Proceedings of the Royal Society of Edinburgh, **140A**, 135–152, *2010*

On travelling wavefronts of Nicholson's blowflies equation with diffusion

Chi-Kun Lin
Department of Applied Mathematics, National Chiao Tung University, Hsinchu 30010, Taiwan (cklin@math.nctu.edu.tw)

Ming Mei
Department of Mathematics, Champlain College Saint-Lambert, Saint-Lambert, Quebec J4P 3P2, Canada, and
Department of Mathematics and Statistics, McGill University, Montreal, Quebec H3G 1M8, Canada
(mmei@champlaincollege.qc.ca; mei@math.mcgill.ca)

(MS received 23 July 2008; accepted 28 April 2009)

This paper is devoted to the study of Nicholson's blowflies equation with diffusion: a kind of time-delayed reaction diffusion. For any travelling wavefront with speed $c > c^*$ (c^* is the minimum wave speed), we prove that the wavefront is time-asymptotically stable when the delay-time is sufficiently small, and the initial perturbation around the wavefront decays to zero exponentially in space as $x \to -\infty$, but it can be large in other locations. The result develops and improves the previous wave stability obtained by Mei *et al.* in 2004. The new approach developed in this paper is the comparison principle combined with the technical weighted-energy method. Numerical simulations are also carried out to confirm our theoretical results.

1. Introduction

We consider a time-delayed reaction–diffusion equation, the so-called Nicholson blowflies equation with diffusion

$$\frac{\partial N(t,x)}{\partial t} - \frac{\partial^2 N(t,x)}{\partial x^2} + dN(t,x) = pf(N(t-r,x)), \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}, \tag{1.1}$$

with the initial datum

$$N(s,x) = N_0(s,x), \quad s \in [-r,0], \quad x \in \mathbb{R}, \tag{1.2}$$

which demonstrates the distribution of the Australian sheep blowfly population, where $N(t,x)$ denotes the population of the blowflies at time t and location x, $d > 0$ is the death rate of the mature population, $r > 0$ is the maturation delay, the time required for a newborn to become matured, $p > 0$ is the impact of the birth on the immature population and $f(N)$ is the birth-rate function, which satisfies the following:

(H$_1$) there exist $N_- = 0$ and $N_+ > 0$ such that $f(0) = 0$, $f'(0) = 1$, $pf(N_+) = dN_+$ and $pf'(N_+) < d$;

(H_2) $0 \leqslant f'(N) \leqslant 1$ and $f''(N) \leqslant 0$ for $0 = N_- \leqslant N \leqslant N_+$.

Here, $N_\pm$ are the equilibria of (1.1). The important prototypes for such a birth-rate function are

$$f_1(N) = N\mathrm{e}^{-aN^q}, \quad f_2(N) = \frac{N}{1+aN^q}, \quad a > 0,\ q > 0,$$

where, for $f_1(N)$, we have

$$N_- = 0 \quad \text{and} \quad N_+ = \left(\frac{1}{a}\ln\frac{p}{d}\right)^{1/q}, \quad 1 < \frac{p}{d} \leqslant \mathrm{e};$$

for $f_2(N)$, we have

$$N_- = 0 \quad \text{and} \quad N_+ = \left(\frac{p-d}{ad}\right)^{1/q}$$

with

$$1 < \frac{p}{d} \leqslant \frac{q}{q-1} \text{ if } q > 1 \quad \text{or} \quad 1 < \frac{p}{d} < \infty \text{ if } q \leqslant 1.$$

The existence of travelling wavefronts for (1.1) with such birth-rate functions are studied in [5]. In particular, when $q = 1$, $f_1(N)$ is just Nicholson's birth-rate function

$$f(N(t-r,x)) = N(t-r,x)\mathrm{e}^{-aN(t-r,x)} \tag{1.3}$$

with $N_- = 0$ and

$$N_+ = \frac{1}{a}\ln\frac{p}{d} > 0 \quad \text{for } 1 < \frac{p}{d} \leqslant \mathrm{e}.$$

Lucilia cuprina is a kind of Australian blowfly. These blowflies lay their eggs on sheep, and soon the eggs become maggots, which feed on the host. As a result, the injured sheep may die. So, in order to eliminate the blowflies, it is useful to investigate changes in their population. In the 1940s, Nicholson [10, 11] made the pioneering study on the distribution of the blowflies' population over time. Based on Nicholson's experimental data, Gurney *et al.* [3] established a dynamical model, the so-called Nicholson blowflies equation,

$$N'(t) + dN(t) = pN(t-r)\mathrm{e}^{-aN(t-r)}.$$

On the other hand, it is necessary and interesting also to consider the spatial diffusion of blowflies. This leads naturally to the study of the time-delayed reaction–diffusion equation, (1.1), which was first studied by So and Yang [15, 17, 25] in 1998 for Hopf bifurcations and the stability of steady-state solutions. Since then, a number of in-depth research works on this model have been published (see, for example, [5, 7, 8, 12, 15–17, 20, 22, 25] and the references therein).

In [16], by the upper–lower solutions method, So and Zou proved the existence of wavefronts (a special solution in the form $\phi(x+ct)$, where c is the speed) for (1.1) related to two constant states $N_\pm$ (the constant equilibria). More precisely, there exists a number $c^* > 0$ (called the critical wave speed, or the minimum wave speed); the wavefront $\phi(x+ct)$ exists for any $c > c^*$. As pointed out in [22], when $r = 0$ (i.e. there is no time delay), the critical wave speed is $c^* = 2\sqrt{p-d}$, and when $r \to \infty$ the minimal wave speed is $c^* = O(1)(1/r) \to 0$.

Regarding the stability of travelling waves of the model (1.1), by using the technical weighted energy method, Mei *et al*. [8] proved that the wavefront is asymptotically stable in time when the wave speed is as large as $c > 2\sqrt{p-d}$ and the initial perturbation around the wavefront is sufficiently small. However, such a stability result is not satisfied; the stability of slower wavefronts (i.e. the wave speed satisfies $c^* < c \leqslant 2\sqrt{p-d}$) is much more interesting, but remains an open question. On the other hand, although we usually need to restrict the initial perturbation around the wavefront to be small for the wave stability proof, the wave stability for a large initial perturbation is much more significant and important from both a mathematical and a physical point of view. This is the so-called large-initial-perturbation problem. Unfortunately, the stability of the wavefronts with a large initial perturbation for (1.1) is also unknown. The solution of these two problems is the main aim of this paper.

Since the usual weighted energy method is no longer applicable, we develop a new approach, combining the comparison principle and the weighted energy method, to prove the stability for all waves (i.e. $c > c^*$) with a large initial perturbation in a weighted Sobolev space. Inspired by Gourley's elegant estimate in [2], we can also show the stability for all wavefronts when the delay time r is sufficiently small. The selection of a suitable weight function is also crucial, which is different from the case in [7,8], and can guarantee stability for all wavefronts. Of course, it is necessary that the initial perturbation around the wavefront decays to zero exponentially in space as $x \to -\infty$, but it can be allowed to be large in other locations. For research related to other population-dynamical models, we refer the reader to [1, 2, 4, 7, 9, 14, 18, 24] and the references therein (see also [19, 21, 23, 26]).

Notation. Throughout the paper, $C > 0$ denotes a generic constant, while $C_i > 0$, $i = 0, 1, 2, \dots$, represents a specific constant. Let I be an interval, typically $I = \mathbb{R}$. $L^2(I)$ is the space of the square integrable functions on I, and $H^k(I)$, $k \geqslant 0$, is the Sobolev space of the L^2-functions $f(x)$ defined on the interval I whose derivatives $\mathrm{d}^i f/\mathrm{d}x^i$, $i = 1, \dots, k$, also belong to $L^2(I)$. Here $L^2_w(I)$ represents the weighted L^2-space with the weight $w(x) > 0$ and its norm is defined by

$$\|f\|_{L^2_w} = \left(\int_I w(x)|f(x)|^2 \,\mathrm{d}x \right)^{1/2};$$

$H^k_w(I)$ is the weighted Sobolev space with the norm

$$\|f\|_{H^k_w} = \left(\sum_{i=0}^{k} \int_I w(x) \left| \frac{\mathrm{d}^i}{\mathrm{d}x^i} f(x) \right|^2 \mathrm{d}x \right)^{1/2}.$$

Let $T > 0$ and let $\mathcal{B}$ be a Banach space. We denote by $C^0([0,T];\mathcal{B})$ the space of the $\mathcal{B}$-valued continuous functions on $[0,T]$, and by $L^2([0,T];\mathcal{B})$ the space of the $\mathcal{B}$-valued L^2-functions on $[0,T]$. The corresponding spaces of the $\mathcal{B}$-valued functions on $[0,\infty)$ are defined similarly.

