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ASYMPTOTIC BEHAVIOR OF SUBSONIC ENTROPY SOLUTIONS OF THE ISENTROPIC EULER-POISSON EQUATIONS

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Abstract. The hydrodynamic model for semiconductors in one dimension is considered. For perturbated Riemann data, global subsonic (weak) entropy solutions, piecewise continuous and piecewise smooth solutions with shock discontinuities are constructed and their asymptotic behavior is analyzed. In subsonic domains, the solution is smooth and, exponentially as $t \to \infty$, tends to the corresponding stationary solution due to the influence of Poisson coupling. Along the shock discontinuity, the shock strength and the difference of derivatives of solutions decay exponentially affected by the relaxation mechanism.

1. Introduction.

Since its introduction by Bløtekjær [3], the hydrodynamic model for semiconductors has recently attracted much attention because of its ability to model hot electron effects which are not described by the classical drift-diffusion model. For further discussion on these models in physics and engineering, and their derivation from the kinetic transport equation, we refer to [30, 34, 22, 35, 36] for details.

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After an appropriate scaling, the one-dimensional time-dependent system in the case of one carrier type, i.e., electrons, reads

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + p(\rho))_x = \rho\phi_x - \dfrac{\rho u}{\tau}, \\ \phi_{xx} = \rho - \mathcal{C}(x), \end{cases} \tag{1.1}$$

where $\rho > 0$ and u denote the electron density and velocity, respectively. $j = \rho u$ is called the current density, $E = \phi_x$ is the electrostatic potential, and $p = p(\rho)$ is the pressure-density relation which satisfies

$$\rho^2 p'(\rho) \text{ is strictly monotonically increasing from } (0,\infty) \text{ into } (0,\infty). \tag{1.2}$$

In the present paper, we assume that

$$p(\rho) = \rho^\gamma, \quad \gamma \geq 1. \tag{1.3}$$

Also, $\tau = \tau(\rho, \rho u) > 0$ is the momentum relaxation time, which is assumed to equal 1 for convenience. The device domain is the real line, and the function $\mathcal{C} = \mathcal{C}(x) > 0$ is the doping profile, which stands for the given background density of changed ions.

Noticing that $j = \rho u$ and $E = \phi_x$, Eq. (1.1) can be written as

$$\begin{cases} \rho_t + j_x = 0, \\ j_t + (\frac{j^2}{\rho} + p(\rho))_x = \rho E - \dfrac{j}{\tau}, \\ E_x = \rho - \mathcal{C}(x), \\ u = j/\rho. \end{cases} \tag{1.4}$$

In the present paper, we consider the following initial value problems (IVP) for the hydrodynamic model (1.1) (or (1.4)), with initial data given by

$$(\rho, u)(x, 0) = (\rho_0, u_0)(x), \tag{1.5}$$

where

$$\lim_{x\to\pm\infty} (\rho_0, u_0)(x) = (\mathcal{V}_\pm, \mathcal{U}_\pm), \quad \mathcal{V}_+\mathcal{U}_+ = \mathcal{V}_-\mathcal{U}_-. \tag{1.6}$$

The goal here is to discuss the influence of the relaxation mechanism and the Poisson coupling on the existence and asymptotic behavior of (weak) entropy solutions.

For the hydrodynamic model for semiconductors, the existence problem has been considered by many authors. For the steady-state system on a strip domain, Degond and Markowich [6, 7] first proved the existence and uniqueness of subsonic solutions in one dimension, and, for irrational flow, in three dimensions, respectively; the existence and uniqueness of subsonic solutions in two dimensions was discussed by Markowich [31]. The corresponding investigations on transonic solutions in one dimension were done in [2, 33, 9]. For the time-dependent system, Marcati and Natalini [28, 29] discussed the existence of weak solutions on the real line and proved the zero relaxation limit to the drift-diffusion model for $1 < \gamma \leq \frac{5}{3}$. Zhang [42, 43] discussed the existence of weak solutions and the relaxation limits for $\gamma > \frac{5}{3}$. Gasser and Natalini [11] discussed the relaxation limit for the non-isentropic hydrodynamic model. On the strip domain, the existence of weak solutions was obtained by Zhang [40] and by Fang and Ito [8], respectively. Hsiao and

K. Zhang [20, 21] discussed the relaxation limit and verified the boundary conditions for weak solutions in the sense of trace. Chen and Wang [5] investigated the existence of weak solutions on compact domains with geometric symmetry. Under assumption of zero-current density at boundaries, Hsiao and Yang [19] discussed the time-asymptotic convergence of the smooth solutions of the hydrodynamic model and those of the drift-diffusion model to the unique steady-state solution. For density and potential boundary conditions, Li, Markowich, and Mei [24] reproved the existence and uniqueness of a subsonic steady-state solution of the hydrodynamic model and established its stability for small perturbations. Regarding other topics on smooth solutions for the time-dependent hydrodynamic models for semiconductor devices, such as time-asymptotic convergence to the stationary solution of the drift-diffusion equation, the stability and instability of the steady-state solutions, initial boundary value problems, and numerical analysis, we refer the reader to [26, 13, 12, 27, 41, 4, 23] and references therein.

However, in the weak solution case, few results are known on the asymptotic behavior of weak solutions. Our interest in the present paper is to investigate the large time behavior of (weak) entropy solutions. As a first step, we consider the asymptotic behavior of piecewise smooth solutions with discontinuities in the subsonic cases. For simplicity, we consider the perturbated Riemann problems, i.e.,

$$(\rho_0, u_0)(x) = \begin{cases} (\rho_l, u_l)(x), & x < 0, \\ (\rho_r, u_r)(x), & x > 0, \end{cases} \tag{1.7}$$

and

$$(\varrho_-, u_-) = \lim_{x\to 0-} (\rho_l, u_l)(x) \neq \lim_{x\to 0+} (\rho_r, u_r)(x) =: (\varrho_+, u_+). \tag{1.8}$$

This problem is of importance in the study of existence and asymptotic behavior of solutions for general initial-value problems. It works as the building block to construct a weak solution [28, 29, 37], and is the first step to investigate the interactions of elementary waves. Unfortunately, due to global effects of the relaxation damping and Poisson terms, the investigation of the Riemann problems for (1.1) causes difficulties. The loss of a self-similar solution also makes it complicated to construct globally-defined solutions.

The corresponding steady-state system for (1.4) reads

$$\begin{cases} \tilde{j} = \text{ const.}, \\ \left(\dfrac{\tilde{j}^2}{\tilde{\rho}} + p(\tilde{\rho})\right)_x = \tilde{\rho}\widetilde{E} - \tilde{j}, \\ \widetilde{E}_x = \tilde{\rho} - \mathcal{C}(x), \\ \tilde{u} = \tilde{j}/\tilde{\rho}. \end{cases} \tag{1.9}$$

It was proved by Macarti and Mei [27] that there is a unique smooth steady-state solution $(\tilde{\rho}, \tilde{u}, \widetilde{E})$ (up to a shift) of system (1.9) satisfying $\tilde{\rho}\widetilde{E}(+\infty) = \tilde{\rho}\widetilde{E}(-\infty) = \tilde{j}$. Therein [27], the restriction of zero current used in [26] was also removed; namely, the current density $\tilde{j}$ may take a nonzero constant value.

In the present paper, we show that the piecewise continuous and piecewise smooth subsonic solution (ρ, u, E) to the IVP (1.4) and (1.7) exists globally and tends to the solution $(\tilde{\rho}, \tilde{u}, \widetilde{E})$ of the steady-state system (1.9) as time t tends to infinity. For simplicity, we first consider the case that the two states (ϱ_-, u_-) and (ϱ_+, u_+) are connected by two shock curves. More precisely, in phase space there is a state (ϱ_c, u_c) such that (ϱ_-, u_-) and (ϱ_c, u_c) are connected by a backward shock curve, and (ϱ_c, u_c) and (ϱ_+, u_+) are connected by a forward shock curve. The methods used in the present paper can be applied to deal with other kinds of connections between (ϱ_-, u_-) and (ϱ_+, u_+). In fact, the main result (see Theorem 3.1) shows that if the initial jump is sufficiently small and the initial value is a small perturbation of $(\tilde{\rho}, \tilde{u}, \widetilde{E})$ with $\mathcal{V}_+\mathcal{U}_+ = \tilde{j}$, then the piecewise continuous and piecewise smooth solution (ρ, u, E) to the IVP (1.4) and (1.7) (or IVP (1.1) and (1.7)) exists globally. The discontinuities consist of two shock curves—a backward one and a forward one. These shock curves never disappear in finite time, but the shock strengths decay exponentially. As time t tends to infinity, the solutions (ρ, u, E) converge to $(\tilde{\rho}, \tilde{u}, \widetilde{E})$ exponentially.

The present paper is organized as follows. Under the restriction conditions on the doping profile (2.1) and (2.4), we first state the existence results in [27] on the solution to the steady-state system (1.9) in Sec. 2, where related properties about the solution are also given. In Sec. 3, some preliminaries on the Riemann problem for the Euler equation are introduced first, and the main result is given. The result is proved in Sec. 4. We first construct the solution locally under the a priori assumptions that $|(\rho-\tilde{\rho}, u-\tilde{u}, E-\widetilde{E})| \ll 1$ (Secs. 4.1–4.2). In Sec. 4.3 the globally-defined solution and its asymptotic behavior are obtained.

NOTATION. Let $L^2(D)$ be the usual space of square integrable functions on domain $D \subseteq R$, and let $H^m(D)$ $(m \geq 1)$ be the usual space of functions f on D satisfying $\partial_x^i f \in L^2(D)$, $i = 1, 2, 3, \ldots, m$. In the present paper, for convenience, C denotes a generic positive constant, and the c_i and a_i with i integral denote positive constants.

2. Steady-state system. In this section, we state the existence and uniqueness of stationary solutions for system (1.4), as well as the properties of these solutions. All of these are shown by Marcati and Mei in [27].

In this paper, we assume that the doping profile satisfies

$$\begin{gathered}\mathcal{C}(x) \in C^2(R), \qquad \mathcal{C}'(x) \in L^1(R) \cap H^1(R), \\ \lim_{x\to\pm\infty} \mathcal{C}(x) = \mathcal{C}_\pm > 0, \qquad \mathcal{C}^* = \sup_{x\in R} \mathcal{C}(x) > 0, \qquad \mathcal{C}_* = \inf_{x\in R} \mathcal{C}(x) > 0,\end{gathered} \tag{2.1}$$

from which one can verify $\mathcal{C}'(x) \in W^{1,4}(R)$.

