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# A More Effective Iteration Method for Solving Algebraic Equations

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**Abstract**

This note is concerned with the iteration numerical scheme for solving nonlinear algebraic equations. A more effective and more rapidly convergent iteration method is proposed, which improves the convergence of the previous numerical schemes and also eliminates the need of careful initial points selection in the previous work given by J.-H. He [*Appl. Math. Comp.* 135 (2003) 81-84].

**Keywords:** Nonlinear algebraic equations. Iteration method. Newton method.

## 1 Introduction

Iteration method is a quite useful and effective approach for solving nonlinear equations, for example, bisection method, secant method, Newton method (cf.[1]), and many other improved iteration methods ([2, 3, 4, 5] and the references therein). These methods have been widely used in scientific computation, engineering and technology.

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For a nonlinear algebraic equation

$$f(x) = 0, \tag{1}$$

the most famous iteration scheme for solving Eq. (1) numerically is Newton method:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{2}$$

equivalently,

$$f(x_n) + f'(x_n)(x_{n+1} - x_n) = 0. \tag{3}$$

In [2], by applying Taylor's expansion

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2}f''(x_0)(x - x_0)^2 + \cdots ,$$

J.-H.He proposed a faster convergent iteration method as follows

$$f(x_n) + f'(x_n)(x_{n+1} - x_n) + \frac{1}{2}f''(x_n)(x_{n+1} - x_n)^2 + g(x_n) = 0, \tag{4}$$

where

$$g(x_n) = f(x_n) - f(x_{n-1}) - f'(x_{n-1})(x_n - x_{n-1}) - \frac{1}{2}f''(x_{n-1})(x_n - x_{n-1})^2. \tag{5}$$

Obviously, Eq. (4) is a quadratic equation for $x_{n+1}$, and can be solved as

$$x_{n+1} = \frac{-B \pm \sqrt{B^2 - 4A(C + g(x_n))}}{2A}, \tag{6}$$

where

$$\begin{cases} A = \frac{1}{2}f''(x_n), \\ B = f'(x_n) - f''(x_n)x_n, \\ C = f(x_n) - f'(x_n)x_n + \frac{1}{2}f''(x_n)x_n^2, \end{cases} \tag{7}$$

(by the way, in [2], $C$ is mistakenly written as $C = f(x_n) - f'(x_n)x_n + \frac{1}{2}f''(x_n)x_n^2 + g(x_n)$, where $g(x_n)$ should be removed).

This method is more quickly convergent than Newton method in (2), but is not so effective and useful, because we have to carefully select some suitable initial data $x_0$ and $x_1$ to guarantee Eq. (6) to have a solution, namely, $B^2 - 4A(C + g(x_n)) \geq 0$ for each step. In fact, as numerical experiments (see Example below), a lot of initial data for $x_0$ and $x_1$ do not satisfy the method (4)-(5). In order to overcome this shortcoming, we propose the following improved scheme. Our new scheme is adapted for all given initial data, just like Newton method, and is much more rapidly convergent than Newton method (2) and the method (4) introduced in [2].

## 2 New iteration scheme and numerical results

Since the quadratic equation (4) may not have a solution for many given initial data $x_0$ and $x_1$, but a cubic equation always has a solution for any given initial data $x_0$ and $x_1$. Thus, instead of (4), we introduce the following cubic equation for $x_{n+1}$

$$\begin{aligned} & f(x_n) + f'(x_n)(x_{n+1} - x_n) + \frac{1}{2} f''(x_n)(x_{n+1} - x_n)^2 \\ & + \frac{1}{3!} f'''(x_n)(x_{n+1} - x_n)^3 + g(x_n) = 0, \end{aligned} \tag{8}$$

where

$$\begin{aligned} g(x_n) \;=\; & f(x_n) - f(x_{n-1}) - f'(x_{n-1})(x_n - x_{n-1}) - \frac{1}{2} f''(x_{n-1})(x_n - x_{n-1})^2 \\ & - \frac{1}{3!} f'''(x_{n-1})(x_n - x_{n-1})^3. \end{aligned} \tag{9}$$

