappa_i e^{-2\lambda(i-c\tau_i)}\right. \\
& \left. + \alpha\kappa_{m+1}e^{2\lambda c\tau_{m+1}} + \sum_{j=1}^l \beta_j\kappa_{m+1+j}e^{2\lambda(j+c\tau_{m+1+j})}\right] \\
& - \left[\sum_{i=1}^m \frac{a_i}{\kappa_i}f'(0) + \frac{\alpha}{\kappa_{m+1}}f'(0) + \sum_{j=1}^l \frac{\beta_j}{\kappa_{m+1+j}}f'(0)\right]. \\
= & -2\lambda c + 2 - 2f'(0)\left[\sum_{i=1}^m a_i e^{-\lambda(i-c\tau_i)} + \alpha e^{\lambda c\tau_{m+1}} + \sum_{j=1}^l \beta_j e^{\lambda(j+c\tau_{m+1+j})}\right] \\
= & 2\Delta(c,\lambda) =: C_5 > 0 \quad \text{for } \lambda \in (\lambda_1, \lambda_2).
\end{aligned} \tag{3.15}$$

This completes the proof. □

Lemma 3.4. *Assume that* (H1)–(H4) *hold. Then, for any* $c < c_*$, *there exists a positive number* $\mu_1 > 0$ *such that* $B_{\mu,v}(t,n) > 0$ *for* $0 < \mu < \mu_1$, *where* μ_1 *is the unique root of the following equation*

$$\begin{aligned}
C_5 - 2\mu - f'(0)&\left[\sum_{i=1}^m a_i\left(e^{2\mu\tau_i} - 1\right)e^{-\lambda(i-c\tau_i)} + \alpha\left(e^{2\mu\tau_{m+1}} - 1\right)e^{\lambda c\tau_{m+1}}\right. \\
&\left. + \sum_{j=1}^l \beta_j\left(e^{2\mu\tau_{m+1+j}} - 1\right)e^{\lambda(j+c\tau_{m+1+j})}\right] = 0.
\end{aligned} \tag{3.16}$$

Proof. As shown in Lemma 3.3, it follows immediately that

$$\begin{aligned}
B_{\mu,v}(t,n) \geq\ & C_5 - 2\mu - f'(0)\Bigg[\sum_{i=1}^{m} a_i\kappa_i\left(e^{2\mu\tau_i}-1\right)\frac{v(\xi(t+\tau_i,n+i))}{v(\xi(t,n))} \\
& + \alpha\kappa_{m+1}\left(e^{2\mu\tau_{m+1}}-1\right)\frac{v(\xi(t+\tau_{m+1},n))}{v(\xi(t,n))} \\
& + \sum_{j=1}^{l}\beta_j\kappa_{m+1+j}\left(e^{2\mu\tau_{m+1+j}}-1\right)\frac{v(\xi(t+\tau_{m+1+j},n-j))}{v(\xi(t,n))}\Bigg] \\
=\ & C_5 - 2\mu - f'(0)\Bigg[\sum_{i=1}^{m} a_i\left(e^{2\mu\tau_i}-1\right)e^{-\lambda(i-c\tau_i)} \\
& + \alpha\left(e^{2\mu\tau_{m+1}}-1\right)e^{\lambda c\tau_{m+1}} + \sum_{j=1}^{l}\beta_j\left(e^{2\mu\tau_{m+1+j}}-1\right)e^{\lambda(j+c\tau_{m+1+j})}\Bigg] \\
>\ & 0 \quad \text{for all } \mu\in(0,\mu_1).
\end{aligned}$$

This completes the proof. □

According to Lemma 3.4 and dropping the positive term

$$\int_0^t \sum_n e^{2\mu s}u_n^2(s)v(\xi(s,n))B_{\mu,v}(s,n)ds$$

in (3.12), we obtain the following basic energy estimate.