Dividing $(1.9)_2$ by $\tilde{\rho}$, then differentiating it with respect to x, one obtains in terms of $(1.1)_3$ that

$$\left(\frac{\partial F}{\partial \tilde{\rho}}(\tilde{\rho}, \tilde{j})\tilde{\rho}_x\right)_x + \tilde{j}\left(\frac{1}{\tilde{\rho}}\right)_x - \tilde{\rho} = -\mathcal{C}(x), \tag{2.2}$$

where

$$F(\tilde{\rho}, \tilde{j}) = \frac{\tilde{j}^2}{2\tilde{\rho}^2} + h(\tilde{\rho}), \qquad h'(\tilde{\rho}) = \frac{p'(\tilde{\rho})}{\tilde{\rho}}.$$

To make Eq. (2.2) uniformly elliptic, we need

$$\frac{\partial F}{\partial \tilde{\rho}}(\tilde{\rho}, \tilde{j}) = \frac{1}{\tilde{\rho}} p'(\tilde{\rho}) - \frac{\tilde{j}^2}{\tilde{\rho}^3} > 0 \Leftrightarrow \tilde{\rho}^2 p'(\tilde{\rho}) > \tilde{j}^2. \tag{2.3}$$

We conclude from (1.3) and (2.3) that there exists a unique $\tilde{\rho}_m = \tilde{\rho}_m(\tilde{j}) \geq 0$ such that

$$p'(\tilde{\rho}) - \frac{\tilde{j}^2}{\tilde{\rho}^2} > 0$$

for $\tilde{\rho} > \tilde{\rho}_m$.

Note that, by (2.3), the minimal point $\tilde{\rho}_m$ of $\tilde{\rho} \to F(\tilde{\rho}, \tilde{j})$ is a strictly increasing function of $\tilde{j}$, and $\tilde{\rho}_m(\tilde{j} = 0) = 0$. One can make sure that Eq. (2.2) is uniformly elliptic for $\tilde{\rho} > \tilde{\rho}_m$. By (2.3) and $\tilde{j} = \tilde{\rho}\tilde{u}$, this condition implies $|\tilde{u}| < c(\tilde{\rho})$, where $c(\tilde{\rho}) = \sqrt{p'(\tilde{\rho})}$ is the speed of sound.

One can prove ([27]) that if

$$\inf_{x \in R} \mathcal{C}(x) = \mathcal{C}_* > \tilde{\rho}_m(\tilde{j}), \tag{2.4}$$

then there is a regular solution up to a shift to (1.9) with $\tilde{\rho}\widetilde{E}(\pm\infty) = \tilde{j}$.

We remark that if $|\tilde{j}|$ is so large or $\mathcal{C}_*$ is so small that

$$\mathcal{C}_* \leq \tilde{\rho}_m(\tilde{j}), \tag{2.5}$$

then the flow may at least be partly supersonic and the occurrence of shocks cannot be excluded.

We expect to prove that for a stationary solution it follows that $|(\tilde{\rho}_x, \widetilde{E}_x)(x)| \to 0$ as $|x| \to +\infty$. The second equation of (1.9) is equal to

$$\left(p'(\tilde{\rho}) - \frac{\tilde{j}^2}{\tilde{\rho}^2}\right) \tilde{\rho}_x = \tilde{\rho}\widetilde{E}_x - \tilde{j}. \tag{2.6}$$

Thus, setting $x \to \pm\infty$, thanks to (2.6), $(1.9)_3$, and (2.3), which implies $p'(\mathcal{C}_\pm) - \frac{\tilde{j}^2}{\mathcal{C}_\pm^2} > 0$, we get

$$\mathcal{C}_\pm \widetilde{E}_\pm = \tilde{j}, \tag{2.7}$$

with $\widetilde{E} = \phi_x$, and $\widetilde{E}_\pm$ and $\tilde{j}$ have the same sign, where $\widetilde{E}_- = \widetilde{E}(-\infty)$. Without loss of generality, we assume $\tilde{j} > 0$, i.e., $\widetilde{E}_\pm > 0$. Note that, if $\tilde{j} = 0$, then $\widetilde{E}_+ = \widetilde{E}_- = 0$ and the potential $\widetilde{E}_+ - \widetilde{E}_- = 0$, which is a trivial case.

We now state the existence result and the properties of the stationary solutions as follows.

THEOREM 2.1 ([27]). Under the assumptions (1.3), (2.1), and (2.4), there exists a unique (up to a shift) smooth solution $(\tilde{\rho}, \tilde{j}, \widetilde{E})(x)$ of problem (1.9), such that

$$\mathcal{C}_* \le \tilde{\rho}(x) \le \mathcal{C}^*, \quad x \in R, \tag{2.8}$$

$$|\tilde{\rho}(x) - \mathcal{C}_\pm| = O(e^{-c_\pm |x|}) \text{ as } x \to \pm\infty, \tag{2.9}$$

$$\|\tilde{\rho} - \mathcal{C}\|_2^2 \le C_1 a_1, \tag{2.10}$$

$$\sup_{x\in R}(|\tilde{\rho}'(x)|^2 + |\tilde{\rho}''(x)|^2) \le C_2 a_2, \tag{2.11}$$

$$\sup_{x\in R}|\widetilde{E}(x)|^2 \le C_3 a_3, \tag{2.12}$$

where C_i $(i = 1, 2, 3)$ are positive constants only depending on $\mathcal{C}^*, \mathcal{C}_*, \mathcal{C}_-$, and $\mathcal{C}_+$, but not on a_i $(i = 1, 2, 3)$. The positive constants c_+ and a_i $(i = 1, 2, 3)$ are given as

$$\begin{cases} c_\pm = \dfrac{\widetilde{E}_\pm}{p'(\mathcal{C}_\pm) - \widetilde{E}_\pm^2} > 0, \\ a_1 = |\log \mathcal{C}_+ - \log \mathcal{C}_-| + \|\mathcal{C}'\|_{L^1} + \|\mathcal{C}'\|_{H^1}^2 + \|\mathcal{C}'\|_{L^4}^4 \\ \qquad + (|\log \mathcal{C}_+ - \log \mathcal{C}_-| + \|\mathcal{C}'\|_{L^1} + \|\mathcal{C}'\|_{L^2}^2)^3, \\ a_2 = a_1^{1/2} + \|\mathcal{C}'\|_{L^\infty} + (a_1^{1/2} + \|\mathcal{C}'\|_{L^\infty})^2, \\ a_3 = [a_1 + \tilde{\rho}_m^2 p'(\tilde{\rho}_m)]/\mathcal{C}_*^2. \end{cases} \tag{2.13}$$

3. Hydrodynamic model and main result. Consider the following Euler equations:

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + p(\rho))_x = 0, \end{cases} \tag{3.1}$$

where the pressure p is given by (1.3). Equations (3.1) can be written as

$$\mathbf{v}_t + f(\mathbf{v})_x = 0, \tag{3.2}$$

where $\mathbf{v} = (\rho, j)^{\mathrm{T}}$ with $j = \rho u$, $f(\mathbf{v}) = (j, \frac{j^2}{\rho} + p(\rho))^{\mathrm{T}}$. The Jacobi matrix of f is

$$\nabla f = \begin{pmatrix} 0 & 1 \\ -\frac{j^2}{\rho^2} + p'(\rho) & \frac{2j}{\rho} \end{pmatrix}. \tag{3.3}$$

The eigenvalues of (3.3) are

$$\lambda_1 = \frac{j}{\rho} - \sqrt{\gamma}\rho^{(\gamma-1)/2}, \quad \lambda_2 = \frac{j}{\rho} + \sqrt{\gamma}\rho^{(\gamma-1)/2}, \tag{3.4}$$

or

$$\lambda_1 = u - \sqrt{\gamma}\rho^{(\gamma-1)/2}, \quad \lambda_2 = u + \sqrt{\gamma}\rho^{(\gamma-1)/2}, \tag{3.5}$$

and the Riemann invariants are

$$\begin{cases} s = u + \psi(\rho), \\ r = u - \psi(\rho), \end{cases} \tag{3.6}$$

where ψ is defined by

$$\psi(\rho)=\begin{cases}\frac{2\sqrt{\gamma}}{\gamma-1}\rho^{(\gamma-1)/2}, & \gamma\neq 1,\\ \ln\rho, & \gamma=1.\end{cases}$$

Via Riemann invariants, the solution (ρ,u) can be represented as

$$u=\frac{1}{2}(s+r),\quad \rho=H(r,s),\tag{3.7}$$

where

$$H(r,s)=\begin{cases}\left(\frac{\gamma-1}{4\sqrt{\gamma}}(s-r)\right)^{\frac{2}{\gamma-1}}, & \gamma\neq 1,\\ e^{\frac{1}{2}(s-r)}, & \gamma=1.\end{cases}$$

An i-shock wave, $i=1,2$, for (3.2), is characterized by the Rankine-Hugoniot condition and Lax entropy condition. Namely, along the discontinuity $x=x_i(t)$, it follows that

$$\begin{cases}\dot{x}_1(t)=-\sqrt{[\rho u^2+p(\rho)]_1/[\rho]_1},\\ [\rho u]_1=\sqrt{[\rho u^2+p(\rho)]_1/[\rho]_1}\cdot[\rho]_1,\\ \lambda_1(x_1(t)-0,t)>\dot{x}_1(t)>\lambda_1(x_1(t)+0,t),\end{cases}\tag{3.8}$$

or

$$\begin{cases}\dot{x}_2(t)=\sqrt{[\rho u^2+p(\rho)]_2/[\rho]_2},\\ [\rho u]_2=-\sqrt{[\rho u^2+p(\rho)]_2/[\rho]_2}\cdot[\rho]_2,\\ \lambda_2(x_2(t)-0,t)>\dot{x}_2(t)>\lambda_2(x_2(t)+0,t).\end{cases}\tag{3.9}$$

Here and afterward, we denote

$$[F]_i=F(x_i(t)+0,t)-F(x_i(t)-0,t),\quad i=1,2.$$

In this section, we consider the IVP (1.1)–(1.5) in the case that the two states (ϱ_-,u_-) and (ϱ_+,u_+) are connected by two shock curves in phase space; i.e., there is a state (ϱ_c,u_c) such that

$$\begin{cases}u_->u_c>u_+,\\ (\varrho_c u_c-\varrho_-u_-)^2=(\varrho_c u_c^2-\varrho_-u_-^2+p(\varrho_c)-p(\varrho_-))(\varrho_c-\varrho_-),\varrho_-<\varrho_c,\\ (\varrho_+u_+-\varrho_c u_c)^2=(\varrho_+u_+^2-\varrho_c u_c^2+p(\varrho_+)-p(\varrho_c))(\varrho_+-\varrho_c),\varrho_+<\varrho_c.\end{cases}\tag{3.10}$$

Set

$$\varepsilon_0(x)=\left\{\int_{-\infty}^{0-}+\int_{0+}^{x}\right\}(\varrho_0(y)-\tilde{\rho}(y))\,dy,\qquad \varepsilon_1(x)=\rho_0u_0(x)-\tilde{j}.$$

Denote

$$\delta_0=|\varrho_c-\varrho_+|+|\varrho_--\varrho_c|,\qquad \eta_0=a_1+a_2+a_3,$$

$$\mu_0=\|(\varepsilon_0,\varepsilon_{0x},\varepsilon_{0xx},\varepsilon_{0xxx})\|+\|(\varepsilon_1,\varepsilon_{1x},\varepsilon_{1xx})\|<+\infty,\tag{3.11}$$

$$\mu_1=\sum_{i=0}^{2}\sup_{x\neq 0}\{|\partial_x^i(\rho_0(x)-\tilde{\rho}(x))|+|\partial_x^i(u_0(x)-\tilde{u}(x))|\}<+\infty,\tag{3.12}$$

$$\mu_2=\sup_{x\neq 0}\{|\partial_x^3(\rho_0(x)-\tilde{\rho}(x))|+|\partial_x^3(u_0(x)-\tilde{u}(x))|\}<+\infty,\tag{3.13}$$

where

$$\|f\| = \sqrt{\int_{-\infty}^{0-} |f(x)|^2\,dx + \int_{0+}^{\infty} |f(x)|^2\,dx}, \tag{3.14}$$

$\tilde{u}(x) = \tilde{j}/\tilde{\rho}(x)$, and a_1, a_2, a_3 are given by (2.13).