In what follows, we shall prove the new stability results. In § 2, we first introduce the travelling wavefronts and their properties, as well as the known nonlinear stability result, then we state our new stability results in two different cases: with or without time delay. In § 3, after establishing the comparison principle and some key energy estimates in the weighted Sobolev spaces, where the suitably selected weight function plays a crucial role, we then prove the nonlinear stability. In § 4, we carry out numerical simulations in two cases that confirm our theoretical results stated in § 2.

2. Preliminaries and main theorems

A *travelling wavefront* of equation (1.1) connecting with the constant states $N_\pm$ is a monotone solution $N(t,x) = \phi(x+ct)$ satisfying the second-order ordinary differential equation

$$\left.\begin{aligned} c\phi'(\xi) - \phi''(\xi) + d\phi(\xi) = pf(\phi(\xi - cr)),\\ \phi(\pm\infty) = N_\pm, \end{aligned}\right\} \tag{2.1}$$

where $\xi = x + ct$.

When $f(N) = N\mathrm{e}^{-aN}$, $f(N) = f_1(N)$ or $f(N) = f_2(N)$, the existence of the wavefront was shown by So and Zou [16] and Liang and Wu [5] using the upper–lower-solutions method which is a corollary of the result for a more general case studied earlier by Schaaf [13]. By the same approach, we can similarly prove the existence of travelling waves for (1.1) with a general birth-rate function $f(N)$ satisfying (H_1) and (H_2). Details of the proof are omitted.

Proposition 2.1. *Under assumptions (H_1) and (H_2), there exists a minimum speed $c^* = c^*(r)$ satisfying*

$$F_{c^*}(\lambda_{c^*}) = G_{c^*}(\lambda_{c^*}), \qquad F'_{c^*}(\lambda_{c^*}) = G'_{c^*}(\lambda_{c^*}), \tag{2.2}$$

where

$$F_c(\lambda) = p\mathrm{e}^{-\lambda cr}, \qquad G_c(\lambda) = c\lambda - \lambda^2 + d, \tag{2.3}$$

and (c^*, λ_{c^*}) is the tangent point of $F_c(\lambda)$ and $G_c(\lambda)$. Then for all $c > c^*$, the travelling wavefront $\phi(x+ct)$ of equation (1.1) connecting $N_\pm$ exists.

In particular, when $r = 0$, equation (1.1) becomes the regular reaction–diffusion equation without time delay, and its critical wave speed is $c^* = 2\sqrt{p-d}$.

Now we define two different weight functions:

$$w_2(x) = \begin{cases} \exp\left\{ \displaystyle\int_{x_2}^{x} \gamma(\xi)\,\mathrm{d}\xi \right\} & \text{for } x \leqslant x_2, \\ 1 & \text{for } x > x_2, \end{cases} \tag{2.4}$$

where $\gamma(\xi)$ is selected such that

$$-c - \sqrt{c^2 - 4[pf'(\phi(\xi)) - d]} < \gamma(\xi) < -c + \sqrt{c^2 - 4[pf'(\phi(\xi)) - d]}, \tag{2.5}$$

$\phi(\xi)$ is the given travelling wavefront, and the number x_2 is selected to be sufficiently large such that (3.28) holds, i.e.

$$d - pf'(\phi(x_2)) > 0.$$

Another weight function is defined as

$$w_3(x) = \begin{cases} \exp\{-kc(x - x_3 - cr)\} & \text{for } x \leqslant x_3 + cr, \\ 1 & \text{for } x > x_3 + cr, \end{cases} \tag{2.6}$$

where the number x_3 satisfies (3.36), i.e. we need

$$d - pf'(\phi(x_3 - cr)) \cosh(c^2 r) > 0.$$

The number k in (2.6) satisfies $1 < k < 2$ and

$$4p \exp\{-\tfrac{1}{2}kc^2 r\} < 4d - k^2c^2 + 2kc^2. \tag{2.7}$$

The above inequality can be verified for some $k \in (1, 2)$, if

$$4p \exp\{-\tfrac{1}{2}c^2 r\} < 4d + c^2 \tag{2.8}$$

holds. In fact, consider two functions

$$g_1(k) := 4p \exp\{-\tfrac{1}{2}kc^2 r\} \quad \text{and} \quad g_2(k) := 4d - k^2c^2 + 2kc^2.$$

Since the maximum value of $g_2(k)$ is

$$g_2(1) = 4d + c^2 > 4p \exp\{-\tfrac{1}{2}c^2 r\} = g_1(1),$$

we can conclude that, for some k around 1, the graph of $g_2(k)$ is above $g_1(k)$. Thus, we can then select some $k \in (1, 2)$ such that (2.7) holds.

We now state our main results.

Theorem 2.2 (the case when $r = 0$). *Let $r = 0$. Under assumptions (H_1) and (H_2), for any given wavefront $\phi(x + ct)$ with speed $c > c^* = 2\sqrt{p - d}$, if the initial data satisfy*

$$N_- = 0 \leqslant N_0(0, x) \leqslant N_+ \quad \textit{for } x \in \mathbb{R}, \tag{2.9}$$

and the initial perturbation is $N_0(s, x) - \phi(x + cs) \in C([-r, 0]; H^1_{w_2}(R))$, then the solution of (1.1) satisfies

$$N_- = 0 \leqslant N(t, x) \leqslant N_+ \quad \textit{for } (t, x) \in \mathbb{R}_+ \times \mathbb{R}, \tag{2.10}$$

and

$$N(t, x) - \phi(x + ct) \in C([0, \infty); H^1_{w_2}(\mathbb{R})). \tag{2.11}$$

In particular, the solution $N(t, x)$ converges to the wavefront $\phi(x+ct)$ exponentially in time:

$$\sup_{x \in \mathbb{R}} |N(t, x) - \phi(x + ct)| \leqslant C\mathrm{e}^{-\mu_2 t}, \quad t \geqslant 0, \tag{2.12}$$

for some positive number μ_2.

Theorem 2.3 (the case when $r > 0$). *Let $r > 0$. Under the assumptions (H_1), (H_2), (2.8) and*

$$\cosh(c^2 r) < \frac{d}{pf'(N_+)}, \tag{2.13}$$

for any given wavefront $\phi(x+ct)$ with speed $c > c^*$, if the initial data satisfy

$$N_- = 0 \leqslant N_0(s,x) \leqslant N_+ \quad \textit{for } (s,x) \in [-r,0] \times \mathbb{R}, \tag{2.14}$$

and the initial perturbation is $N_0(s,x) - \phi(x+cs) \in C([-r,0]; H^1_{w_3}(\mathbb{R}))$, then the solution of (1.1) satisfies

$$N_- = 0 \leqslant N(t,x) \leqslant N_+ \quad \textit{for } (t,x) \in \mathbb{R}_+ \times \mathbb{R} \tag{2.15}$$

and

$$N(t,x) - \phi(x+ct) \in C([0,\infty); H^1_{w_3}(\mathbb{R})). \tag{2.16}$$

In particular, the solution $N(t,x)$ converges to the wavefront $\phi(x+ct)$ time-exponentially

$$\sup_{x\in\mathbb{R}} |N(t,x) - \phi(x+ct)| \leqslant C\mathrm{e}^{-\mu_3 t}, \quad t \geqslant 0, \tag{2.17}$$

for some positive number μ_3.

Remark 2.4.

(i) Theorem 2.2 shows that the stability holds for *all* wavefronts if the initial perturbation around the wavefront is exponentially decaying as $x \to -\infty$ (namely, $N_0(s,x) - \phi(x+cs) \sim \mathrm{e}^{-c|x|/2}$ as $x \to -\infty$). This decay can be obtained by the Sobolev embedding inequality and the definition of $w_2(x)$. Although when $r = 0$ the stability shown in [8] also holds for the wavefronts with speed $c > 2\sqrt{p-d} = c^*$, the initial perturbation around the wavefront in a weighted Sobolev space is restricted to be sufficiently small. However, here we do not need such a condition for wavefront stability.