Lemma 3.5. *Assume that* (H1)–(H4) *hold. Then for any $c < c_*$, it holds*

$$e^{2\mu t}\|u(t)\|_{l_v^2}^2 \leq \|u^0(0)\|_{l_v^2}^2 + C_4\int_{-r}^{0}\|u^0(s)\|_{l_v^2}^2 ds,\ t\geq 0,\ \mu\in(0,\mu_1). \tag{3.17}$$

Notice that the standard Sobolev's embedding result is $l^2 \hookrightarrow l^\infty$. However, we cannot apply the standard Sobolev's embedding inequality $l_v^2 \hookrightarrow l^\infty$ since $v(\xi_n)\to 0$ as $n\to\infty$ and $\xi_n\to\infty$. For any integer interval $I|_{\mathbb{Z}} = (-\infty, N]\subset\mathbb{Z}$ for some large integer $N\gg 1$, we may have the Sobolev's embedding result $l_v^2(I|_{\mathbb{Z}})\hookrightarrow l^\infty(I|_{\mathbb{Z}})$, which gives the following l^∞-estimate.

Lemma 3.6. *Assume that* (H1)–(H4) *hold. Then, for any $c < c_*$, it holds*

$$\sup_{n\in I|_{\mathbb{Z}}}|u_n(t)| \leq C_6 e^{-\mu t}\left(\|u^0(0)\|_{l_v^2}^2 + \int_{-r}^{0}\|u^0(s)\|_{l_v^2}^2 ds\right)^{\frac{1}{2}},\quad t\geq 0,\ \mu\in(0,\mu_1). \tag{3.18}$$

Proof. According to the standard Sobolev's embedding inequality $l^2 \hookrightarrow l^\infty$, it follows that

$$\sup_{n \in I|_{\mathbb{Z}}} |u_n(t)| \le \|u(t)\|_{l^2(I|_{\mathbb{Z}})}.$$

Since $v(\xi_n) = e^{-2\lambda(\xi_n - \xi_0)} \ge 1$ for $\xi_n \le \xi_0$ as $n \in I|_{\mathbb{Z}}$, there exists some positive constant C such that

$$\|u(t)\|_{l^2(I|_{\mathbb{Z}})} \le C\|u(t)\|_{l^2_v(I|_{\mathbb{Z}})}.$$

According to the above inequalities and (3.17), one has immediately the conclusion. □

In order to extend the time-exponential decay in (3.18) to the whole space $\mathbb{Z}$, the key step is to prove the decay at $n = \infty$.

Lemma 3.7. *Assume that* (H1)–(H4) *hold. Then, for any $c < c_*$, it holds*

$$\lim_{n\to\infty} u_n(t) \le C_7 e^{-\mu_2 t}, \quad t \ge 0. \tag{3.19}$$

Proof. According to $Q_n(t) \le 0$, (3.5) can be reduced to

$$\begin{aligned}\frac{du_n(t)}{dt} \le & -u_n(t) + \sum_{i=1}^{m} a_i f'(\phi(n - ct - i + c\tau_i))u_{n-i}(t - \tau_i) \\ & + \alpha f'(\phi(n - ct + c\tau_{m+1}))u_n(t - \tau_{m+1}) \\ & + \sum_{j=1}^{l} \beta_j f'(\phi(n - ct + j + c\tau_{m+1+j}))u_{n+j}(t - \tau_{m+1+j}).\end{aligned} \tag{3.20}$$

Taking limits as $n \to \infty$ and letting $\lim_{n\to\infty} u_n(t) := u_\infty(t)$, it follows from (3.20) that

$$\begin{aligned}\frac{du_\infty(t)}{dt} \le & -u_\infty(t) + \sum_{i=1}^{m} a_i f'(K)u_\infty(t - \tau_i) + \alpha f'(K)u_\infty(t - \tau_{m+1}) \\ & + \sum_{j=1}^{l} \beta_j f'(K)u_\infty(t - \tau_{m+1+j}).\end{aligned} \tag{3.21}$$