We have the following main result.

THEOREM 3.1. Let $(\rho_0, u_0) \in C^3(R - \{0\})$, $\varepsilon_0 \in L^2(R - \{0\})$, and $(\varepsilon_0, \varepsilon_1) \in H^3(R - \{0\}) \times H^2(R - \{0\})$. Let (3.10) and (3.11)–(3.13) hold. Then there exists a $\beta_0 > 0$ such that if $\delta_0 + \mu_0 + \mu_1 + \eta_0 < \beta_0$, then the global weak entropy solution (ρ, u, E) of the IVP (1.1) and (1.5) uniquely exists. It is piecewise continuous and piecewise smooth with two shock discontinuities—a forward shock curve $x = x_2(t)$ and a backward shock curve $x = x_1(t)$ satisfying $x_1(0) = x_2(0) = 0$ and $x_1(t) < x_2(t)$ for $t > 0$. Away from the discontinuities, $(\rho, u, E_x)(\cdot, t) \in C^3$. In addition, as t tends to infinity,

$$\sum_{i=0}^{2}(|[\partial_x^i(\rho, u)]_1)| + |[\partial_x^i(\rho, u)]_2|) \sim O(1)e^{-\kappa_1 t} \to 0, \tag{3.15}$$

and

$$\sum_{i=0}^{2} \|\partial_x^i(\rho - \tilde{\rho}, \rho u - \tilde{j}, E - \widetilde{E})(\cdot, t)\| \sim O(1)e^{-\kappa_2 t} \to 0, \tag{3.16}$$

with two positive constants κ_1 and κ_2, where

$$\|f(\cdot, t)\| = \sqrt{\fint_{-\infty}^{+\infty} f(y,t)^2\,dy}$$

and

$$\fint_{-\infty}^{+\infty} f(y,t)^2\,dy = \left\{\int_{-\infty}^{x_1(t)-0} + \int_{x_1(t)+0}^{x_2(t)-0} + \int_{x_2(t)+0}^{+\infty}\right\} f(y,t)^2\,dy.$$

REMARK 3.2. 1) A similar result is true for general pressure $p(v)$ with $p'(v) < 0 < p''(v)$.

2) For other kinds of connections of (ϱ_-, u_-) and (ϱ_+, u_+) in phase space, such as a rarefaction wave and a shock wave, or two rarefaction waves, similar results can be proved by using the same approach as the present paper.

4. Proof of main result. In this section, we construct the global weak entropy solutions for the IBVP (1.1), (1.5), and (3.10), and investigate their asymptotic behavior.

4.1. *Shock waves and geometric structure.* Define

$$\begin{cases} D_- = \partial_t + \lambda_1 \partial_x, \\ D_+ = \partial_t + \lambda_2 \partial_x. \end{cases}$$

Then the system (1.1) (or (1.4)) can be written, via Riemann invariants, as

$$\begin{cases} D_- r = -\frac{1}{2}(r+s) + E, \\ D_+ s = -\frac{1}{2}(r+s) + E, \end{cases} \tag{4.1}$$

and the corresponding initial value is

$$(r,s)(x,0) = (r_0, s_0)(x) = \begin{cases} (r_l, s_l)(x), & x < 0, \\ (r_r, s_r)(x), & x > 0, \end{cases} \tag{4.2}$$

where

$$(r_l, s_l) = (u_l - \psi(\rho_l), u_l + \psi(\rho_l)), \tag{4.3}$$

$$(r_r, s_r) = (u_r - \psi(\rho_r), u_r + \psi(\rho_r)), \tag{4.4}$$

and

$$(r_-, s_-) = \lim_{x\to 0-} (r_l, s_l)(x) \neq \lim_{x \to 0+} (r_r, s_r)(x) =: (r_+, s_+). \tag{4.5}$$

By the argument used by Li and Yu in [25] to establish the local existence theorem, one can prove that the discontinuous initial value problem (4.1)–(4.5) admits a unique discontinuous solution (r,s) for $0 < t_0 \ll 1$ in the class of piecewise continuous and piecewise smooth functions. This solution contains a forward shock $x = x_2(t)$ and a backward shock $x = x_1(t)$, both passing through $(0,0)$. It is known, due to the entropy condition, that $x = x_2(t)$ must be located on the right side of $x = x_+(t)$, given by

$$\dot{x}_+(t) = \lambda_2(x_+, (t), t), \quad x_+(0) = 0,$$

and $x = x_1(t)$ must be located on the left side of $x = x_-(t)$, given by

$$\dot{x}_-(t) = \lambda_1(x_-, (t), t), \quad x_-(0) = 0.$$

Moreover, it can be shown that $s - r > 0$ is bounded, $|(r_x, s_x)| \ll 1$ for $x \neq x_i(t)$ and $|[\rho]_i| \ll 1$ $(i = 1, 2)$, provided that $\rho_0(x) > 0$ and $|(\rho_0, u_0)_x|_{C^0} + \delta_0 \ll 1$.

For any $T \geq t_0 \geq 0$, denote

$$\begin{aligned} \Omega_0(T) &= \{(x,t) | x_1(t) < x < x_2(t), 0 \leq t < T\}, \\ \Omega_-(T) &= \{(x,t) | x < x_1(t), 0 \leq t < T\}, \\ \Omega_+(T) &= \{(x,t) | x_2(t) < x, 0 \leq t < T\}. \end{aligned}$$

Without loss of generality, we assume that the piecewise continuous and piecewise smooth solution exists on

$$\Omega_s(T) =: \Omega_+(T) \cup \Omega_-(T) \cup \Omega_0(T),$$

and assume that on domain $\Omega_s(T)$ it follows that

$$|(\rho(x,t) - \tilde{\rho}, \rho u - \tilde{j})| + |\rho_x(x,t)| + |u_x(x,t)| \leq \eta \ll 1, \tag{4.6}$$

and along $x = x_i(t)$ $(i = 1, 2)$ it follows that

$$|[\rho]_i| \ll 1, \quad i = 1, 2. \tag{4.7}$$

From (4.6) and Theorem 2.1, one can verify that the subsonic condition holds for the solution (ρ, u, E) on $\Omega_s(T)$; i.e.,

$$\max_{x \in \Omega_s(T)} |u| \rho^{\frac{1-\gamma}{2}}(x,t) < \sqrt{\gamma}, \tag{4.8}$$

and there are two constants ρ_* and ρ^* such that

$$0 < \rho_* < \overline{\rho} < \rho^*. \tag{4.9}$$

In this subsection, we investigate the qualitative behaviors of the piecewise smooth solution $(r,s)(x,t)$ with shock discontinuity to (1.1) (or (1.4)), under the assumptions (4.6)–(4.7).

Define, for $i=1,2$,

$$A^-_{i,1}(t)=\left(\frac{\lambda^-_{i,1}-\dot{x}_1}{\lambda^+_{2,1}-\dot{x}_1}\right)^3\left(\frac{\rho^+_1}{\rho^-_1}\right)^{\frac{\gamma-3}{2}}(x_1(t),t),$$
$$A^+_{i,1}(t)=\left(\frac{\lambda^+_{i,1}-\dot{x}_1}{\lambda^+_{2,1}-\dot{x}_1}\right)^3(x_1(t),t),$$
$$A^+_{i,2}(t)=\left(\frac{\lambda^+_{i,2}-\dot{x}_2}{\lambda^-_{1,2}-\dot{x}_2}\right)^3\left(\frac{\rho^+_2}{\rho^-_2}\right)^{\frac{3-\gamma}{2}}(x_2(t),t),$$
$$A^-_{i,2}(t)=\left(\frac{\lambda^-_{i,2}-\dot{x}_2}{\lambda^-_{1,2}-\dot{x}_2}\right)^3(x_2(t),t),$$

and set

$$K_1(t)=-\frac{(\dot{x}_1-\lambda^+_{2,1})^2}{\dot{x}_1}(x_1(t),t),$$
$$K_2(t)=\frac{(\dot{x}_2-\lambda^-_{1,2})^2}{\dot{x}_2}(x_2(t),t)$$
$$M^\pm_{1,1}(t)=\frac{1}{\dot{x}_1\sqrt{\gamma}}(\rho^\pm_1)^{\frac{3-\gamma}{2}}(\lambda^+_{2,1}-\lambda^\pm_{1,1})(\lambda^\pm_{1,1}+\lambda^+_{2,1}-2\dot{x}_1)(x_1(t),t),$$
$$M^\pm_{1,2}(t)=\frac{1}{\dot{x}_2\sqrt{\gamma}}(\rho^\pm_2)^{\frac{3-\gamma}{2}}(\lambda^\pm_{2,2}-\lambda^-_{1,2})(\lambda^\pm_{2,2}+\lambda^-_{1,2}-2\dot{x}_2)(x_2(t),t),$$
$$M_{2,1}(t)=\frac{1}{\dot{x}_1\sqrt{\gamma}}(\rho^-_1)^{\frac{3-\gamma}{2}}(\dot{x}_1-\lambda^-_{2,1})(\lambda^+_{2,1}+\lambda^-_{2,1}-2\dot{x}_1)(x_1(t),t),$$
$$M_{2,2}(t)=\frac{1}{\dot{x}_2\sqrt{\gamma}}(\rho^+_2)^{\frac{3-\gamma}{2}}(\lambda^+_{1,2}-\dot{x}_2)(\lambda^+_{1,2}+\lambda^-_{1,2}-2\dot{x}_2)(x_2(t),t),$$

where and from now on

$$\lambda^\pm_{j,i}=\lambda_j(x_i(t)\pm 0,t),\quad f^\pm_i=f(x_i(t)\pm 0,t),\quad i=1,2,\ j=1,2.$$

Differentiating $(3.9)_2$ and $(3.8)_2$ with respect to t, respectively, and using (3.6), we have, after tedious calculations, the following lemma (cf. [15, 17, 18]).