(ii) For theorem 2.3, under the two sufficient conditions (2.8) and (2.13), the stability holds for all wavefronts, including the slower wavefronts with $c^* < c \leqslant 2\sqrt{p-d}$, if the initial perturbation around the wavefront is exponentially decaying as $x \to -\infty$ (namely, $N_0(s,x) - \phi(x+cs) \sim \mathrm{e}^{-kc|x|/2}$ as $x \to -\infty$). No smallness condition is needed for the initial perturbation $N_0(s,x) - \phi(x+cs)$ in $H^1_{w_3}(\mathbb{R})$. When $r \ll 1$, from assumption (H_1), it can be verified that the sufficient conditions (2.8) and (2.13) are satisfied automatically.

(iii) Both theorems 2.2 and 2.3 essentially improve the previous stability results obtained in [8].

Regarding the smallness of the delay time $r > 0$, biologically, the time to maturity (i.e. the delay time r) for the blowflies is *ca.* 10–25 days (see http://ipm.ncsu.edu/AG369/notes/blow_flies.html), so we believe that r actually is small. Thus, from theorem 2.3, we immediately obtain the following corollary.

Corollary 2.5. *Under assumptions (H_1) and (H_2), when $r > 0$ is small enough, for any given wavefront $\phi(x+ct)$ with speed $c > c^*$, if the initial data satisfy (2.14) and the initial perturbation is $N_0(s,x) - \phi(x+cs) \in C([-r,0]; H^1_{w_3}(\mathbb{R}))$, then the solution $N(t,x)$ of (1.1) converges to the wavefront $\phi(x+ct)$ exponentially in time:*

$$\sup_{x\in\mathbb{R}} |N(t,x) - \phi(x+ct)| \leqslant C\mathrm{e}^{-\mu_3 t}, \quad t \geqslant 0, \tag{2.18}$$

for some positive number μ_3.

3. Proof of nonlinear stability

The existence and uniqueness of the solution for the initial-value problem (1.1), (1.2) has been proved in [8] (see also a different method in [23]). In order to prove theorems 2.2 and 2.3, we first need to establish the comparison principle for equation (1.1) and then, by using the comparison principle together with the weighted energy method, we can prove our new stability results in three different cases, according to the signs of the initial perturbation.

As shown by Martin and Smith in [6] (see also [4, 7]), we have the following two lemmas.

LEMMA 3.1 (boundedness). *Under assumptions (H_1) and (H_2), let*

$$N_- = 0 \leqslant N_0(s,x) \leqslant N_+ \quad \text{for } (s,x) \in [-r,0] \times \mathbb{R}. \tag{3.1}$$

Then the solution $N(t,x)$ of the Cauchy problem (1.1), (1.2) satisfies

$$N_- \leqslant N(t,x) \leqslant N_+ \quad \text{for } (t,x) \in [0,\infty) \times \mathbb{R}. \tag{3.2}$$

LEMMA 3.2 (comparison principle). *Under assumptions (H_1) and (H_2), let $\bar{N}(t,x)$ and $\underline{N}(t,x)$ be the solutions of (1.1) and (1.2) with the initial data $\bar{N}_0(s,x)$ and $\underline{N}_0(s,x)$, respectively. If*

$$N_+ \geqslant \bar{N}_0(s,x) \geqslant \underline{N}_0(s,x) \geqslant N_- \quad \text{for } (s,x) \in [-r,0] \times \mathbb{R}, \tag{3.3}$$

then

$$N_+ \geqslant \bar{N}(t,x) \geqslant \underline{N}(t,x) \geqslant N_- \qquad \text{for } (t,x) \in \mathbb{R}_+ \times \mathbb{R}. \tag{3.4}$$

In what follows, we prove theorems 2.2 and 2.3 by means of the comparison principle together with the weighed energy method. In the proof, we will also show how to select suitable weight functions that play a key role in the stability proof by the weighted energy method, and how to obtain stability with a large initial perturbation by using the comparison principle.

For given initial data $N_0(s,x)$ satisfying

$$N_- \leqslant N_0(s,x) \leqslant N_+ \quad \text{for } (s,x) \in [-r,0] \times \mathbb{R},$$

let

$$\left.\begin{aligned} V_0^+(s,x) &= \max\{N_0(s,x), \phi(x+cs)\} \\ V_0^-(s,x) &= \min\{N_0(s,x), \phi(x+cs)\} \end{aligned}\right\} \quad \text{for } (s,x) \in [-r,0] \times \mathbb{R}, \tag{3.5}$$

so

$$N_- \leqslant V_0^-(s,x) \leqslant N_0(s,x) \leqslant V_0^+(s,x) \leqslant N_+ \qquad \text{for } (s,x) \in [-r,0] \times \mathbb{R}, \tag{3.6}$$

$$N_- \leqslant V_0^-(s,x) \leqslant \phi(x+cs) \leqslant V_0^+(s,x) \leqslant N_+ \quad \text{for } (s,x) \in [-r,0] \times \mathbb{R}. \tag{3.7}$$

Denote by $V^+(t,x)$ and $V^-(t,x)$ the corresponding solutions of (1.1), (1.2) with respect to the above-mentioned initial data $V_0^+(s,x)$ and $V_0^-(s,x)$, i.e.

$$\left.\begin{aligned} V_t^\pm - V_{xx}^\pm + dV^\pm &= pf(V^\pm(t-r,x)), \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}, \\ V^\pm(s,x) &= V_0^\pm(s,x), \qquad\qquad\qquad x \in \mathbb{R}, \quad s \in [-r,0]. \end{aligned}\right\} \tag{3.8}$$

By the comparison principle, we have

$$N_- \leqslant V^-(t,x) \leqslant N(t,x) \leqslant V^+(t,x) \leqslant N_+ \quad \text{for } (t,x) \in \mathbb{R}_+ \times \mathbb{R}, \tag{3.9}$$

$$N_- \leqslant V^-(t,x) \leqslant \phi(x+ct) \leqslant V^+(t,x) \leqslant N_+ \quad \text{for } (t,x) \in \mathbb{R}_+ \times \mathbb{R}. \tag{3.10}$$

We now prove the stability of the travelling wavefronts presented in theorems 2.2 and 2.3 in three cases.

Case 1 (convergence of $V^+(t,x)$ to $\phi(x+ct)$). Let $\xi := x + ct$ and

$$v(t,\xi) := V^+(t,x) - \phi(x+ct), \qquad v_0(s,\xi) = V_0^+(s,x) - \phi(x+cs). \tag{3.11}$$

Then by (3.6) and (3.9) we have

$$v(t,\xi) \geqslant 0, \qquad v_0(s,\xi) \geqslant 0. \tag{3.12}$$

From (1.1), it can be verified that $v(t,\xi)$ defined in (3.11) satisfies

$$\left.\begin{aligned} v_t + cv_\xi - v_{\xi\xi} + dv - pf'(\phi(\xi - cr))v(t-r,\xi-cr) & \\ = pQ(t-r,\xi-cr), \qquad & (t,x) \in \mathbb{R}_+ \times \mathbb{R}, \\ v(s,\xi) = v_0(s,\xi), \qquad & (s,x) \in [-r,0] \times \mathbb{R}, \end{aligned}\right\} \tag{3.13}$$

where

$$Q(t-r,\xi-cr) = f(\phi+v) - f(\phi) - f'(\phi)v \tag{3.14}$$

with $\phi = \phi(\xi - cr)$ and $v = v(t-r,\xi-cr)$.

