Appl. Math.-JCU
10B (1995), 411–418

ANALYSES FOR A MATHEMATICAL MODEL OF THE PATTERN FORMATION ON SHELLS OF MOLLUSCS

MEI MING* AND XIAO YINGKUN**

Abstract. This paper analyses a mathematical model of the pattern formation on the shell of molluscs which is actually a kind of reaction–diffusion system. The existence and uniqueness of a global smooth solution of this system with Cauchy problem and its stability and time decay rate are studied by means of an elementary energy method.

1. Introduction

How to understand the pattern formation of a shell of molluscs is an interesting problem for bio–mathematicians. In the nature, some molluscs are quite different in the sense of their sorts, but their pattern formations are so similar. This shows us that they have the same reaction function. In 1969, C. H. Wadington and J. Cowe[7] proposed a concept of the tent–like pattern formation. In 1982, D. Lingdsay[2] proposed one called as pattern formation of bivalved molluscs (for example, shell, etc.). After that, H. Meinhardt and H. Klingler[5] explained that the pattern formation of shell of molluscs is a reaction–diffusion process. We believe that this explanation is reasonable due to that, for example, the formation of oblique–line is just one: a pigment–producing cell can affect its neighbour cell, and that, this neighbour cell will become a pigment–producing cell and influence its neighbour in turn, then step by step, an obligue–line of pigment–producing cells will be formed. For this phenomenon, H. Meinhardt and H. Klingler[5] gave a model as follows:

$$a_t - D_a \Delta a = -\mu a + \frac{a^2 s}{1 + \kappa a^2 + \rho_0}, \tag{1.1}$$

$$s_t - D_s \Delta s = -\gamma s + \sigma - \frac{a^2 s}{1 + \kappa a^2 + \rho_0}. \tag{1.2}$$

Here, $a(x,t)$ the activator density, $s(x,t)$ the substrate density, $x \in R^3$, $t \geq 0$, $D_a > 0$ and $D_s > 0$ are the diffusion coefficients of $a(x,t)$ and $s(x,t)$, respectively. ρ,

Received December 23, 1992.
1991 *MR Subject Classification*: 35K55, 58F39, 92B20.
Keywords: Reaction–diffusion equations, smooth solutions, asymptotic stability.
*Supported by a Sasakawa Science Research Grant 7-087 of the Japan Science Society.
**Supported by the Natural Science Foundation of Jiangxi Province, China.

σ, ρ_0, μ, γ and κ are positive constants, μ and γ denote the decay rates of activator and substrate, respectively. Δ is the Laplacian operator in R^3. T. Jiang[1] studied the existence and uniquess of the solution for (1.1), (1.2) with Neumann boundary condition, and discussed the branch of the solutions.

In this paper, we study the Cauchy problem for system (1.1), (1.2) with the initial data

$$t=0: \quad a=a_0(x), \quad s=s_0(x), \quad x\in R^3. \tag{1.3}$$

Our plan contains the following. After stating the notions, we prove the existence and uniqueness of global smooth solutions of (1.1)-(1.3) in Sect.2, and show the stability and time decay rate of the solutions in Sect.3. These results will show us that, in this autocatalytic molecular reaction process, the densities of activator and substrate stably extend to a steady station in exponential decay, and there is no pattern to form after the end of this process for a long time.

Notations We denote the norm and the product of $L^2(R^3)$ by $\|\cdot\|$ and $(,)$, the norms of $H^k(R^3)$ and $C^k(R^3)$ by $\|\cdot\|_k$ and $\|\cdot\|_{C^k}$, respectively. Let

$$D^k=\partial^k/\partial x_1^{k_1}\partial x_2^{k_2}\partial x_3^{k_3},\ k=k_1+k_2+k_3.$$

For any fixed positive constants N_1, M_i, $i=1,2$, we define a functional set($0<N<N_1$, $T>0$) as the following

$$\begin{aligned}
&\Omega(0,T\,|\,N,M_1^2,M_2^2)\\
=&\{(u,v)\in R^2\,|\,(u,v)\in (C^1(0,T;H^2)\cap C^0(0,T;H^4)\cap L^2(0,T;H^5))^2,\\
&\sup_{0\le t\le T}(\|u(t)\|_4^2+\|v(t)\|_4^2)\le N^2\\
&\sup_{0\le t\le T}(\|u_t(t)\|_4^2+\|v_t(t)\|_4^2)\le M_1^2\\
&\sup_{0\le t\le T}\int_0^T((\|u(t)\|_4^2+\|v(t)\|_4^2))dt\le M_2^2\},
\end{aligned}$$

which will be used later. We can well understand that $\Omega(0,T\,|\,N,M_1^2,M_2^2)$ is a closed convex set in the Banach space $(C^1(0,T;H^2)\cap C^0(0,T;H^4)\cap L^2(0,T;H^5))^2$.