LEMMA 4.1. *Let (ρ,u,E) be a piecewise smooth subsonic solution of the IVP (1.4) and (1.5) with shock discontinuity $x=x_i(t)$ $(i=1,2)$ on $\Omega_s(T)$. Then, along $x=x_2(t)$, it follows that*

$$(r^-_2)_x=A^+_{1,2}(r^+_2)_x-A^+_{2,2}(s^+)_x+A^-_{2,2}(s^-_2)_x+B_0(t)\tag{4.10}$$

and

$$-K_2\frac{1}{[\rho]^2}D_t^+[\rho]_2 = 2\left(\dot{x}_2 - \frac{[\rho u^2]_2}{[\rho u]_2}\right) - \frac{1}{2}M_{1,2}^+\frac{\lambda_2^+ - \dot{x}_2}{[\rho]_2}(s^+)_x \\ + \frac{1}{2}M_{1,2}^-\frac{\lambda_2^- - \dot{x}_2}{[\rho]_2}(s^-)_x + \frac{1}{2}M_{2,2}\frac{[\lambda_1]_2}{[\rho]_2}(r^+)_x, \tag{4.11}$$

where $D_t^+ =: \partial_t + \dot{x}_2(t)\partial_x$ and

$$B_0(t) = 4\frac{\sqrt{\gamma}(\rho_2^-)^{\frac{\gamma-3}{2}}}{(\lambda_{1,2}^- - \dot{x}_2)^3}\left(\dot{x}_2 - \frac{[\rho u^2]_2}{[\rho u]_2}\right)\dot{x}_2[\rho]_2;$$

and along $x = x_1(t)$, it follows that

$$(s_1^+)_x = A_{2,1}^-(s_1^-)_x + A_{1,1}^+(r_1^+)_x - A_{1,1}^-(r_1^-)_x + B_1(t) \tag{4.12}$$

and

$$-K_1\frac{1}{[\rho]_1}D_t^-[\rho]_1 = -2\left(\dot{x}_1 - \frac{[\rho u^2]_1}{[\rho u]_1}\right) + \frac{1}{2}M_{1,1}^+\frac{\lambda_1^+ - \dot{x}_1}{[\rho]_1}(r^+)_x \\ - \frac{1}{2}M_{1,1}^-\frac{\lambda_1^- - \dot{x}_1}{[\rho]_1}(r^-)_x - \frac{1}{2}M_{2,1}\frac{[\lambda_2]_1}{[\rho]_1}(s^+)x, \tag{4.13}$$

where $D_t^- =: \partial_t + \dot{x}_1(t)\partial_x$ and

$$B_1(t) = 4\frac{\sqrt{\gamma}(\rho_1^+)^{\frac{\gamma-3}{2}}}{(\lambda_{2,1}^+ - \dot{x}_1)^3}\left(\dot{x}_1 - \frac{[\rho u^2]_1}{[\rho u]_1}\right)\dot{x}_1[\rho]_1.$$

Then, the exponential decay of shock strength follows from Lemma 4.1.

LEMMA 4.2. Let (ρ, u, E) be a piecewise smooth subsonic solution of the IVP (1.4) and (1.5) with shock discontinuity $x = x_i(t)$ $(i = 1, 2)$ on $\Omega_s(T)$. Then, along $x = x_2(t)$, it follows for $t \in [0, T]$ that

$$-\frac{1}{[\rho]_2}D_t^+([\rho]_2) \in [\beta_1, \beta_2], \tag{4.14}$$

$$\delta_0 e^{-\beta_2 t} \leq \rho(x_2(t) + 0, t) - \rho(x_2(t) - 0, t) \leq \delta_0 e^{-\beta_1 t}, \tag{4.15}$$

and along $x = x_1(t)$, it follows for $t \in [0, T]$ that

$$-\frac{1}{[\rho]_1}D_t^-([\rho]_1) \in [\beta_1, \beta_2], \tag{4.16}$$

$$\delta_0 e^{-\beta_2 t} \leq \rho(x_1(t) + 0, t) - \rho(x_1(t) - 0, t) \leq \delta_0 e^{-\beta_1 t}, \tag{4.17}$$

with constants $\beta_i > 0$ $(i = 1, 2)$, provided that (4.6)–(4.7) hold.

Proof. We only prove (4.14)–(4.15). By $(3.9)_1$, one can verify that along $x = x_2(t)$, it follows that

$$\sqrt{\frac{[\rho u^2 + p(\rho)]_2}{[\rho]_2}} = \lambda_{2,2} \pm O(1)[\rho]_2, \tag{4.18}$$

$$\sqrt{\frac{[\rho u^2 + p(\rho)]_2}{[\rho]_2}} = \frac{[\rho u^2]_2}{[\rho u]_2} + \frac{\gamma(\rho_2^{\pm})^{\gamma-1}}{u^{\pm} + \sqrt{\gamma}(\rho_2^{\pm})^{\frac{\gamma-1}{2}}} \pm O(1)[\rho]_2, \tag{4.19}$$

$$[\lambda_1]_2 = O(1)[\rho]_2, \tag{4.20}$$

provided that $[\rho]_2 \ll 1$.

Then, it follows from the entropy condition and (4.18) that

$$0 < \alpha_1 =: C^{-1}\sqrt{\gamma}(\rho_*)^{\frac{\gamma-1}{2}} \le K_2 \le C\sqrt{\gamma}(\rho^*)^{\frac{\gamma-1}{2}} =: \alpha_2, \tag{4.21}$$

$$0 < \alpha_3 =: C^{-1}\rho_* \le |M_{1,2}^{\pm}| + |M_{2,2}| \le C\rho^* =: \alpha_4, \tag{4.22}$$

where $\alpha_i > 0$ $(i = 1, 2, 3, 4)$ is constant, provided that (4.6) holds. Substituting (4.19)–(4.22) into (4.11) yields (4.14)–(4.15) for two constants β_1 and β_2, provided that (4.6)–(4.7) hold.

Similarly, one can prove (4.16)–(4.17). Then, the proof of Lemma 4.2 is completed. □

4.2. *The a priori estimates.* In this subsection, we obtain the a priori estimates in order to extend the solution, more precisely, to obtain the bounds of $(r_{xx}, s_{xx})(x,t)$ and $(r_{xxx}, s_{xxx})(x,t)$ by solving initial value problems on $\Omega_{\pm}(T)$ and solving initial boundary value problems on $\Omega_0(T)$, and to estimate the decay rates of $[r_x]_i, [s_x]_i, [r_{xx}]_i$, and $[s_{xx}]_i$ under the a priori assumptions (4.6)–(4.7).

Define

$$a(\rho) = \tfrac{1}{4}(\gamma+1)\rho^{(3-\gamma)/4}, \quad d(\rho) = \rho^{(\gamma-3)/4}, \tag{4.23}$$

$$h(\rho) = \begin{cases} \frac{2}{3-\gamma}\rho^{(\gamma-3)/4}, & \gamma \ne 3, \\ -\ln \rho^{1/2}, & \gamma = 3, \end{cases} \tag{4.24}$$

and set

$$Y_1 = d(\rho)s_x + h(\rho), \qquad Z_1 = d(\rho)r_x + h(\rho), \tag{4.25}$$

$$Y_2 = (Y_1)_x, \qquad Z_2 = (Z_1)_x. \tag{4.26}$$

It follows from (4.1) that for $(x,t) \in \Omega_s(T)$,

$$D_+Y_1 = -a(\rho)Y_1^2 - b_1(x,t)Y_1 + f_1(x,t) + d(\rho)(\rho - \mathcal{C}(x)), \tag{4.27}$$

$$D_-Z_1 = -a(\rho)Z_1^2 - b_2(x,t)Z_1 + f_2(x,t) + d(\rho)(\rho - \mathcal{C}(x)), \tag{4.28}$$

where

$$b_1(x,t) = b_2(x,t) = \tfrac{1}{2} - 2a(\rho)h(\rho), \tag{4.29}$$

$$f_1(x,t) = f_2(x,t) = h(\rho)(\tfrac{1}{2} - a(\rho)h(\rho)), \tag{4.30}$$

and that

$$D_+Y_2 = -b_3(x,t)Y_2 + f_3(x,t) + (d(\rho - \mathcal{C}(x)))_x(x,t), \tag{4.31}$$

$$D_-Z_2 = -b_4(x,t)Z_2 + f_4(x,t) + (d(\rho - \mathcal{C}(x)))_x(x,t), \tag{4.32}$$

where

$$b_3(x,t) = \tfrac{1}{2} + \tfrac{1}{2}(\gamma+3)u_x + \gamma\psi'\rho_x, \tag{4.33}$$

$$b_4(x,t) = \tfrac{1}{2} + \tfrac{1}{2}(\gamma+3)u_x + \gamma\psi'\rho_x, \tag{4.34}$$

$$f_3(x,t) = -[a'(\rho)(d(\rho)s_x + h(\rho))^2 + b_1'(\rho)(d(\rho)s_x + h(\rho)) - f_1'(\rho)]\rho_x, \tag{4.35}$$

$$f_4(x,t) = -[a'(\rho)(d(\rho)r_x + h(\rho))^2 + b_1'(\rho)(d(\rho)r_x + h(\rho)) - f_1'(\rho)]\rho_x. \tag{4.36}$$