Let $w(\xi) > 0$ be a weight function which will be specified later. Multiplying (3.13) by $\mathrm{e}^{2\mu t}w(\xi)v(t,\xi)$, where $\mu > 0$ will also be determined later, we obtain

$$\begin{aligned} \{\tfrac{1}{2}\mathrm{e}^{2\mu t}wv^2\}_t + \{\tfrac{1}{2}\mathrm{e}^{2\mu t}cwv^2 - \mathrm{e}^{2\mu t}wvv_\xi\}_\xi & \\ + \mathrm{e}^{2\mu t}wv_\xi^2 + \mathrm{e}^{2\mu t}w'v_\xi v & \\ + \left\{-\frac{1}{2}c\frac{w'}{w} + d - \mu\right\}\mathrm{e}^{2\mu t}wv^2 & \\ - p\mathrm{e}^{2\mu t}wvf'(\phi(\xi-cr))v(t-r,\xi-cr) & \\ = p\mathrm{e}^{2\mu t}wvQ(t-r,\xi-cr). & \end{aligned} \tag{3.15}$$

By the Cauchy inequality

$$|xy| \leqslant \frac{\varepsilon}{2}x^2 + \frac{1}{2\varepsilon}y^2 \quad \text{for any } \varepsilon > 0,$$

we have, by taking $\varepsilon = 2$, that

$$|\mathrm{e}^{2\mu t}w'vv_\xi| = \mathrm{e}^{2\mu t}w\left|v_\xi\frac{w'}{w}v\right| \leqslant \mathrm{e}^{2\mu t}wv_\xi^2 + \tfrac{1}{4}\mathrm{e}^{2\mu t}\left(\frac{w'}{w}\right)^2 wv^2.$$

Substituting the above inequality into (3.15), we obtain

$$\{\tfrac{1}{2}\mathrm{e}^{2\mu t}wv^2\}_t + \{\tfrac{1}{2}\mathrm{e}^{2\mu t}cwv^2 - \mathrm{e}^{2\mu t}wvv_\xi\}_\xi + \left\{-\frac{1}{2}c\frac{w'}{w} + d - \mu - \frac{1}{4}\left(\frac{w'}{w}\right)^2\right\}\mathrm{e}^{2\mu t}wv^2$$
$$- p\mathrm{e}^{2\mu t}wvf'(\phi(\xi - cr))v(t-r,\xi-cr)$$
$$= p\mathrm{e}^{2\mu t}wvQ(t-r,\xi-cr). \tag{3.16}$$

Integrating (3.16) over $[0,t]\times\mathbb{R}$ yields

$$\mathrm{e}^{2\mu t}\|v(t)\|^2_{L^2_w} + \int_0^t\int_R \mathrm{e}^{2\mu\tau}\left\{-c\frac{w'(\xi)}{w(\xi)} + 2d - 2\mu - \frac{1}{2}\left(\frac{w'(\xi)}{w(\xi)}\right)^2\right\}w(\xi)v^2(\tau,\xi)\,\mathrm{d}\xi\,\mathrm{d}\tau$$
$$- 2p\int_0^t\int_R \mathrm{e}^{2\mu\tau}w(\xi)f'(\phi(\xi-cr))v(\tau,\xi)v(\tau-r,\xi-cr)\,\mathrm{d}\xi\,\mathrm{d}\tau$$
$$\leqslant \|v_0(0)\|^2_{L^2_w} + 2p\int_0^t\int_R \mathrm{e}^{2\mu\tau}w(\xi)v(\tau,\xi)Q(\tau-r,\xi-cr). \tag{3.17}$$

Notice that, from (H_2), $f'(\phi) > 0$ for $N_- < \phi(\xi) < N_+$. Again, using the Cauchy inequality we obtain

$$|2p\mathrm{e}^{2\mu\tau}w(\xi)f'(\phi(\xi-cr))v(\tau,\xi)v(\tau-r,\xi-cr)|$$
$$\leqslant p\mathrm{e}^{2\mu\tau}w(\xi)f'(\phi(\xi-cr))\left[\eta v^2(\tau,\xi) + \frac{1}{\eta}v^2(\tau-r,\xi-cr)\right]$$

for any positive constant η, which will be specified later. Thus, the third term on the left-hand-side of (3.17) is reduced to

$$\left|2p\int_0^t\int_R \mathrm{e}^{2\mu\tau}w(\xi)f'(\phi(\xi-cr))v(\tau,\xi)v(\tau-r,\xi-cr)\,\mathrm{d}\xi\,\mathrm{d}\tau\right|$$
$$\leqslant p\int_0^t\int_R \mathrm{e}^{2\mu\tau}w(\xi)f'(\phi(\xi-cr))\left[\eta v^2(\tau,\xi) + \frac{1}{\eta}v^2(\tau-r,\xi-cr)\right]\mathrm{d}\xi\,\mathrm{d}\tau$$
$$= p\eta\int_0^t\int_R \mathrm{e}^{2\mu\tau}w(\xi)f'(\phi(\xi-cr))v^2(\tau,\xi)\,\mathrm{d}\xi\,\mathrm{d}\tau$$
$$+ \frac{p}{\eta}\int_0^t\int_R \mathrm{e}^{2\mu\tau}w(\xi)f'(\phi(\xi-cr))v^2(\tau-r,\xi-cr)\,\mathrm{d}\xi\,\mathrm{d}\tau$$
$$[\text{change of variables: } \xi - cr \to \xi,\ \tau - r \to \tau]$$
$$= p\eta\int_0^t\int_R \mathrm{e}^{2\mu\tau}w(\xi)f'(\phi(\xi-cr))v^2(\tau,\xi)\,\mathrm{d}\xi\,\mathrm{d}\tau$$
$$+ \frac{p}{\eta}\mathrm{e}^{2\mu r}\int_{-r}^{t-r}\int_R \mathrm{e}^{2\mu\tau}w(\xi+cr)f'(\phi(\xi))v^2(\tau,\xi)\,\mathrm{d}\xi\,\mathrm{d}\tau$$
$$\leqslant p\eta\int_0^t\int_R \mathrm{e}^{2\mu\tau}w(\xi)f'(\phi(\xi-cr))v^2(\tau,\xi)\,\mathrm{d}\xi\,\mathrm{d}\tau$$

$$
\begin{aligned}
&+\frac{p}{\eta}\mathrm{e}^{2\mu r}\int_0^t\int_R \mathrm{e}^{2\mu\tau}w(\xi+cr)f'(\phi(\xi))v^2(\tau,\xi)\,\mathrm{d}\xi\,\mathrm{d}\tau\\
&+\frac{p}{\eta}\mathrm{e}^{2\mu r}\int_{-r}^0\int_R \mathrm{e}^{2\mu\tau}w(\xi+cr)f'(\phi(\xi))v_0^2(\tau,\xi)\,\mathrm{d}\xi\,\mathrm{d}\tau. \qquad (3.18)
\end{aligned}
$$

Substituting (3.18) into (3.17) leads to

$$
\begin{aligned}
\mathrm{e}^{2\mu t}\|v(t)\|^2_{L^2_w} &+ \int_0^t\int_R \mathrm{e}^{2\mu\tau}B_{\eta,\mu,w}(\xi)w(\xi)v^2(\tau,\xi)\,\mathrm{d}\xi\,\mathrm{d}\tau\\
&\leqslant \|v_0(0)\|^2_{L^2_w} + 2p\int_0^t\int_R \mathrm{e}^{2\mu\tau}w(\xi)v(\tau,\xi)Q(\tau-r,\xi-cr)\,\mathrm{d}\xi\,\mathrm{d}\tau\\
&\qquad + \frac{p}{\eta}\mathrm{e}^{2\mu r}\int_{-r}^0\int_R \mathrm{e}^{2\mu\tau}w(\xi+cr)f'(\phi(\xi))v_0^2(s,\xi)\,\mathrm{d}\xi\,\mathrm{d}\tau. \qquad (3.19)
\end{aligned}
$$

where

$$
\begin{aligned}
B_{\eta,\mu,w}(\xi) := -c\frac{w'(\xi)}{w(\xi)} + 2d - 2\mu - \frac{1}{2}\left(\frac{w'(\xi)}{w(\xi)}\right)^2&\\
- p\eta f'(\phi(\xi-cr)) - \frac{p}{\eta}\mathrm{e}^{2\mu r}\frac{w(\xi+cr)}{w(\xi)}f'(\phi(\xi)).& \qquad (3.20)
\end{aligned}
$$