By the Sobolev's inequality, there exists $N_1>0$ such that for any $T>0$ the inequality

$$\sup_{0\le t\le T}(\|u(t)\|_4^2+\|v(t)\|_4^2)\le N^2$$

implies that

$$\sup_{0\le t\le T}(\|u(t)\|_{L^\infty}^2+\|v(t)\|_{L^\infty}^2)\le r^2.$$

Therefore we choose $R_1=R_1(r)$ as such constant satisfying above.

2. Global Existence and Uniqueness

Letting
$$u(x,t) = a(x,t), \quad v(x,t) = s(x,t) - \sigma\gamma^{-1},$$
then (1.1)–(1.3) can be reduced to
$$u_t - D_a \Delta u = -\mu u + f(u,v), \tag{2.1}$$
$$v_t - D_s \Delta v = -\gamma v - f(u,v), \tag{2.2}$$
$$t = 0: \quad u = a_0(x), \quad v = s_0(x) - \sigma\gamma^{-1}, \quad x \in R^3. \tag{2.3}$$
where $f(u,v) = u^2(v + \sigma\gamma^{-1})/(1 + \kappa u^2 + \rho_0)$.

The differential inequalities as the following lemmas can be found in [3,6], see also [4].

Lemma 2.1[3,6] *Suppose that $f \in C^5(R^2)$ and $(u,v) \in \Omega(0,T \mid N, M_1^2, M_2^2)$, then*
$$\|D^p\{f(u,v)w(x,t)\} - f(u,v)D^p w(x,t)\|$$
$$\leq C(\|u(t)\|_4 + \|v(t)\|_4)\|w(t)\|_4, \quad 0 \leq p < 4 \tag{2.4}$$
holds for $t \in [0,T]$ and $w(x,t) \in C^0(0,T; H^4)$, where $C > 0$ only depends on N_1.

Lemma 2.2[3,6] *Suppose that $h(u,v) \in C^5(R^2)$, $h(0,0) = 0$ and $(u,v) \in \Omega(0,T \mid N, M_1^2, M_2^2)$, then*
$$\|D^p h(u,v)\| \leq C(\|u(t)\|_4 + \|v(t)\|_4), \quad 0 \leq p \leq 4 \tag{2.5}$$
holds for $t \in [0,T]$, where $C > 0$ only depends on N_1.

By a standard energy method, we can prove that there exists the unique local solution of (2.1)–(2.3) in $\Omega(0, t_0 \mid N, M_1^2, M_2^2)$ for some $t_0 > 0$. For the details, we may refere to [4]. Thus, our local existence is stated as follows without proof.

Proposition 2.3. (*Local Existence*) *Suppose that $a_0(x)$, $s_0(x) - \sigma\gamma^{-1} \in H^4(R^3)$ and satisfy*
$$\|a_0\|_4^2 + \|s_0(x) - \sigma\gamma^{-1}\|_4^2 \leq N_0^2, \quad (N_0 < N/2). \tag{2.6}$$
Then there exists $t_0 = t_0(N_0) > 0$ such that there is a unique pair of solutions for (2.1)–(2.3) satisfying
$$(u,v) \in \Omega(0, t_0 \mid 2N_0, C'N_0^2, C''N_0^2),$$
where C', $C'' > 0$ are constants independent of N_0.

As we know, to show the global existence, the *a priori estimate* plays an important role in the procedure by the energy method. Our basic energy estimate is the following.

Proposition 2.4. (*A Priori Estimate*) *Under the assumptions in Proposition 2.3. Suppose that* $(u,v)\in\Omega(0,T\,|\,2N,C'N^2,C''N^2)(T>0,N<N_1/2)$ *is the solution of* (2.1)–(2.3). *Then there exists a sufficiently small* $N_0>0$, *such that*

$$\|a_0\|_4^2+\|s_0(x)-\sigma\gamma^{-1}\|_4^2\le N_0^2,\quad (N_0<N/2), \tag{2.7}$$

when $N\le N_0$, *then*

$$\|u(t)\|_4^2+\|v(t)\|_4^2\le N_0^2 \tag{2.8}$$

holds for $t\in[0,T]$.