The next step is to obtain the relation for Y_2 and Z_2 along $x = x_i(t)$ $(i = 1,2)$. Differentiating (4.10) with respect to t, one obtains that along $x = x_2(t)$,

$$\begin{aligned}(r_2^-)_{xx} = {} & A_{1,2}^+ \frac{\lambda_{1,2}^+ - \dot{x}_2}{\lambda_{1,2}^- - \dot{x}_2}(r_2^+)_{xx} - A_{2,2}^+ \frac{\lambda_{2,2}^+ - \dot{x}_2}{\lambda_{1,2}^- - \dot{x}_2}(s_2^+)_{xx} \\ & + A_{2,2}^- \frac{\lambda_{2,2}^- - \dot{x}_2}{\lambda_{1,2}^- - \dot{x}_2}(s_2^-)_{xx} + B_2(t),\end{aligned} \tag{4.37}$$

where

$$\begin{aligned}B_2(t) = \frac{1}{\lambda_{1,2}^- - \dot{x}_2}\{ & -(\lambda_{1,2}^-)_x(r_2^-)_x - \tfrac{1}{2}(s_2^- + r_2^-)_x - \dot{B}_0(t) \\ & + A_{1,2}^+((\lambda_{1,2}^+)_x(r_2^+)_x + \tfrac{1}{2}(s_2^+ + r_2^+)_x) - \dot{A}_{1,2}^+(r_2^+)_x \\ & - A_{2,2}^+((\lambda_{2,2}^+)_x(s_2^+)_x - \tfrac{1}{2}(s_2^+ + r_2^+)_x) + \dot{A}_{2,2}^+(s_2^+)_x \\ & + A_{2,2}^-((\lambda_{2,2}^-)_x(s_2^-)_x + \tfrac{1}{2}(s_2^- + r_2^-)_x) - \dot{A}_{2,2}^-(s_2^-)_x \\ & + (\phi_2^-)_{xx} - A_{1,2}^+(\phi_2^+)_{xx} + A_{2,2}^+(\phi_2^+)_{xx} - A_{2,2}^-(\phi_2^-)_{xx}\}.\end{aligned} \tag{4.38}$$

It follows from (4.37) that

$$\begin{aligned}Z_2^-(x_2(t)-0,t) = {} & B_{1,2}^+ Z_2^+(x_2(t)+0,t) - B_{2,2}^+ Y_2^+(x_2(t)+0,t) \\ & + B_{2,2}^- Y_2^-(x_2(t)-0,t) + B_3(t),\end{aligned} \tag{4.39}$$

where

$$B_{1,2}^+(t) = \left(\frac{\lambda_{1,2}^+ - \dot{x}_2}{\lambda_{1,2}^- - \dot{x}_2}\right)^4 \left(\frac{\rho_2^+}{\rho_2^-}\right)^{\frac{9-3\gamma}{4}} (x_2(t),t),$$

$$B_{2,2}^+(t) = \left(\frac{\lambda_{2,2}^+ - \dot{x}_2}{\lambda_{1,2}^- - \dot{x}_2}\right)^4 \left(\frac{\rho_2^+}{\rho_2^-}\right)^{\frac{9-3\gamma}{4}} (x_2(t),t),$$

$$B_{2,2}^-(t) = \left(\frac{\lambda_{2,2}^- - \dot{x}_2}{\lambda_{1,2}^- - \dot{x}_2}\right)^4 (x_2(t),t),$$

and

$$\begin{aligned}B_3(t) = {} & (h_2^-)_x + (d_2^-)_x(r_2^-)_x - B_{1,2}^+((h_2^+)_x + (d_2^+)_x(r_2^+)_x) \\ & + B_{2,2}^+((h_2^+)_x + (d_2^+)_x(s_2^+)_x) \\ & - B_{2,2}^-((h_2^-)_x + (d_2^-)_x(s_2^-)_x) + d_2^- B_2(t).\end{aligned} \tag{4.40}$$

Differentiating (4.12) with respect to t, we have that along $x = x_1(t)$,

$$
\begin{aligned}
(s_1^+)_{xx} = A_{2,1}^- \frac{\lambda_{2,1}^- - \dot{x}_1}{\lambda_{2,1}^+ - \dot{x}_1}(s_2^-)_{xx} + A_{1,1}^+ \frac{\lambda_{1,1}^+ - \dot{x}_1}{\lambda_{2,1}^+ - \dot{x}_1}(r_2^+)_{xx} \\
- A_{1,1}^- \frac{\lambda_{1,1}^- - \dot{x}_1}{\lambda_{2,1}^+ - \dot{x}_1}(r_2^-)_{xx} + B_4(t),
\end{aligned}
\tag{4.41}
$$

where

$$
\begin{aligned}
B_4(t) \frac{1}{\lambda_{2,1}^+ - \dot{x}_1} \{ & -(\lambda_{2,1}^+)_x (s_1^+)_x - \tfrac{1}{2}(s_1^+ + r_1^+)_x - \dot{B}_1(t) \\
& + A_{2,1}^- ((\lambda_{2,1}^-)_x (s_1^-)_x + \tfrac{1}{2}(s_1^- + r_1^-)_x) - \dot{A}_{2,1}^- (s_1^-)_x \\
& + A_{1,1}^+ ((\lambda_{1,1}^+)_x (r_1^+)_x + \tfrac{1}{2}(s_1^+ + r_1^+)_x) - \dot{A}_{1,1}^+ (r_2^+)_x \\
& - A_{1,1}^- ((\lambda_{1,1}^-)_x (r_1^-)_x + \tfrac{1}{2}(s_1^- + r_1^-)_x) + \dot{A}_{1,1}^- (s_r^-)_x \\
& + (\phi_1^+)_{xx} - A_{2,1}^- (\phi_1^-)_{xx} - A_{1,1}^+ (\phi_1^+)_{xx} + A_{1,1}^- (\phi_1^-)_{xx} \}.
\end{aligned}
\tag{4.42}
$$

It follows from (4.41) that

$$
\begin{aligned}
Y_2^+(x_1(t)+0,t) = B_{2,1}^- Y_2^-(x_1(t)-0,t) + B_{1,1}^+ Z_2^+(x_1(t)+0,t) \\
- B_{1,1}^- Z_2^-(x_1(t)-0,t) + B_5(t),
\end{aligned}
\tag{4.43}
$$

where

$$
\begin{aligned}
B_{2,1}^-(t) &= \left(\frac{\lambda_{2,1}^- - \dot{x}_1}{\lambda_{2,1}^+ - \dot{x}_1} \right)^4 \left(\frac{\rho_2^+}{\rho_2^-} \right)^{\frac{3\gamma-9}{4}} (x_1(t),t), \\
B_{1,1}^+(t) &= \left(\frac{\lambda_{1,1}^+ - \dot{x}_1}{\lambda_{2,1}^- - \dot{x}_1} \right)^4 (x_1(t),t), \\
B_{1,1}^-(t) &= \left(\frac{\lambda_{1,1}^- - \dot{x}_1}{\lambda_{2,1}^- - \dot{x}_1} \right)^4 \left(\frac{\rho_2^+}{\rho_2^-} \right)^{\frac{3\gamma-9}{4}} (x_1(t),t),
\end{aligned}
$$

and

$$
\begin{aligned}
B_5(t) = (h_1^+)_x + (d_1^+)_x (s_1^+)_x - B_{2,1}^- ((h_1^-)_x + (d_1^-)_x (s_1^-)_x) \\
- B_{1,1}^+ ((h_1^+)_x + (d_1^+)_x (r_1^+)_x) \\
+ B_{1,1}^- ((h_1^-)_x + (d_1^-)_x (r_1^-)_x) + d_1^- B_4(t).
\end{aligned}
\tag{4.44}
$$

LEMMA 4.3. Under the assumption (4.6)–(4.7), it follows that

$$
\begin{aligned}
|A_{1,2}^+ - 1| + |B_{1,2}^+ - 1| + |A_{2,2}^\pm| + |B_{2,2}^\pm| = O(1)[\rho]_2, \\
|A_{2,1}^- - 1| + |B_{2,1}^- - 1| + |A_{1,1}^\pm| + |B_{1,1}^\pm| = O(1)[\rho]_1,
\end{aligned}
\tag{4.45}
$$

$$
\begin{aligned}
|B_2(t)| + |B_3(t)| \le C|[\rho]_2| + \max_{(x,t)\in\Omega_s(T)} (|\rho_x(x,t)| + |u_x(x,t)|), \\
|B_4(t)| + |B_5(t)| \le C|[\rho]_1| + \max_{(x,t)\in\Omega_s(T)} (|\rho_x(x,t)| + |u_x(x,t)|).
\end{aligned}
\tag{4.46}
$$

Proof. By (4.18)–(4.19) and the following,

$$|B_0(t)| + |\dot{B}_0(t)| \le C|[\rho]_2|, \qquad |B_1(t)| + |\dot{B}_1(t)| \le C|[\rho]_1|, \tag{4.47}$$

$$|\dot{A}^+_{1,2}| + |\dot{A}^+_{2,2}| + |\dot{A}^-_{2,2}| + |\dot{A}^-_{2,1}| + |\dot{A}^+_{1,1}| + |\dot{A}^-_{1,1}| \le C, \tag{4.48}$$

one can prove (4.45)–(4.46). □

LEMMA 4.4. Assume that the piecewise smooth solution of the IVP (1.1) and (1.5) exists on $\Omega_s(T)$. Then, under the assumptions of Theorem 3.1, it follows that on $\Omega_s(T)$,

$$\begin{aligned} |s_{xx}| + |r_{xx}| &\le C \left(\mu_0 + \mu_1 + \delta_0 + \eta_0 + \max_{(x,t)\in\Omega_s(T)} (|\rho_x| + |u_x|) \right), \\ |s_{xxx}| + |r_{xxx}| &\le C, \end{aligned} \tag{4.49}$$

provided that (4.6)–(4.7) hold.

Proof. We can estimate the terms on the left-hand side of (4.49) on $\Omega_\pm(T)$ by solving initial value problems due to the entropy conditions. On $\Omega_0(T)$, the bounds of $|s_{xx}|, |r_{xx}|, |s_{xxx}|$, and $|r_{xxx}|$ can be obtained by solving the corresponding boundary value problems for them. In fact, on $\Omega_\pm(T)$, the systems (4.31) and (4.32) are linear for (Z_2, Y_2). Integrating (4.31) and (4.32) along 1-characteristics and 2-characteristics respectively on $[0,t]$, noticing, due to (4.6), that

$$b_3(x,t) \ge \frac{1}{3}, \quad b_4(x,t) \ge \frac{1}{3}, \quad (x,t) \in \Omega_\pm(T), \tag{4.50}$$

and

$$|f_3(x,t)| + |f_4(x,t)| + |(d(\rho - \mathcal{C}(x)))_x| \le C \left(\eta_0 + \max_{(x,t)\in\Omega_\pm(T)} |\rho_x(x,t)| \right), \tag{4.51}$$

we can show, for any $(x,t) \in \Omega_\pm(T)$, that

$$|Y_2(x,t)| + |Z_2(x,t)| \le C \left(\mu_1 + \mu_0 + \eta_0 + \max_{(x,t)\in\Omega_\pm(T)} |\rho_x(x,t)| \right). \tag{4.52}$$

Then, by (3.6), we gain, for any $(x,t) \in \Omega_\pm(T)$, that

$$|s_{xx}| + |r_{xx}| \le C \left(\mu_1 + \mu_0 + \eta_0 + \max_{(x,t)\in\Omega_\pm(T)} |\rho_x(x,t)| \right). \tag{4.53}$$