For the nonlinearity $Q(t-r,\xi-cr) = f(\phi+v) - f(\phi) - f'(\phi)v$, using Taylor's formula, we have

$$
Q(t-r,\xi-cr) = f(\phi+v) - f(\phi) - f'(\phi)v = \frac{f''(\tilde{\phi})}{2!}v^2, \qquad (3.21)
$$

where $\tilde{\phi}$ is some function between ϕ and $\phi+v = V^+(t,x)$, i.e. $\phi \leqslant \tilde{\phi} \leqslant V^+(t,x)$. Since both ϕ and $V^+(t,x)$ satisfy $0 \leqslant \phi \leqslant N_+$ and $0 \leqslant V^+(t,x) \leqslant N_+$, we then have $0 \leqslant \tilde{\phi} \leqslant N_+$, which combines with assumption (H_2) to ensure that

$$
f''(\tilde{\phi}) \leqslant 0, \quad \text{i.e. } Q(t-r,\xi-cr) \leqslant 0 \text{ for } (t,x) \in \mathbb{R}_+ \times \mathbb{R}.
$$

Thus, using the facts that $Q(t-r,\xi-cr) \leqslant 0$ and $v(t,\xi) \geqslant 0$ (see (3.12)), we obtain

$$
2p\int_0^t\int_R \mathrm{e}^{2\mu\tau}w(\xi)v(\tau,\xi)Q(\tau-r,\xi-cr)\,\mathrm{d}\xi\,\mathrm{d}\tau \leqslant 0. \qquad (3.22)
$$

Substituting (3.22) into (3.19), we obtain

$$
\begin{aligned}
\mathrm{e}^{2\mu t}\|v(t)\|^2_{L^2_w} &+ \int_0^t\int_R \mathrm{e}^{2\mu\tau}B_{\eta,\mu,w}(\xi)w(\xi)v^2(\tau,\xi)\,\mathrm{d}\xi\,\mathrm{d}\tau\\
&\leqslant \|v_0(0)\|^2_{L^2_w} + \frac{p}{\eta}\mathrm{e}^{2\mu r}\int_{-r}^0\int_R \mathrm{e}^{2\mu\tau}w(\xi+cr)f'(\phi(\xi))v_0^2(s,\xi)\,\mathrm{d}\xi\,\mathrm{d}\tau\\
&\leqslant C_1\left(\|v_0(0)\|^2_{L^2_w} + \int_{-r}^0\|v_0(\tau)\|^2_{L^2_w}\,\mathrm{d}\tau\right) \qquad (3.23)
\end{aligned}
$$

for some positive constant C_1.

Let

$$A_{\eta,w}(\xi) := -c\frac{w'(\xi)}{w(\xi)} + 2d - \frac{1}{2}\left(\frac{w'(\xi)}{w(\xi)}\right)^2 - p\eta f'(\phi(\xi - cr)) - \frac{p}{\eta}\frac{w(\xi + cr)}{w(\xi)} f'(\phi(\xi)). \tag{3.24}$$

Then

$$B_{\eta,\mu,w}(\xi) = A_{\eta,w}(\xi) - 2\mu - \frac{p}{\eta}(\mathrm{e}^{2\mu r} - 1) f'(\phi(\xi))\frac{w(\xi + cr)}{w(\xi)}. \tag{3.25}$$

Next we prove $B_{\eta,\mu,w}(\xi) > 0$ by selecting some suitable weight function $w(\xi)$ and the exponent μ. This is one of key steps in the proof of the wavefront stability.

LEMMA 3.3 (the case when $r = 0$). *Under assumptions (H_1) and (H_2), for $r = 0$, let $w(\xi) = w_2(\xi)$, as defined in (2.4), and $\eta = \eta_2 = 1$. Then*

$$A_{\eta_2,w_2}(\xi) \geqslant C_2 > 0, \quad \xi \in \mathbb{R}, \tag{3.26}$$

for some positive constant C_2.

Proof.

CASE 1 ($\xi \leqslant x_2$). We have

$$\begin{aligned} A_{\eta_2,w_2}(\xi) &= -c\frac{w_2'}{w_2} + 2d - \frac{1}{2}\left(\frac{w_2'}{w_2}\right)^2 - 2pf'(\phi(\xi)) \\ &= -\frac{1}{2}\left[\left(\frac{w_2'}{w_2}\right)^2 + 2c\frac{w_2'}{w_2} + 4pf'(\phi(\xi)) - 4d\right] \\ &= -\frac{1}{2}\left[\frac{w_2'}{w_2} - (\sqrt{c^2 - 4(pf'(\phi(\xi)) - d)} - c)\right] \\ &\quad \times \left[\frac{w_2'}{w_2} + (\sqrt{c^2 - 4(pf'(\phi(\xi)) - d)} + c)\right] \\ &> 0, \end{aligned} \tag{3.27}$$

by selecting

$$w_2(\xi) = \exp\left\{\int_{x_2}^{\xi} \gamma(\zeta)\,\mathrm{d}\zeta\right\}$$

with

$$-c - \sqrt{c^2 - 4[pf'(\phi(\xi)) - d]} < \gamma(\xi) < -c + \sqrt{c^2 - 4[pf'(\phi(\xi)) - d]}.$$

Here, $c > c^* = 2\sqrt{p - d}$ and $0 < f'(\phi) < 1$ (see (H_2)) imply that $c^2 - 4[pf'(\phi(\xi)) - d] > 0$.

CASE 2 ($\xi > x_2$). In this case, we have $w_2(\xi) = w_2(\xi + cr) = 1$. From ($\mathrm{H}_2$), $f'(\phi)$ is decreasing in ϕ. Since $\phi(\xi)$ is increasing in ξ, this means that $f'(\phi(\xi))$ is decreasing in ξ, and satisfies $0 < f'(N_+) < f'(\phi(\xi)) \leqslant f'(\phi(x_2))$ for $\infty > \xi > x_2$. Notice that $d - pf'(N_+) > 0$ (see (H_1)). We can select x_2 to be sufficiently large such that $|\phi(x_2) - N_+| \ll 1$ and

$$d - pf'(\phi(x_2)) > 0. \tag{3.28}$$

Thus, we obtain

$$A_{\eta_2,w_2}(\xi) = 2[d - pf'(\phi(\xi))] \geqslant 2[d - pf'(\phi(x_2))] > 0. \tag{3.29}$$

Let

$$C_2 := \min_{\xi\in(-\infty,\infty)} A_{\eta_2,w_2}(\xi) > 0. \tag{3.30}$$

Combining (3.27) and (3.29), we prove (3.26). □

LEMMA 3.4 (the case when $r > 0$). *Under assumptions (H_1) and (H_2), for $r > 0$, let (2.8) and (2.13) hold, and let $w(\xi) = w_3(\xi)$, as defined in (2.6), and $\eta = \eta_3 = \exp\{-\frac{1}{2}kc^2r\}$. Then*

$$A_{\eta_3,w_3}(\xi) \geqslant C_3 > 0, \quad \xi \in \mathbb{R}, \tag{3.31}$$

for some positive constant C_3.

Proof.

CASE 1 ($\xi \leqslant x_3$). In this case, we have

$$w_3(\xi) = \exp\{-kc(\xi - x_3 - cr)\}, \qquad w_3(\xi + cr) = \exp\{-kc(\xi - x_3)\}$$

and, inspired by [2], we set $\eta = \eta_3 = \exp\{-\frac{1}{2}kc^2r\}$. From assumptions ($H_1$) and ($H_2$), we realize that $0 < f'(\phi(x_3)) \leqslant f'(\phi(\xi)) < f'(\phi(\xi - cr)) < f'(\phi(-\infty)) = f'(0) = 1$ for $\xi \in (-\infty, x_3]$. Thus, we obtain

$$\begin{aligned} A_{\eta_3,w_3}(\xi) &= kc^2 + 2d - \tfrac{1}{2}k^2c^2 - pf'(\phi(\xi - cr))\exp\{-\tfrac{1}{2}kc^2r\} \\ &\quad - pf'(\phi(\xi))\exp\{-\tfrac{1}{2}kc^2r\} \\ &\geqslant kc^2 + 2d - \tfrac{1}{2}k^2c^2 - 2p\exp\{-\tfrac{1}{2}kc^2r\} \\ &= \tfrac{1}{2}[4d - k^2c^2 + 2kc^2 - 4p\exp\{-\tfrac{1}{2}kc^2r\}] > 0, \end{aligned} \tag{3.32}$$

where we used (2.7) for the last step of (3.32), which can be guaranteed under the condition (2.8), as shown earlier.