Proof. We first denote that

$$L_1(u,v;U,V)\equiv U_t-D_a\Delta U+\mu U-\frac{\sigma u}{\gamma(1+\kappa u^2+\rho_0)}U-\frac{u^2}{(1+\kappa u^2+\rho_0)}V, \tag{2.9}$$

$$L_2(u,v;U,V)\equiv V_t-D_s\Delta V+\gamma V+\frac{\sigma u}{\gamma(1+\kappa u^2+\rho_0)}U+\frac{u^2}{(1+\kappa u^2+\rho_0)}V, \tag{2.10}$$

and $u_\varepsilon=J_\varepsilon u$, $v_\varepsilon=J_\varepsilon v$, where $J_\varepsilon u=\varepsilon^{-3}\int_{R^3}j(\frac{x-y}{\varepsilon})u(y,t)dy$.

Differentiating $L_i(u,v;u_\varepsilon,v_\varepsilon)$ $(i=1,2)$ as D^k $(0\le k\le 4)$, making their products with D^ku_ε and D^kv_ε, respectively, adding them and integrating the result yield

$$\begin{aligned}I_k&\equiv\int_0^t((D^ku_\varepsilon,D^kL_1(u,v;u_\varepsilon,v_\varepsilon))+(D^kv_\varepsilon,D^kL_2(u,v;u_\varepsilon,v_\varepsilon)))d\tau\\&=\frac{1}{2}(\|D^ku_\varepsilon(t)\|^2+\|D^kv_\varepsilon(t)\|^2-\|D^ku_\varepsilon(0)\|^2-\|D^kv_\varepsilon(0)\|^2)\\&\quad+\int_0^t(D_a\|D^{k+1}u_\varepsilon(\tau)\|^2+D_s\|D^{k+1}v_\varepsilon(\tau)\|^2)d\tau\\&\quad+\int_0^t(\mu\|D^ku_\varepsilon(\tau)\|^2+\gamma\|D^kv_\varepsilon(\tau)\|^2)d\tau\\&\quad+\int_0^t(-D^ku_\varepsilon+D^kv_\varepsilon,D^kg(u,v;u_\varepsilon,v_\varepsilon))d\tau,\end{aligned} \tag{2.11}$$

where

$$g(u,v;u_\varepsilon,v_\varepsilon)=\frac{\sigma u}{\gamma(1+\kappa u^2+\rho_0)}u_\varepsilon+\frac{u^2}{(1+\kappa u^2+\rho_0)}v_\varepsilon. \tag{2.12}$$

From Lemmas 2.1 and 2.2 we obtain

$$\|D^kg(u,v;u_\varepsilon,v_\varepsilon)\|\le C_1R(\|u_\varepsilon(t)\|_4^2+\|v_\varepsilon(t)\|_4^2). \tag{2.13}$$

Moreover, by Schwarz's inequality and Cauchy's inequality ($ab \le \frac{\eta}{2}a^2 + \frac{1}{2\eta}b^2$, for any $\eta > 0$), using (2.13), we have

$$\begin{aligned} I_k \ge & \frac{1}{2}(\|D^k u_\varepsilon(t)\|^2 + \|D^k v_\varepsilon(t)\|^2 - \|D^k u_\varepsilon(0)\|^2 - \|D^k v_\varepsilon(0)\|^2) \\ & + \int_0^t (D_a\|D^{k+1}u_\varepsilon(\tau)\|^2 + D_s\|D^{k+1}v_\varepsilon(\tau)\|^2)d\tau \\ & + \int_0^t [(\mu - \frac{\eta}{2})\,\|\,D^k u_\varepsilon(\tau)\|^2 + (\gamma - \frac{\eta}{2})\|D^k v_\varepsilon(\tau)\|^2]d\tau \\ & - \frac{C_1 R}{\eta}\int_0^t (\|u_\varepsilon(\tau)\|_4^2 + \|v_\varepsilon(\tau)\|_4^2)d\tau. \end{aligned} \tag{2.14}$$

Choosing $\eta < \min\{2\mu, 2\gamma\}$ and summing (2.14) with k yield

$$\begin{aligned} \sum_{k=0}^4 I_k \ge & \frac{1}{2}(\|u_\varepsilon(t)\|_4^2 + \|v_\varepsilon(t)\|_4^2 - \|u_\varepsilon(0)\|_4^2 - \|v_\varepsilon(0)\|_4^2) \\ & + \int_0^t (D_a\|u_\varepsilon(\tau)\|_5^2 + D_s\|v_\varepsilon(\tau)\|_5^2)d\tau \\ & + \int_0^t [(\mu - \frac{\eta}{2})\|u_\varepsilon(\tau)\|_4^2 + (\gamma - \frac{\eta}{2})\|v_\varepsilon(\tau)\|_4^2]d\tau \\ & - \frac{5C_1 R}{\eta}\int_0^t (\|u_\varepsilon(\tau)\|_4^2 + \|v_\varepsilon(\tau)\|_4^2)d\tau. \end{aligned} \tag{2.15}$$