Similarly, we can yield the corresponding IVP for $(Y_2)_x$ and $(Z_2)_x$ on $\Omega_\pm(T)$ and then obtain, for any $(x,t) \in \Omega_\pm(T)$, that

$$|s_{xxx}| + |r_{xxx}| \le C \left(\mu_2 + \mu_1 + \mu_0 + \eta_0 + \max_{(x,t)\in\Omega_\pm(T)} |\rho_x(x,t)| \right).$$

Now we estimate them on $\Omega_0(T)$. By (3.10), (3.8)–(3.9), (4.10), and (4.12), one can show

$$\lim_{t\to 0} (|(r_x, s_x)(x_1(t)+0, t)| + |(r_x, s_x)(x_2(t)-0, t)|) \le C(\mu_1 + \mu_0 + \delta_0). \tag{4.54}$$

By (4.54) and the assumptions of Theorem 3.1, we find

$$\lim_{t\to 0} |B_7(t)| \le C(\mu_1 + \mu_0 + \delta_0). \tag{4.55}$$

Thus, we conclude, in terms of (4.54), (4.39), (4.43), (4.55), and (4.45), that

$$\lim_{t\to 0}(|(Z_2,Y_2)(0,t)|)\le C(\mu_1+\mu_0+\delta_0). \tag{4.56}$$

On $\Omega_0(T)$, solving the boundary value problems (4.31), (4.32), (4.39), and (4.43) for (Y_2,Z_2), we obtain, in view of Lemmas 4.2–4.3, (4.56) and (4.26), that

$$|s_{xx}|+|r_{xx}|\le C\left(\mu_1+\mu_0+\delta_0+\eta_0+\max_{(x,t)\in\Omega_s(T)}(|\rho_x(x,t)|+|u_x(x,t)|)\right), \tag{4.57}$$

for any $(x,t)\in\Omega_0(T)$ with $t>0$, provided that (4.6) and (4.7) hold. Similarly, we can obtain, for $(x,t)\in\Omega_0(T)$ with $t>0$, that

$$|s_{xxx}|+|r_{xxx}|\le C\left(\mu_2+\mu_1+\mu_0+\delta_0+\eta_0+\max_{(x,t)\in\Omega_s(T)}(|\rho_x(x,t)|+|u_x(x,t)|)\right).$$

Thus, the proof is completed. □

LEMMA 4.5. Assume that the piecewise smooth solution of the IVP (1.1) and (1.5) exists on $\Omega_s(T)$. Then, it follows that along $x=x_i(t)$ $(i=1,2)$,

$$|[s_x]_i|+|[r_x]_i|+|[s_{xx}]_i|+|[r_{xx}]_i|\le C\delta_0e^{-\beta_3t},\quad 0\le t<T, \tag{4.58}$$

$$|[\rho_x]_i|+|[u_x]_i|+|[\rho_{xx}]_i|+|[u_{xx}]_i|\le C\delta_0e^{-\beta_3t},\quad 0\le t<T, \tag{4.59}$$

provided that (4.6)–(4.7) hold.

Proof. We first estimate $[s_x]_2$. Due to Lemma 4.2 and

$$|[s_x]_2|\le C(|[Y_1]_2|+|[v]_2|),$$

we only need to estimate $[Y_1]_2$. By (4.27) it follows, along $x=x_2(t)$, that

$$D_t^+[Y_1]_2=-b_5(t)[Y_1]_2+f_7(t), \tag{4.60}$$

where

$$b_5(t)=\tfrac12-a(\rho_2^+)[h(\rho)]_2+a(\rho_2^+)d(\rho_2^-)(s_2^-)_x+a(\rho_2^+)d(\rho_2^+)(s_2^+)_x, \tag{4.61}$$

$$\begin{aligned}f_7(t)&=(\dot x_2-\lambda_{2,2}^+)(Y_1^+)_x+(\lambda_{2,2}^--\dot x_2)(Y_1^-)_x\\&\quad-[a(\rho)]_2(Y_1^-)^2-[b_1(\rho)]_2Y_1^-+[f_1]_2+[d(\rho)(\rho-C(x))].\end{aligned} \tag{4.62}$$

By (4.23), (4.24), (4.18), Lemma 4.2, and Lemma 4.4, we have

$$b_5(t)\ge\frac13 \tag{4.63}$$

and

$$\begin{aligned}&|[h(\rho)]_2|+|[a(\rho)]_2|+|[b_1(\rho)]_2|+|[f_1(\rho)]_2|\\&\quad+|\dot x_2-\lambda_{2,2}^\pm|+|[d(\rho)(\rho-C(x))]_2|\le C\delta_0e^{-\beta_1t},\end{aligned} \tag{4.64}$$

provided that (4.6)–(4.7) hold.

Then, it follows from (4.64) and Lemma 4.4 that

$$|f_7(t)|\le C\delta_0e^{-\beta_1t}. \tag{4.65}$$

Multiplying (4.60) by $e^{\int_0^t b_5(\tau)\,d\tau}$ and integrating it over $[0,t]$, we have, in terms of (4.63) and (4.65), that

$$|[Y_1]_2| \leq C\delta_0 e^{-\beta_3 t}, \tag{4.66}$$

where

$$\beta_3 = \begin{cases} \frac{1}{3}, & \text{if } B_1 > \frac{1}{3}, \\ \beta_1, & \text{if } B_1 < \frac{1}{3}, \\ \frac{1}{4}, & \text{otherwise.} \end{cases}$$

Similarly, we can estimate the other terms on the left-hand side of (4.58) in view of Lemmas 4.2–4.4, (4.28)–(4.32). The estimate (4.59) follows from (4.58) and (3.7). $\square$

4.3. *Energy estimates.* In this subsection, we need to estimate the bounds of ρ, ρ_x, and u_x on $\Omega_s(T)$. In fact, what we do is to obtain the following energy estimates.

LEMMA 4.6. Under the assumptions of Theorem 3.1, it follows that

$$\|(\varepsilon, \varepsilon_x, \varepsilon_t)(t)\|^2 + \int_0^t \|(\varepsilon, \varepsilon_x, \varepsilon_t)(\tau)\|^2\,d\tau \leq C(\mu_0 + \mu_1 + \delta_0 + \eta_0)e^{-\beta_4 t}, \tag{4.67}$$

$$\|(\varepsilon_x, \varepsilon_{xx}, \varepsilon_{xt})(t)\|^2 + \int_0^t \|(\varepsilon_x, \varepsilon_{xx}, \varepsilon_{xt})(\tau)\|^2\,d\tau \leq C(\mu_0 + \mu_1 + \delta_0 + \eta_0)e^{-\beta_5 t}, \tag{4.68}$$

$$\|(\varepsilon_{xx}, \varepsilon_{xxx}, \varepsilon_{xxt})(t)\|^2 + \int_0^t \|(\varepsilon_{xx}, \varepsilon_{xxx}, \varepsilon_{xxt})(\tau)\|^2\,d\tau \leq C(\mu_0 + \mu_1 + \delta_0 + \eta_0)e^{-\beta_6 t}, \tag{4.69}$$

provided that (4.6)–(4.7) hold.

Proof. From $(1.1)_1$ and $(1.9)_1$, one has

$$(\rho - \tilde{\rho})_t + (\rho u - \tilde{j})_x = 0. \tag{4.70}$$

Define

$$\varepsilon(x,t) = \int_{-\infty}^{x} (\rho(y,t) - \tilde{\rho}(y))\,dy. \tag{4.71}$$

Then from (4.70), (4.71), and $(1.1)_3$, one obtains on $\Omega_s(T)$ that

$$\varepsilon_{xt} + (\rho u - \tilde{j})_x = 0, \tag{4.72}$$

$$\varepsilon_x = \rho(x,t) - \tilde{\rho}(x), \quad \varepsilon_t = -(j(x,t) - \tilde{j}), \quad \varepsilon = E - \widetilde{E}. \tag{4.73}$$

Integrating (4.72) with respect to x over $(-\infty, x)$ together with the shock properties (3.8) and (3.9) leads to

$$\varepsilon_t + (\rho u - \tilde{j}) = 0, \tag{4.74}$$

for $(x,t) \in \Omega_s(T)$.

Applying the relations (4.73) and (4.74), the IVP (1.1)–(1.5) can be reformulated, via (1.9), into

$$\begin{cases} \varepsilon_{tt} + \varepsilon_t - \left(\left(p'(\tilde{\rho}) - \dfrac{\tilde{j}^2}{\tilde{\rho}^2} \right) \varepsilon_x \right)_x + \left(\dfrac{2\tilde{j}}{\tilde{\rho}} \varepsilon_t \right)_x + \tilde{\rho}\varepsilon + \widetilde{E}\varepsilon_x \\ \qquad = g_{1x} + g_{2x} - \varepsilon\varepsilon_x, \\ (\varepsilon, \varepsilon_t)|_{t=0} = (\varepsilon_0, \varepsilon_1)(x) =: \left(\displaystyle\int_{-\infty}^{x} (\rho_0 - \tilde{\rho})(y)\, dy, -(\rho_0 u_0 - \tilde{j})(x) \right), \end{cases} \tag{4.75}$$

for $(x,t) \in \Omega_s(T)$, where

$$g_1 = \frac{(\tilde{j} - \varepsilon_t)^2}{\tilde{\rho} + \varepsilon_x} - \frac{\tilde{j}^2}{\tilde{\rho}} + \frac{2\tilde{j}}{\tilde{\rho}}\varepsilon_t + \frac{\tilde{j}^2}{\tilde{\rho}^2}\varepsilon_x, \tag{4.76}$$

$$g_2 = p(\tilde{\rho} + \varepsilon_x) - p(\tilde{\rho}) - p'(\tilde{\rho})\varepsilon_x. \tag{4.77}$$

Proof of (4.67). Multiplying (4.75) by $(\frac{1}{2}\varepsilon + \varepsilon_t)$, we have

$$\{G_1(\varepsilon, \varepsilon_x, \varepsilon_t)\}_t + G_2(\varepsilon, \varepsilon_x, \varepsilon_t) - \{G_1(\varepsilon, \varepsilon_x, \varepsilon_t)\}_x = (g_{1x} + g_{2x} - \varepsilon\varepsilon_x)(\tfrac{1}{2}\varepsilon + \varepsilon_t), \tag{4.78}$$

where

$$G_1(\varepsilon, \varepsilon_x, \varepsilon_t) = (\frac{1}{4} + \frac{\tilde{\rho}}{2})\varepsilon^2 + \frac{1}{2}\varepsilon\varepsilon_t + \frac{1}{2}\varepsilon_t^2 + \frac{1}{2}\left(p'(\tilde{\rho}) - \frac{\tilde{j}^2}{\tilde{\rho}^2} \right) \varepsilon_x^2, \tag{4.79}$$

$$G_2(\varepsilon, \varepsilon_x, \varepsilon_t) = \frac{1}{2}\tilde{\rho}\varepsilon^2 + \frac{1}{2}\widetilde{E}\varepsilon\varepsilon_x + \left(\frac{1}{2} - \frac{\tilde{j}\tilde{\rho}_x}{\tilde{\rho}^2} \right) \varepsilon_t^2 + \left(\widetilde{E} - \frac{\tilde{j}}{\tilde{\rho}} \right) \varepsilon_x\varepsilon_t + \frac{1}{2}\left(p'(\tilde{\rho}) - \frac{\tilde{j}^2}{\tilde{\rho}^2} \right) \varepsilon_x^2, \tag{4.80}$$

$$G_3(\varepsilon, \varepsilon_x, \varepsilon_t) = \frac{1}{2}\left(p'(\tilde{\rho}) - \frac{\tilde{j}^2}{\tilde{\rho}^2} \right) \varepsilon\varepsilon_x + \frac{\tilde{j}}{\tilde{\rho}}\varepsilon\varepsilon_t + \left(p'(\tilde{\rho}) - \frac{\tilde{j}^2}{\tilde{\rho}^2} \right) \varepsilon_t\varepsilon_x + \frac{\tilde{j}}{\tilde{\rho}}\varepsilon_t^2. \tag{4.81}$$