CASE 2 ($x_3 < \xi \leqslant x_3 + cr$). In this case, we have

$$w_3(\xi) = \exp\{-kc(\xi - x_3 - cr)\} \quad w_3(\xi + cr) = 1 \quad \text{and} \quad \eta = \eta_3 = \exp\{-\tfrac{1}{2}kc^2r\}.$$

As shown before, $f'(\phi(\xi))$ is decreasing in ξ, namely, $f'(\phi(x_3 + cr)) < f'(\phi(\xi)) \leqslant f'(\phi(x_3)) < f'(\phi(\xi - cr)) \leqslant f'(\phi(x_3 - cr))$. Using the fact that $\exp\{kc(\xi - x_3 - cr)\} \leqslant e^0 = 1$ due to $\xi - x_3 - cr \leqslant 0$, we then obtain

$$\begin{aligned} A_{\eta_3,w_3}(\xi) &= kc^2 + 2d - \tfrac{1}{2}k^2c^2 - p\eta_3 f'(\phi(\xi - cr)) - \frac{p}{\eta_3} f'(\phi(\xi))e^{kc(\xi - x_3 - cr)} \\ &\geqslant kc^2 + 2d - \tfrac{1}{2}k^2c^2 - p\eta_3 f'(\phi(x_3 - cr)) - \frac{p}{\eta_3} f'(\phi(x_3 - cr)) \\ &= kc^2 + 2d - \tfrac{1}{2}k^2c^2 - 2pf'(\phi(x_3 - cr))\cosh(\tfrac{1}{2}kc^2r). \end{aligned} \tag{3.33}$$

Notice that $1 < k < 2$, and $\cosh x$ is increasing for $x > 0$, which implies that $0 < \cosh(\frac{1}{2}c^2 r) < \cosh(\frac{1}{2}kc^2 r) < \cosh(c^2 r)$. From (3.33) we then have

$$\begin{aligned} A_{\eta_3,w_3}(\xi) &\geqslant kc^2 + 2d - \tfrac{1}{2}k^2c^2 - 2pf'(\phi(x_3 - cr))\cosh(\tfrac{1}{2}kc^2 r) \\ &= \tfrac{1}{2}k(2-k)c^2 + 2d - 2pf'(\phi(x_3 - cr))\cosh(\tfrac{1}{2}kc^2 r) \\ &\geqslant 2d - 2pf'(\phi(x_3 - cr))\cosh(c^2 r). \end{aligned} \tag{3.34}$$

From (2.13), we have

$$d - pf'(N_+)\cosh(c^2 r) > 0. \tag{3.35}$$

Let x_3 be sufficiently large, i.e. $|\phi(x_3 - cr) - N_+| \ll 1$, such that (3.35) can imply

$$d - pf'(\phi(x_3 - cr))\cosh(c^2 r) > 0. \tag{3.36}$$

Applying this to (3.34), we then prove that

$$A_{\eta_3,w_3}(\xi) \geqslant 2[d - pf'(\phi(x_3 - cr))\cosh(c^2 r)] > 0. \tag{3.37}$$

Case 3 ($\xi > x_3 + cr$). In this case,

$$w_3(\xi) = w_3(\xi + cr) = 1 \quad \text{and} \quad \eta = \eta_3 = \exp\{-\tfrac{1}{2}kc^2 r\}.$$

Then

$$\begin{aligned} A_{\eta_3,w_3}(\xi) &= 2d - \eta_3 pf'(\phi(\xi - cr)) - \frac{p}{\eta_3} f'(\phi(\xi)) \\ &\geqslant 2d - \eta_3 pf'(\phi(x_3 - cr)) - \frac{p}{\eta_3} f'(\phi(x_3 - cr)) \\ &= 2d - 2pf'(\phi(x_3 - cr))\cosh(\tfrac{1}{2}kc^2 r) \\ &\geqslant 2d - 2pf'(\phi(x_3 - cr))\cosh(c^2 r) > 0, \end{aligned} \tag{3.38}$$

due to (3.36).

Thus, let

$$C_3 := \min\{\tfrac{1}{2}[4d - k^2c^2 + 2kc^2 - 4p\exp\{-\tfrac{1}{2}kc^2 r\}], 2d - 2pf'(\phi(x_3 - cr))\cosh(c^2 r)\}. \tag{3.39}$$

Combining (3.32), (3.37) and (3.38), we prove (3.31). □

Lemma 3.5. *There exist some small numbers $\mu_2 > 0$ and $\mu_3 > 0$ such that*

$$C_2 - 2\mu_2 - \frac{p}{\eta_2}(e^{2\mu_2 r} - 1) > 0 \quad \textit{and} \quad C_3 - 2\mu_3 - \frac{p}{\eta_3}(e^{2\mu_3 r} - 1) > 0. \tag{3.40}$$

Then

$$B_{\eta_2,\mu_2,w_2}(\xi) > 0 \quad \textit{and} \quad B_{\eta_3,\mu_3,w_3}(\xi) > 0 \quad \textit{for all } \xi \in \mathbb{R}. \tag{3.41}$$

Proof. Notice that

$$\frac{w_i(\xi + cr)}{w_i(\xi)} \leqslant 1, \quad i = 2, 3, \quad \text{and} \quad 0 < f'(\phi(\xi)) < 1 \quad \text{for all } \xi \in \mathbb{R}.$$

From (3.26), (3.27) and (3.31), it can be verified that there exist some small numbers $\mu_2 > 0$ and $\mu_3 > 0$, such that, for $r = 0$,

$$B_{\eta_2,\mu_2,w_2}(\xi) = A_{\eta_2,w_2}(\xi) - 2\mu_2 - \frac{p}{\eta_2}(\mathrm{e}^{2\mu_2 r} - 1)f'(\phi(\xi))\frac{w_2(\xi + cr)}{w_2(\xi)}$$
$$\geqslant C_2 - 2\mu_2 - \frac{p}{\eta_2}(\mathrm{e}^{2\mu_2 r} - 1) > 0 \quad \text{for all } \xi \in \mathbb{R}, \tag{3.42}$$

and, for $r > 0$,

$$B_{\eta_3,\mu_3,w_3}(\xi) = A_{\eta_3,w_3}(\xi) - 2\mu_3 - \frac{p}{\eta_3}(\mathrm{e}^{2\mu_3 r} - 1)f'(\phi(\xi))\frac{w_3(\xi + cr)}{w_3(\xi)}$$
$$\geqslant C_3 - 2\mu_3 - \frac{p}{\eta_3}(\mathrm{e}^{2\mu_3 r} - 1) > 0 \quad \text{for all } \xi \in \mathbb{R}. \tag{3.43}$$

The proof is complete. □

Applying (3.42) and (3.43) to (3.23), and dropping the positive term

$$\int_0^t \int_R \mathrm{e}^{2\mu\tau} B_{\eta,\mu,w}(\xi) w(\xi) v^2(\tau, \xi)\,\mathrm{d}\xi\,\mathrm{d}\tau,$$

we have

$$\mathrm{e}^{2\mu_2 t}\|v(t)\|^2_{L^2_{w_2}} \leqslant C_1\left(\|v_0(0)\|^2_{L^2_{w_2}} + \int_{-r}^0 \|v_0(\tau)\|^2_{L^2_{w_2}}\,\mathrm{d}\tau\right) \quad \text{for } r = 0 \tag{3.44}$$

and

$$\mathrm{e}^{2\mu_3 t}\|v(t)\|^2_{L^2_{w_3}} \leqslant C_1\left(\|v_0(0)\|^2_{L^2_{w_3}} + \int_{-r}^0 \|v_0(\tau)\|^2_{L^2_{w_3}}\,\mathrm{d}\tau\right) \quad \text{for } r > 0. \tag{3.45}$$