In particular, selecting $N_0 < \min\{\frac{\eta(2\mu-\eta)}{10C_1}, \frac{\eta(2\gamma-\eta)}{10C_1}, \frac{N_1}{2}\}$, when $N \le N_0$, we get

$$\begin{aligned} \sum_{k=0}^4 I_k \ge & \frac{1}{2}(\|u_\varepsilon(t)\|_4^2 + \|v_\varepsilon(t)\|_4^2 - \|u_\varepsilon(0)\|_4^2 - \|v_\varepsilon(0)\|_4^2) \\ & + \theta\int_0^t (\|u_\varepsilon(\tau)\|_4^2 + \|v_\varepsilon(\tau)\|_4^2)d\tau, \end{aligned} \tag{2.16}$$

where $\theta = \min\{\mu - \frac{\eta}{2} - \frac{5}{\eta}C_1N_0, \gamma - \frac{\eta}{2} - \frac{5}{\eta}C_1N_0\}$.

Letting $\varepsilon \to 0$, we know that the following uniformly holds for $t \in [0, T]$

$$\|u_\varepsilon(t)\|_4 \to \|u(t)\|_4, \quad \|v_\varepsilon(t)\|_4 \to \|v(t)\|_4. \tag{2.17}$$

Notice that (u, v) is the smooth solution of (2.1)–(2.3), namely, $L_1(u, v; u, v) = L_2(u, v; u, v) = 0$, then we have

$$\begin{aligned} I_k \equiv \int_0^t & ((D^k u_\varepsilon, D^k(L_1(u, v; u_\varepsilon, v_\varepsilon) - J_\varepsilon L_1(u, v; u, v))) \\ & + (D^k v_\varepsilon, D^k(L_2(u, v; u_\varepsilon, v_\varepsilon) - J_\varepsilon L_2(u, v; u, v))))d\tau. \end{aligned}$$

Due to the properties of J_ε (see [6]) we have

$$\|D^k(L_i(u,v;u_\varepsilon,v_\varepsilon) - J_\varepsilon L_i(u,v;u,v))\| \to 0, \quad i=1,2, \quad 0 \le k \le 3,$$

$$\begin{aligned}
&\Big|\int_0^t ((D^k u_\varepsilon, D^k(L_1(u,v;u_\varepsilon,v_\varepsilon) - J_\varepsilon L_1(u,v;u,v)))d\tau\Big| \\
\le&\Big(\int_0^t \|D^5 u_\varepsilon(\tau)\|^2 d\tau\Big)^{\frac12}\Big(\int_0^t \|D^3(L_1(u,v;u_\varepsilon,v_\varepsilon) - J_\varepsilon L_1(u,v;u,v))\|^2 d\tau\Big)^{\frac12} \to 0, \\
&\Big|\int_0^t ((D^k v_\varepsilon, D^k(L_2(u,v;u_\varepsilon,v_\varepsilon) - J_\varepsilon L_2(u,v;u,v)))d\tau\Big| \\
\le&\Big(\int_0^t \|D^5 v_\varepsilon(\tau)\|^2 d\tau\Big)^{\frac12}\Big(\int_0^t \|D^3(L_2(u,v;u_\varepsilon,v_\varepsilon) - J_\varepsilon L_2(u,v;u,v))\|^2 d\tau\Big)^{\frac12} \to 0,
\end{aligned}$$

as $\varepsilon \to 0$. Therefore, we obtain $I_k \to 0$ as $\varepsilon \to 0$. Above results imply that (let $\varepsilon \to 0$)

$$\begin{aligned}
\|u(t)\|_4^2 + \|v(t)\|_4^2 + 2\theta\int_0^t (\|u(\tau)\|_4^2 + \|v(\tau)\|_4^2)d\tau& \\
\le \|u(0)\|_4^2 + \|v(0)\|_4^2.& \qquad (2.18)
\end{aligned}$$

Thus (2.7) and (2.18) yield (2.8). The proof is completed. □

Up to now, we can prove the global existence by *a priori estimate* together with the local existence.