It is known that the error estimates for Newton method (2) (or (3)) and the improved method (4) are $O(1)|x_{n+1} - x_n|^2$ and $O(1)|x_{n+1} - x_n|^3$, respectively, but our new improved method (8) has a much better error estimate $O(1)|x_{n+1} - x_n|^4$. So, the convergence of ours is much faster. On the other hand, we don't have any restriction on the selection of the initial data. Therefore, our method improves (4) essentially.

Now we are going to carry out some numerical computations. For the comparability, we still use the same example as shown in [2] (Example 2 therein).

**Example**. For the algebraic equation

$$f(x) = x^3 - e^{-x} = 0, \tag{10}$$

it has a unique solution $x = 0.7728829591 \cdots$. In order to find its numerical solution for error within $10^{-10}$, we use the above-mentioned three methods for different initial data $x_0$ and $x_1$ to solve Eq. (10), and then compare their convergence as well as their effectiveness.

1). For the initial points selected as $x_0 = x_1 = 0$, or $x_0 = 1$ and $x_1 = 2$, respectively, we find that Newton method (2) and our method (8) work out very well, but the iteration scheme (4) doesn't run out anymore, even at the first step. See the following numerical results, which indicate that the effectiveness of Newton method (2) and ours (8) is better than that of the method (4), and furthermore, the convergence of our method (8) is the fastest one among these three iteration methods.

| | Newton Method (2) | Method (4) | Method (8) |
|---|---|---|---|
| initial point(s) | $x_1 = 0$ | $x_0 = 0,\ \ x_1 = 0$ | $x_0 = 0,\ \ x_1 = 0$ |
| $x_2$ | 1 | doesn't exist | 0.7673157381 |
| $x_3$ | 0.8123090301 | | 0.7778393341 |
| $x_4$ | 0.7742765490 | | 0.7728829591 |
| $x_5$ | 0.7728847562 | | |
| $x_6$ | 0.7728829480 | | |
| $x_7$ | 0.7728829591 | | |

and

| | Newton Method (2) | Method (4) | Method (8) |
|---|---|---|---|
| initial point(s) | $x_1 = 2$ | $x_0 = 1,\ \ x_1 = 2$ | $x_0 = 1,\ \ x_1 = 2$ |
| $x_2$ | 1.351920278 | doesn't exist | 0.7710623232 |
| $x_3$ | 0.9666503345 | | 0.7802885533 |
| $x_4$ | 0.8024033817 | | 0.7728829591 |
| $x_5$ | 0.7736707548 | | |
| $x_6$ | 0.7728835337 | | |
| $x_7$ | 0.7728829591 | | |

2). For the initial points selected as $x_0 = 0$ and $x_1 = 0.5$, the mentioned three methods all work excellently. The fastest convergent method is our iteration scheme (8). In fact, as showed below, our iteration method (8) needs only 3 steps to get the numerical solution for error within $10^{-10}$, while Newton method (2) needs 6 steps, and the method (4) needs 5 steps.

| | Newton Method (2) | Method (4) | Method (8) |
|---|---|---|---|
| initial point(s) | $x_1 = 0.5$ | $x_0 = 0, x_1 = 0.5$ | $x_0 = 0, x_1 = 0.5$ |
| $x_2$ | 0.8549721904 | 0.7102225862 | 0.7738712000 |
| $x_3$ | 0.7787105282 | 0.7684413700 | 0.7729427372 |
| $x_4$ | 0.7729142691 | 0.7727883640 | 0.7728829591 |
| $x_5$ | 0.7728829601 | 0.7728829197 | |
| $x_6$ | 0.7728829581 | 0.7728829591 | |
| $x_7$ | 0.7728829591 | | |

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