Integrating (4.78) over R, one has

$$\begin{aligned} &\frac{d}{dt} \fint_{-\infty}^{+\infty} G_1(\varepsilon, \varepsilon_x, \varepsilon_t)\, dx + \fint_{-\infty}^{+\infty} G_2(\varepsilon, \varepsilon_x, \varepsilon_t)\, dx + \dot{x}_2[G_1(\varepsilon, \varepsilon_x, \varepsilon_t)]_2 \\ &\quad + [G_3(\varepsilon, \varepsilon_x, \varepsilon_t)]_2 + \dot{x}_1[G_1(\varepsilon, \varepsilon_x, \varepsilon_t)]_1 + [G_3(\varepsilon, \varepsilon_x, \varepsilon_t)]_1 \\ &\quad = \fint_{-\infty}^{+\infty} (g_{1x} + g_{2x} - \varepsilon\varepsilon_x) \left(\frac{1}{2}\varepsilon + \varepsilon_t \right) dx. \end{aligned} \tag{4.82}$$

For $a_1 + \tilde{\rho}_m/\rho_* \ll 1$, one can verify that

$$c_4(\varepsilon^2 + \varepsilon_x^2 + \varepsilon_t) \le G_1(\varepsilon, \varepsilon_x, \varepsilon_t) \le c_4'(\varepsilon^2 + \varepsilon_x^2 + \varepsilon_t), \tag{4.83}$$

$$c_5(\varepsilon^2 + \varepsilon_x^2 + \varepsilon_t) \le G_2(\varepsilon, \varepsilon_x, \varepsilon_t) \le c_5'(\varepsilon^2 + \varepsilon_x^2 + \varepsilon_t), \tag{4.84}$$

where

$$0 < c_4 = \frac{1}{2}\min\left\{\rho_*, \frac{1}{2}, P_*\right\}, \qquad 0 < c_4' = \frac{1}{2}\max\left\{\rho^* + 1, \frac{3}{2}, P^*\right\},$$
$$0 < c_5 = \frac{1}{2}\min_{x\in R}\left\{\frac{\rho_*}{2}, \frac{1}{2} - \frac{2\tilde{j}\tilde{\rho}_x}{\tilde{\rho}^2}, \qquad P_*\left(1 - P^*\frac{2\tilde{\rho}_x^2}{\tilde{\rho}^2}\right) - \frac{\widetilde{E}^2}{2\tilde{\rho}}\right\},$$
$$0 < c_5' = \frac{1}{2}\max_{x\in R}\left\{\frac{3\rho^*}{2}, \frac{3}{2} - \frac{2\tilde{j}\tilde{\rho}_x}{\tilde{\rho}_2^2}, \qquad P^* + (P^*)^2\frac{2\tilde{\rho}_x^2}{\tilde{\rho}^2} + \frac{\widetilde{E}^2}{2\tilde{\rho}}\right\},$$

$$P^* = \max_{x\in R}\{p'(\tilde{\rho}) - \frac{\tilde{j}^2}{\tilde{\rho}^2}\}, \qquad P_* = \min_{x\in R}\left\{p'(\tilde{\rho}) - \frac{\tilde{j}^2}{\tilde{\rho}^2}\right\}.$$

It follows from (4.83)–(4.84) that

$$\frac{c_5}{c_4'}G_1(\varepsilon, \varepsilon_x, \varepsilon_t) \le G_2(\varepsilon, \varepsilon_x, \varepsilon_t). \tag{4.85}$$

By Theorem 2.1, Lemma 4.5, (4.73), $(3.8)_3$, and $(3.9)_3$, one has

$$\begin{aligned} &|\dot{x}_2[G_1(\varepsilon, \varepsilon_x, \varepsilon_t)]_2| + |\dot{x}_1[G_1(\varepsilon, \varepsilon_x, \varepsilon_t)]_1| \\ &\qquad + |[G_3(\varepsilon, \varepsilon_x, \varepsilon_t)]_2| + |[G_3(\varepsilon, \varepsilon_x, \varepsilon_t)]_1| \\ &\le C(|[(\rho, \rho_x, u, u_x)]_1| + |[(\rho, \rho_x, u, u_x)]_2|) \\ &\le C\delta_0 e^{-\beta_3 t}. \end{aligned} \tag{4.86}$$

From (4.76)–(4.77), a straightforward calculation yields

$$|\partial_x g_1| = O(1)(|\varepsilon_{xx}| + |\varepsilon_{xt}| + |\tilde{\rho}_x|)(|\varepsilon_t| + |\varepsilon_x|), \tag{4.87}$$
$$|\partial_x g_2| = O(1)|\varepsilon_{xx}\varepsilon_x|. \tag{4.88}$$

Thus, the right-hand side of (4.82) can be estimated as

$$\begin{aligned} &\left|\fint_{-\infty}^{+\infty}(g_{1x} + g_{2x} - \varepsilon\varepsilon_x)(\tfrac{1}{2}\varepsilon + \varepsilon_t)\,dx\right| \\ &\qquad \le C\eta\fint_{-\infty}^{+\infty}(\varepsilon^2 + \varepsilon_x^2 + \varepsilon_t^2)(x,t)\,dx \\ &\qquad \le C\eta\fint_{-\infty}^{+\infty}G_2(\varepsilon, \varepsilon_x, \varepsilon_t)\,dx. \end{aligned} \tag{4.89}$$

Substituting (4.89), (4.86), and (4.85) into (4.82) and integrating it over $[0,t]$, one obtains (4.67) in terms of Lemma 4.2.

Proof of (4.68). Differentiating (4.75) with respect to x, we have

$$\begin{aligned} \varepsilon_{xtt} + \varepsilon_{xt} - \left(\left(p'(\tilde{\rho}) - \frac{\tilde{j}^2}{\tilde{\rho}^2}\right)\varepsilon_{xx}\right)_x + \left(\frac{2\tilde{j}}{\tilde{\rho}}\varepsilon_{xt}\right)_x + \tilde{\rho}\varepsilon_x + \widetilde{E}\varepsilon_{xx} \\ = g_3 + g_{1xx} + g_{2xx} + g_{3x} + g_4, \end{aligned} \tag{4.90}$$

where

$$g_3 = \left(\frac{2\tilde{j}\tilde{\rho}_x}{\tilde{\rho}^2} \varepsilon_{xt} \right)_x, \tag{4.91}$$

$$g_4 = \left(\left(p'(\tilde{\rho}) - \frac{\tilde{j}^2}{\tilde{\rho}^2} \right)_x \varepsilon_x \right)_x + \tilde{\rho}_x \varepsilon + \widetilde{E}_x \varepsilon_x - (\varepsilon\varepsilon_x)_x. \tag{4.92}$$

Multiplying (4.90) by $(\frac{1}{2}\varepsilon_x + \varepsilon_{xt})$ and integrating with respect to x over R, we have

$$\begin{aligned} &\frac{d}{dt} \fint_{-\infty}^{+\infty} G_1(\varepsilon_x, \varepsilon_{xx}, \varepsilon_{xt})\, dx + \fint_{-\infty}^{+\infty} G_2(\varepsilon_x, \varepsilon_{xx}, \varepsilon_{xt})\, dx + \dot{x}_2 [G_1(\varepsilon_x, \varepsilon_{xx}, \varepsilon_{xt})]_2 \\ &\qquad + [G_3(\varepsilon_x, \varepsilon_{xx}, \varepsilon_{xt})]_2 + \dot{x}_1 [G_1(\varepsilon_x, \varepsilon_{xx}, \varepsilon_{xt})]_1 + [G_3(\varepsilon_x, \varepsilon_{xx}, \varepsilon_{xt})]_1 \\ &\qquad = \fint_{-\infty}^{+\infty} g_4 \left(\frac{1}{2}\varepsilon_x + \varepsilon_{xt} \right) dx + \fint_{-\infty}^{+\infty} (g_{1x} + g_{2x} + g_3)_x \left(\frac{1}{2}\varepsilon_x + \varepsilon_{xt} \right) dx. \end{aligned} \tag{4.93}$$

Similarly, in terms of Theorem 2.1, Lemma 4.5, (4.73), $(3.8)_3$, and $(3.9)_3$, one can verify that

$$\begin{aligned} &|\dot{x}_2 [G_1(\varepsilon_x, \varepsilon_{xx}, \varepsilon_{xt})]_2| + |\dot{x}_1 [G_1(\varepsilon_x, \varepsilon_{xx}, \varepsilon_{xt})]_1| \\ &\qquad + |[G_3(\varepsilon_x, \varepsilon_{xx}, \varepsilon_{xt})]_2| + |[G_3(\varepsilon_x, \varepsilon_{xx}, \varepsilon_{xt})]_1| \\ &\qquad \le C(|[(\rho, \rho_x, \rho_{xx}, u, u_x, u_{xx})]_1| + |[(\rho, \rho_x, \rho_{xx}, u, u_x, u_{xx})]_2|) \\ &\qquad \le C\delta_0 e^{-\beta_3 t}. \end{aligned} \tag{4.94}$$