Furthermore, by differentiating (3.13) with respect to ξ, and multiplying it by $\mathrm{e}^{2\mu_i t} w_i(\xi) v(t, \xi)$ ($i = 2, 3$ corresponding to $r = 0$ or $r > 0$), then integrating the resultant equation with respect to (t, x) over $[0, t] \times \mathbb{R}$, and using the basic energy estimates (3.44) and (3.45), we can prove

$$\mathrm{e}^{2\mu_2 t}\|v_\xi(t)\|^2_{L^2_{w_2}} \leqslant C_4\left(\|v_0(0)\|^2_{H^1_{w_2}} + \int_{-r}^0 \|v_0(\tau)\|^2_{H^1_{w_2}}\,\mathrm{d}\tau\right) \quad \text{for } r = 0 \tag{3.46}$$

and

$$\mathrm{e}^{2\mu_3 t}\|v(t)\|^2_{L^2_{w_3}} \leqslant C_4\left(\|v_0(0)\|^2_{H^1_{w_3}} + \int_{-r}^0 \|v_0(\tau)\|^2_{H^1_{w_3}}\,\mathrm{d}\tau\right) \quad \text{for } r > 0 \tag{3.47}$$

for some positive constant C_4. The detail of the proof is omitted. Combining (3.44)–(3.47) yields

$$\|v_\xi(t)\|^2_{H^1_{w_2}} \leqslant C_5\mathrm{e}^{-2\mu_2 t}\left(\|v_0(0)\|^2_{H^1_{w_2}} + \int_{-r}^0 \|v_0(\tau)\|^2_{H^1_{w_2}}\,\mathrm{d}\tau\right) \quad \text{for } r = 0 \tag{3.48}$$

and

$$\|v(t)\|^2_{H^1_{w_3}} \leqslant C_5\mathrm{e}^{-2\mu_3 t}\left(\|v_0(0)\|^2_{H^1_{w_3}} + \int_{-r}^0 \|v_0(\tau)\|^2_{H^1_{w_3}}\,\mathrm{d}\tau\right) \quad \text{for } r > 0 \tag{3.49}$$

provided with some positive constant C_5.

Since $w_i(\xi) \geqslant 1$ for all $\xi \in \mathbb{R}$, $i = 2, 3$, from the definition of the weighted Sobolev space H^1_w, we have

$$\|v(t)\|_{H^1} \leqslant \|v(t)\|_{H^1_{w_i}} \leqslant \sqrt{C_5}\mathrm{e}^{-\mu_i t}\biggl(\|v_0(0)\|^2_{H^1_{w_i}} + \int_{-r}^{0} \|v_0(\tau)\|^2_{H^1_{w_i}}\,\mathrm{d}\tau\biggr)^{1/2}, \quad (3.50)$$

where $i = 2$ or 3 when $r = 0$ or $r > 0$, respectively. Furthermore, by the Sobolev embedding theorem $H^1(R) \hookrightarrow C^0(R)$, we finally prove the following stability result with exponential time decay.

LEMMA 3.6. *It holds that*

$$\sup_{x\in\mathbb{R}} |V^+(t,x) - \phi(x+ct)| = \sup_{\xi\in\mathbb{R}} |v(t,\xi)| \leqslant C_6\mathrm{e}^{-\mu_i t}, \quad t \geqslant 0, \quad (3.51)$$

for some positive constant C_6, where $i = 2$ or 3 when $r = 0$ or $r > 0$, respectively.

CASE 2 (convergence of $V^-(t,x)$ to $\phi(x+ct)$). Let

$$\xi = x+ct, \quad v(t,\xi) = \phi(x+ct)-V^-(t,x), \quad v_0(s,\xi) = \phi(x+cs)-V_0^-(s,x). \quad (3.52)$$

As shown above, we can similarly prove that $V^-(t,x)$ converges to $\phi(x+ct)$, according to the following lemma.

LEMMA 3.7. *It holds that*

$$\sup_{x\in\mathbb{R}} |V^-(t,x) - \phi(x+ct)| = \sup_{\xi\in\mathbb{R}} |v(t,\xi)| \leqslant C_7\mathrm{e}^{-\mu_i t}, \quad t \geqslant 0, \quad (3.53)$$

for some positive constant C_7, where $i = 2$ or 3 when $r = 0$ or $r > 0$, respectively.

CASE 3 (convergence of $N(t,x)$ to $\phi(x+ct)$). In this step, we prove theorems 2.2 and 2.3, as follows.

LEMMA 3.8. *It holds that*

$$\sup_{x\in\mathbb{R}} |N(t,x) - \phi(x+ct)| \leqslant C\mathrm{e}^{-\mu_i t}, \quad t \geqslant 0, \quad (3.54)$$

where $i = 2$ or 3 for $r = 0$ or $r > 0$, respectively.

Proof. Since the initial data are $V_0^-(x,s) \leqslant N_0(x,s) \leqslant V_0^+(x,s)$, from lemma 3.2, the corresponding solutions of (1.1) and (1.2) satisfy

$$V^-(t,x) \leqslant N(t,x) \leqslant V^+(t,x), \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}.$$

Owing to lemmas 3.6 and 3.7, we have the following convergence results:

$$\sup_{x\in\mathbb{R}} |V^-(t,x) - \phi(x+ct)| \leqslant C\mathrm{e}^{-\mu_i t}, \qquad \sup_{x\in\mathbb{R}} |V^+(t,x) - \phi(x+ct)| \leqslant C\mathrm{e}^{-\mu_i t}$$

for $i = 2$ or 3 when $r = 0$ or $r > 0$, respectively. Then, by using the squeeze theorem, we finally prove that

$$\sup_{x\in\mathbb{R}} |N(t,x) - \phi(x+ct)| \leqslant C\mathrm{e}^{-\mu_i t}, \quad t > 0$$

for $i = 2$ or 3 when $r = 0$ or $r > 0$, respectively. This completes the proof. □

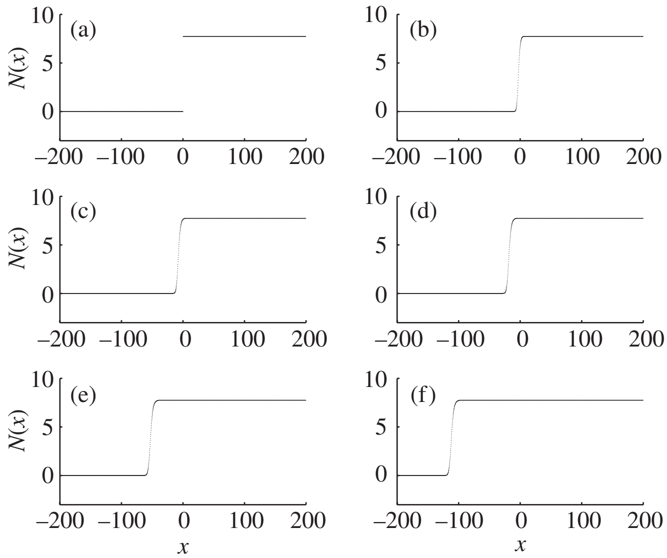

Figure 1. The initial data within $[N_-, N_+]$. The graphs shown are for the solution $N(t,x)$ at (a) $t=0$, (b) $t=5$, (c) $t=10$, (d) $t=20$, (e) $t=50$ and (f) $t=100$.

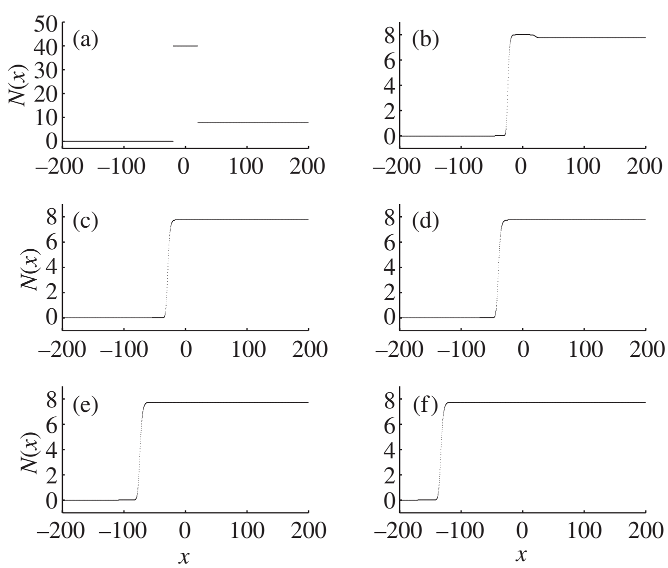

Figure 2. The initial data exceed N_+ at some points. The graphs shown are for the solution $N(t,x)$ at (a) $t=0$, (b) $t=5$, (c) $t=10$, (d) $t=20$, (e) $t=50$ and (f) $t=100$.