When $f'(K) \le 0$, it follows from (3.4) and (3.21) that $\frac{du_\infty(t)}{dt} \le -u_\infty(t)$. Thus, we have

$$u_\infty(t) \le u^0_\infty(0)e^{-t}. \tag{3.22}$$

When $f'(K) > 0$, integrating (3.21) over $[0, t]$, we can obtain

$$\int_0^t u_\infty(s - \tau_i)ds = \int_{-\tau_i}^{t-\tau_i} u_\infty(s)ds \le \int_0^t u_\infty(s)ds + \int_{-\tau_i}^{0} u^0_\infty(s)ds, \; i = 1, \cdots, m + l + 1$$

and

$$u_\infty(t) \le -\int_0^t u_\infty(s)ds + \sum_{i=1}^m a_i f'(K)\int_0^t u_\infty(s-\tau_i)ds$$
$$+\alpha f'(K)\int_0^t u_\infty(s-\tau_{m+1})ds + \sum_{j=1}^l \beta_j f'(K)\int_0^t u_\infty(s-\tau_{m+1+j})ds$$
$$\le \Big[(a+\alpha+\beta)f'(K)-1\Big]\int_0^t u_\infty(s)ds + M, \tag{3.23}$$

where

$$M := \sum_{i=1}^m a_i f'(K)\int_{-\tau_i}^0 u_\infty^0(s)ds + \alpha f'(K)\int_{-\tau_{m+1}}^0 u_\infty^0(s)ds + \sum_{j=1}^l \beta_j f'(K)\int_{-\tau_{m+1+j}}^0 u_\infty^0(s)ds.$$

By the Gronwall's inequality, (3.23) yields

$$u_\infty(t) \le C_8 e^{-\mu_3 t}, \tag{3.24}$$

where $\mu_3 = 1-(a+\alpha+\beta)f'(K) > 0$. Taking $\mu_2 = \min\{1, \mu_3\}$, (3.22) and (3.24) imply that the conclusion holds. This completes the proof. □

According to Lemmas 3.6 and 3.7, it immediately has the following result.

Lemma 3.8. *Assume that* (H1)–(H4) *hold. Then for any $c < c_*$, it holds*

$$\sup_n |\overline{W}_n(t) - \phi(n-ct)| = \sup_n |v_n(t)| \le C_9 e^{-\mu t},\ \ \mu \in (0, \min\{\mu_1, \mu_2\}) \tag{3.25}$$

for all $t \ge 0$ and some positive constant C_9.

Thanks to the above arguments, Step 1 has been proven.

Step 2. $\underline{W}_n(t)$ converges to $\phi(n-ct)$. Let

$$v_n(t) = \underline{W}_n(t) - \phi(n-ct)$$

and

$$v_n^0(s) = \underline{W}_n^0(s) - \phi(n-cs).$$

Similar to all processes in Step 1, we have the following proposition.

Lemma 3.9. *Assume that* (H1)–(H4) *hold. Then for any $c < c_*$, it holds*

$$\sup_n |\underline{W}_n(t) - \phi(n - ct)| = \sup_n |v_n(t)| \le C_{10} e^{-\mu t} \tag{3.26}$$

for all $t \ge 0$ and some positive constant C_{10}.

Step 3. $W_n(t)$ converges to $\phi(n - ct)$.

Lemma 3.10. *Assume that* (H1)–(H4) *hold. Then, for any $c < c_*$, it holds*

$$\sup_n |W_n(t) - \phi(n - ct)| \le C e^{-\mu t} \tag{3.27}$$

for all $t \ge 0$ and some positive constant C.

Proof. Since the initial data satisfy $\underline{W}_n(s) \le w_n(s) \le \overline{W}_n(s)$, $s \in [-r, 0]$, it follows from Lemma 3.2 that the corresponding solutions of (1.1) and (1.2) satisfy

$$\underline{W}_n(t) \le w_n(t) \le \overline{W}_n(t), \quad \text{for all } t \ge 0,\ n \in \mathbb{Z}.$$

According to Lemmas 3.8 and 3.9, the squeezing argument yields

$$\sup_n |W_n(t) - \phi(n - ct)| \le C e^{-\mu t} \tag{3.28}$$

for all $t \ge 0$ and some positive constant C. This completes the proof. □

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