The right-hand side terms of (4.93) can be estimated as follows:

$$\begin{aligned} &\left| \fint_{-\infty}^{+\infty} g_4 \left(\frac{1}{2}\varepsilon_x + \varepsilon_{xt} \right) dx \right| \\ &\qquad \le C\eta \fint_{-\infty}^{+\infty} G_2(\varepsilon_x, \varepsilon_{xx}, \varepsilon_{xt})\, dx + C(\|(\varepsilon_0, \varepsilon_{0x}, \varepsilon_1)\|^2 + \delta) e^{-\beta_4 t}, \end{aligned} \tag{4.95}$$

$$\begin{aligned} &\left| \frac{1}{2} \fint_{-\infty}^{+\infty} (g_{1x} + g_{2x} + g_3)_x \varepsilon_x \, dx \right| \\ &\qquad \le \frac{1}{2} |[(g_{1x} + g_{2x} + g_3)_x \varepsilon_x]_1| + \frac{1}{2} |[(g_{1x} + g_{2x} + g_3)_x \varepsilon_x]_2| \\ &\qquad\quad + \left| \frac{1}{2} \fint_{-\infty}^{+\infty} (g_{1x} + g_{2x} + g_3) \varepsilon_{xx}\, dx \right| \\ &\qquad \le C\eta \fint_{-\infty}^{+\infty} G_2(\varepsilon_x, \varepsilon_{xx}, \varepsilon_{xt})\, dx \\ &\qquad\quad + C(\|(\varepsilon_0, \varepsilon_{0x}, \varepsilon_1)\|^2 + \delta) e^{-\beta_4 t} + C(\delta_1 + \delta_2) e^{\beta_3 t}, \end{aligned} \tag{4.96}$$

$$
\begin{aligned}
&\fint_{-\infty}^{+\infty} g_{3x}\varepsilon_{xt}\,dx \\
&= \fint_{-\infty}^{+\infty} \left(\frac{2\tilde{j}\tilde{\rho}_x}{\tilde{\rho}^2}\varepsilon_{xt}\right)_x \varepsilon_{xt}\,dx \\
&\le C(\eta+\delta_0)\fint_{-\infty}^{+\infty} G_2(\varepsilon_x,\varepsilon_{xx},\varepsilon_{xt})\,dx + \left|\left[\left(\frac{2\tilde{j}\tilde{\rho}_x}{\tilde{\rho}^2}\varepsilon_{xt}\right)\varepsilon_{xt}\right]_1\right| + \left|\left[\left(\frac{2\tilde{j}\tilde{\rho}_x}{\tilde{\rho}^2}\varepsilon_{xt}\right)\varepsilon_{xt}\right]_2\right| \\
&\quad + \frac{1}{2}\left|\left[\frac{2\tilde{j}\tilde{\rho}_x}{\tilde{\rho}^2}\varepsilon_{xt}^2\right]_1\right| + \frac{1}{2}\left|\left[\frac{2\tilde{j}\tilde{\rho}_x}{\tilde{\rho}^2}\varepsilon_{xt}^2\right]_2\right| \\
&\le C(\eta_0+\delta_0)\fint_{-\infty}^{+\infty} G_2(\varepsilon_x,\varepsilon_{xx},\varepsilon_{xt})\,dx + C\delta_0 e^{\beta_3 t},
\end{aligned}
\tag{4.97}
$$

$$
\begin{aligned}
&\fint_{-\infty}^{+\infty} g_{2xx}\varepsilon_{xt}\,dx \\
&\le C\eta\fint_{-\infty}^{+\infty} G_2(\varepsilon_x,\varepsilon_{xx},\varepsilon_{xt})\,dx + \fint_{-\infty}^{+\infty} ((p'(\tilde{\rho}+\varepsilon_x)-p'(\tilde{\rho}))\varepsilon_{xx})_x\varepsilon_{xt}\,dx \\
&\le C\eta\fint_{-\infty}^{+\infty} G_2(\varepsilon_x,\varepsilon_{xx},\varepsilon_{xt})\,dx \\
&\quad + |[(p'(\tilde{\rho}+\varepsilon_x)-p'(\tilde{\rho}))\varepsilon_{xx}\varepsilon_{xt}]_1| + |[(p'(\tilde{\rho}+\varepsilon_x)-p'(\tilde{\rho}))\varepsilon_{xx}\varepsilon_{xt}]_2| \\
&\quad + \frac{1}{2}|[p''(\tilde{\rho}+\varepsilon_x)\varepsilon_{xt}\varepsilon_{xx}^2]_1| + \frac{1}{2}|[p''(\tilde{\rho}+\varepsilon_x)\varepsilon_{xt}\varepsilon_{xx}^2]_2| \\
&\quad - \frac{d}{dt}\fint_{-\infty}^{+\infty} (p'(\tilde{\rho}+\varepsilon_x)-p'(\tilde{\rho})) \\
&\le C\eta\fint_{-\infty}^{+\infty} G_2(\varepsilon_x,\varepsilon_{xx},\varepsilon_{xt})\,dx + C(\delta_1+\delta_2)e^{-\beta_3 t} \\
&\quad - \frac{d}{dt}\fint_{-\infty}^{+\infty} (p'(\tilde{\rho}+\varepsilon_x)-p'(\tilde{\rho}))\varepsilon_{xx}^2\,dx,
\end{aligned}
\tag{4.98}
$$

$$
\begin{aligned}
&\fint_{-\infty}^{+\infty} g_{1xx}\varepsilon_{xt}\,dx \\
&\le C\eta \fint_{-\infty}^{+\infty} G_2(\varepsilon_x,\varepsilon_{xx},\varepsilon_{xt})\,dx + \fint_{-\infty}^{+\infty} \left(\frac{2}{\tilde\rho}\varepsilon_x + \frac{2\tilde j}{\tilde\rho^2}\varepsilon_x\right)\varepsilon_{xxt}\varepsilon_{xt}\,dx \\
&\quad + \fint_{-\infty}^{+\infty} 2(\tilde j - \varepsilon_t)\left(\frac{1}{\tilde\rho-\varepsilon_x} - \frac{1}{\tilde\rho} + \frac{1}{\tilde\rho^2}\varepsilon_x\right)\varepsilon_{xxt}\varepsilon_{xt}\,dx \\
&\quad + \fint_{-\infty}^{+\infty} \left(\frac{2\tilde j}{\tilde\rho^2}\varepsilon_t\varepsilon_{xx} + (\tilde j-\varepsilon_t)^2\left(\frac{1}{(\tilde\rho-\varepsilon_x)^2} - \frac{1}{\tilde\rho^2}\right)\varepsilon_{xx}\right)_x \varepsilon_{xt}\,dx \\
&\le C\eta \fint_{-\infty}^{+\infty} G_2(\varepsilon_x,\varepsilon_{xx},\varepsilon_{xt})\,dx \\
&\quad + \frac{1}{2}\left(\left|\left[\left(\frac{2}{\tilde\rho}\varepsilon_x + \frac{2\tilde j}{\tilde\rho^2}\varepsilon_x\right)\varepsilon_{xt}^2\right]_1\right| + \frac{1}{2}\left|\left[\left(\frac{2}{\tilde\rho}\varepsilon_x + \frac{2\tilde j}{\tilde\rho^2}\varepsilon_x\right)\varepsilon_{xt}^2\right]_2\right|\right) \\
&\quad + \left|\left[(\tilde j-\varepsilon_t)\left(\frac{1}{\tilde\rho-\varepsilon_x} - \frac{1}{\tilde\rho} + \frac{1}{\tilde\rho^2}\varepsilon_x\right)\varepsilon_{xt}^2\right]_1\right| + \left|\left[(\tilde j-\varepsilon_t)\left(\frac{1}{\tilde\rho-\varepsilon_x} - \frac{1}{\tilde\rho} + \frac{1}{\tilde\rho^2}\varepsilon_x\right)\varepsilon_{xt}^2\right]_2\right| \\
&\quad + \left|\left[(\tilde j-\varepsilon_t)^2\left(\frac{1}{(\tilde\rho-\varepsilon_x)^2} - \frac{1}{\tilde\rho^2}\right)\varepsilon_{xx}\varepsilon_{xt}\right]_1\right| + \left|\left[(\tilde j-\varepsilon_t)^2\left(\frac{1}{(\tilde\rho-\varepsilon_x)^2} - \frac{1}{\tilde\rho^2}\right)\varepsilon_{xx}\varepsilon_{xt}\right]_2\right| \\
&\quad - \fint_{-\infty}^{+\infty} (\tilde j-\varepsilon_t)^2\left(\frac{1}{(\tilde\rho-\varepsilon_x)^2} - \frac{1}{\tilde\rho^2}\right)\varepsilon_{xx}\varepsilon_{xxt}\,dx \\
&\le C\eta \fint_{-\infty}^{+\infty} G_2(\varepsilon_x,\varepsilon_{xx},\varepsilon_{xt})\,dx + |[(\rho,\rho_x,u,u_x)]_1| + |[(\rho,\rho_x,u,u_x)]_2| \\
&\quad - \frac{d}{dt}\fint_{-\infty}^{+\infty} (\tilde j-\varepsilon_t)^2\left(\frac{1}{(\tilde\rho-\varepsilon_x)^2} - \frac{1}{\tilde\rho^2}\right)\varepsilon_{xx}^2\,dx. \qquad (4.99)
\end{aligned}
$$

Then, integrating (4.93) over $[0,t]$, one obtains (4.68) in terms of (4.95)–(4.99), (4.83)–(4.84), and (4.67).

Similarly, one can get the estimate (4.69). Thus, the proof of Lemma 4.6 is completed. □

Proof of Theorem 3.1. With the help of Lemma 4.2 and Lemma 4.6, we can prove that (4.6) on $\Omega_s(T)$ and (4.7) really hold if we choose the initial-boundary data and the initial jump small enough such that $C(\mu_1+\mu_0+\delta_0+\eta_0) < \frac{1}{16}\eta$. Therefore, by the standard continuity argument, we can prove the existence of piecewise smooth subsonic solutions for the discontinuous IVP (1.1)–(1.5) on $\Omega_s(T)$ with any $T>0$. The asymptotic behavior of solutions on a smooth domain and the decays along shock discontinuities then follow. In fact, we have the following theorem.

THEOREM 4.7. Let $T > 0$. Under the assumptions of Theorem 3.1, it follows that for $0 < t < T$,

$$\begin{aligned}
\| (\rho - \tilde{\rho}, u, E - \widetilde{E})(\cdot, t)\|_2^2 + \int_0^t \| (\rho - \tilde{\rho}, u, E - \widetilde{E})_x(\cdot, t)\|^2 \, d\tau \\
\leq C(\mu_1 + \mu_0 + \delta_0 + \eta_0)e^{-\beta_7 t}, \\
|[(\rho, \rho_x, \rho_{xx}, u, u_x, u_{xx})]_1| + |[(\rho, \rho_x, \rho_{xx}, u, u_x, u_{xx})]_2| \leq C\delta_0 e^{-\beta_3 t},
\end{aligned} \tag{4.100}$$

and for $x \neq x(t)$,

$$|\rho_{xx}| + |u_{xx}| \leq C(\mu_1 + \mu_0 + \delta_0 + \eta_0), \qquad |\rho_{xxx}| + |u_{xxx}| < C. \tag{4.101}$$

Theorem 3.1 follows from Theorem 4.7.

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