4. Numerical simulations

In this section, we present some numerical simulations. We find that these numerical results in many cases match and confirm our theoretical results shown in § 2. The computational results reported in this section are based on the following finite-difference approximation with a forward scheme for the time derivative and a central scheme for the spatial derivative:

$$\frac{N_j^n - N_j^{n-1}}{\Delta t} = \frac{N_{j-1}^{n-1} - 2N_j^{n-1} + N_{j+1}^{n-1}}{(\Delta x)^2} - dN_j^{n-1} + pf(N_j^{n-r}), \tag{4.1}$$

with Nicholson's birth-rate function $f(N) = N\mathrm{e}^{-aN}$, where Δt and Δx denote the step size in time and space, respectively, and are chosen as $\Delta t = 0.01$ and $\Delta x = 0.2$, so that the stable condition for the finite-difference scheme $\Delta t/(\Delta x)^2 < \frac{1}{2}$ is satisfied. Although the original model assumes the spatial domain in $(-\infty, \infty)$, a finite computational domain $(-L, L)$ is imposed. Here, we let $L = 400$ so that the computational domain is sufficiently large and no artificial numerical reflection is introduced in the computed solution. The parameters are taken as $r = 0.5$, $a = 0.1$, $p = 2.17$ and $d = 1$. Thus, we have

$$N_+ = \frac{1}{a} \ln \frac{p}{d} \approx 7.747\,271\,7.$$

We choose two different types of initial data. One is

$$N_0(s, x) = \begin{cases} 0, & x \leqslant 0, \\ N_+, & x > 0, \end{cases} \tag{4.2}$$

which satisfies $0 = N_- \leqslant N_0(s, x) \leqslant N_+$, and the other is

$$N_0(s, x) = \begin{cases} 0, & x \leqslant -20, \\ 40, & -20 < x \leqslant 20, \\ N_+, & x > 20, \end{cases} \tag{4.3}$$

which exceeds N_+ for $x \in (-20, 20]$.

As shown in figures 1 and 2, whatever the initial data within $[N_-, N_+]$ or exceeding N_+, it perfectly demonstrates that the solution of equation (1.1) converges time-asymptotically to a wavefront; in other words, it demonstrates the nonlinear stability of the wavefront. When the initial data $N_0(s, x)$ are taken as in (4.2), figure 1 directly confirms our theoretical result. But, for the initial data $N_0(s, x)$ chosen as in (4.3), which does not satisfy the condition $N_- \leqslant N_0(s, x) \leqslant N_+$ needed in theorem 2.3, as shown in figure 2, we still numerically obtain that the solution behaves as a travelling wavefront time-asymptotically.

Acknowledgements

The authors express their sincere thanks to the referee for a valuable suggestion that led them to further study the stability of the wavefronts with a general birth-rate function. The authors also thank Dr Guangrui Li for the help in numerical computations. The work was partly done in 2005 when M.M. visited the National Center for Theoretical Sciences (South), Taiwan. He expresses his sincere thanks for their great hospitality. The research of C.-K.L. was supported in part by the National Science Council of Taiwan under Grant no. 95-2115-M-009-MY3, and the research of M.M. was supported in part by the Natural Sciences and Engineering Research Council of Canada under NSERC Grant no. RGPIN 354724-08.

References

1. J. Al-Omari and S. A. Gourley. Monotone travelling fronts in an age-structured reaction–diffusion model of a single species. *J. Math. Biol.* **45** (2002), 294–312.
2. S. A. Gourley. Linear stability of travelling fronts in an age-structured reaction–diffusion population model. *Q. J. Mech. Appl. Math.* **58** (2005), 257–268.
3. W. S. C. Gurney, S. P. Blythe and R. M. Nisbet. Nicholson's blowflies revisited. *Nature* **287** (1980), 17–21.
4. G. Li, M. Mei and Y. S. Wong. Nonlinear stability of traveling wavefronts in an age-structured reaction–diffusion population model. *Math. Biosci. Engng* **5** (2008), 85–100.
5. D. Liang and J. Wu. Travelling waves and numerical approximations in a reaction advection diffusion equation with nonlocal delayed effects. *J. Nonlin. Sci.* **13** (2003), 289–310.
6. R. H. Martin and H. L. Smith. Abstract functional differential equations and reaction–diffusion systems. *Trans. Am. Math. Soc.* **321** (1990), 1–44.
7. M. Mei and J. W.-H. So. Stability of strong travelling waves for a non-local time-delayed reaction–diffusion equation. *Proc. R. Soc. Edinb.* A **138** (2008), 551–568.
8. M. Mei, J. W.-H. So, M. Y. Li and S. S. P. Shen. Asymptotic stability of travelling waves for Nicholson's blowflies equation with diffusion. *Proc. R. Soc. Edinb.* A **134** (2004), 579–594.
9. J. A. J. Metz and O. Diekmann. *The dynamics of physiologically structured populations* (ed. J. A. J. Mets and O. Diekmann) (Springer, 1986).
10. A. J. Nicholson. Competition for food amongst *Lucilia cuprina* larvae. In *Proc. VIII Int. Congr. of Entomology, Stockholm*, pp. 277–281 (Stockholm: Axel R Elfströms Boktryckeri AB, 1948).
11. A. J. Nicholson. An outline of the dynamics of animal populations. *Aust. J. Zool.* **2** (1954), 9–65.
12. C. Ou and J. Wu. Persistence of wavefronts in delayed nonlocal reaction–diffusion equations. *J. Diff. Eqns* **235** (2007), 219–261.
13. K. W. Schaaf. Asymptotic behavior and traveling wave solutions for parabolic functional differential equations. *Trans. Am. Math. Soc.* **302** (1987), 587–615.
14. H. L. Smith and X.-Q. Zhao. Global asymptotic stability of traveling waves in delayed reaction–diffusion equations. *SIAM J. Math. Analysis* **31** (2000), 514–534.
15. J. W.-H. So and Y. Yang. Dirichlet problem for the diffusive Nicholson's blowflies equation. *J. Diff. Eqns* **150** (1998), 317–348.
16. J. W.-H. So and X. Zou. Traveling waves for the diffusive Nicholson's blowflies equation. *Appl. Math. Computat.* **122** (2001), 385–392.
17. J. W.-H. So, J. Wu and Y. Yang. Numerical steady state and Hopf bifurcation analysis on the diffusive Nicholson's blowflies equation. *Appl. Math. Computat.* **111** (2000), 53–69.
18. J. W.-H. So, J. Wu and X. Zou. A reaction–diffusion model for a single species with age structure. I. Travelling wavefronts on unbounded domains. *Proc. R. Soc. Lond.* A **457** (2001), 1841–1853.
19. H. R. Thieme. *Mathematics in population biology* (Princeton University Press, 2003).
20. H. R. Thieme and X.-Q. Zhao. Asymptotic speeds of spread and traveling waves for integral equations and delayed reaction–diffusion models. *J. Diff. Eqns* **195** (2003), 430–370.
21. A. Volpert, V. A. Volpert and Vl. A. Volpert. *Traveling wave solutions of parabolic systems*, Translations of Mathematical Monographs, vol. 140 (Providence, RI: American Mathematical Society, 1994).
22. D. Wei, J.-Y. Wu and M. Mei. Remark on critical speed of traveling wavefronts for Nicholson's blowflies equation with diffusion. *Acta Math. Sci.* **30**(5) (2010). (In the press.)
23. J.-H. Wu. *Theory and applications of partial functional differential equations*, Applied Mathematical Sciences, vol. 119 (Springer, 1996).
24. J.-Y. Wu, D. Wei and M. Mei. Analysis on critical speed of traveling waves. *Appl. Math. Lett.* **20** (2007), 712–718.
25. Y. Yang and J. W.-H. So. Dynamics for the diffusive Nicholson's blowflies equation. In *Dynamical systems and differential equations*, vol. II (ed. W. Chen and S. Hu), pp. 333–352 (Springfield, MO: Southwest Missouri Sate University Press, 1998).
26. X.-Q. Zhao. *Dynamical systems in population biology*, CMS Books in Mathematics, vol. 16 (Springer, 2003).

(*Issued 20 February 2010*)