Theorem 2.5. (*Global Existence*) *Suppose that* $a_0(x)$, $s_0(x) - \sigma\gamma^{-1} \in H^4(R^3)$. *Then there exists a suitably small positive constant* N_0, *such that, when*

$$\|a_0\|_4^2 + \|s_0(x) - \sigma\gamma^{-1}\|_4^2 \le N_0^2, \qquad (2.19)$$

then there exists a unique pair of global solutions for (2.1)–(2.3) $(u,v) \in \Omega(0,\infty \mid 2N_0, C'N_0^2, C''N_0^2)$ *satisfying*

$$\|u(t)\|_4^2 + \|v(t)\|_4^2 \le N_0^2, \quad \text{for} \quad t \in [0,\infty]. \qquad (2.20)$$

Proof. Let N_0 be the selected positive constant in Proposition 2.4. According to the local existence result (Proposition 2.3), there is a $t_0 > 0$ such that the problem (2.1)–(2.3) has a unique local smooth solution in $R^2 \times [0,t_0]$. Due to the *a priori* estimates (Proposition 2.4), we know the local solution satisfies (2.20) for $t \in [0,t_0]$. Now we consider the system (2.1), (2.2) with the "initial data" $(u(x,t_0), v(x,t_0))$. Since this "initial data" satisfies (2.20), applying Proposition 2.3 again, we can extend the solvable interval of the solution $(u(x,t), v(x,t))$ to $R \times [0,2t_0]$, and can also show the estimate (2.20) for $t \in [0,2t_0]$ by Proposition 2.4. Repeating this procedure, we prove that (u,v) is the global smooth solution and satisfies (2.20) for $t \in [0,\infty)$. The proof is completed. □

3. Global Stability and Asymptotic Decay

Before stating our stability theorem, we give a basic estimate on the global smooth solution as follows.

Lemma 3.1. *Under the assumptions in Theorem 2.5,*

$$\|u(t)\|_4^2 + \|v(t)\|_4^2 + 2\theta \int_0^t (\|u(\tau)\|_4^2 + \|v(\tau)\|_4^2) d\tau$$
$$\le \|a_0\|_4^2 + \|s_0 - \frac{\sigma}{\gamma}\|_4^2 \tag{3.1}$$

holds for $t \in [0, \infty)$.

Proof. Firstly, we consider (2.1), (2.2) with this initial data

$$(u, v)|_{t=0} = (J_\varepsilon a_0, J_\varepsilon(s_0 - \frac{\sigma}{\gamma}))(x),$$

and denote the solution as $(u_\varepsilon(x,t), v_\varepsilon(x,t))$. By the same procedure in Proposition 2.4, we can obtain a similar estimate corresponding (2.18) as

$$\|u_\varepsilon(t)\|_4^2 + \|v_\varepsilon(t)\|_4^2 + 2\theta \int_0^t (\|u_\varepsilon(\tau)\|_4^2 + \|v_\varepsilon(\tau)\|_4^2) d\tau$$
$$\le \|u_\varepsilon(0)\|_4^2 + \|v_\varepsilon(0)\|_4^2, \quad t \in [0, \infty). \tag{3.2}$$

Considering now $(u, v)|_{t=0} = (a_0, s_0 - \frac{\sigma}{\gamma}) \in H^4(R^3)$, remarking the relative solution as (u, v), since $C_0^\infty(R^3)$ is dense in $H^4(R^3)$, and using Banach–Saks theorem, we have

$$(u_\varepsilon, v_\varepsilon) \to (u, v) \quad \text{in} \quad H^4(R^3), \quad \text{as} \quad \varepsilon \to 0.$$

Taking the limit, we obtain (3.1). □

Theorem 3.2. *(Global Stability) Under the assumptions in Theorem 2.5, the solution of* (2.1)–(2.3) *is globally stable, i.e., when*

$$\|a_0\|_4^2 + \|s_0 - \frac{\sigma}{\gamma}\|_4^2 \le \varepsilon, \tag{3.3}$$

where $\varepsilon > 0$ *is any constant, then*

$$\|u(t)\|_{L^\infty} + \|v(t)\|_{L^\infty} \le C\varepsilon \tag{3.4}$$

holds for $t \in [0, \infty)$.

Proof. The assertion can be verified by Lemma 3.1 and Sobolev's inequality. □

Theorem 3.3. (*Asymptotic Decay Rate*) *Under the assumptions in Theorem* 2.5, *the asymptotic decay rate of the solution for* (2.1)–(2.3)

$$\|u(t)\|_{L^\infty} + \|v(t)\|_{L^\infty} \le CN_0 e^{-\theta t} \tag{3.5}$$

holds for $t \in [0, \infty)$.

Proof. Due to (3.1) in Lemma 3.1 and Gronwall's inequality, we have

$$\|u(t)\|_4^2 + \|v(t)\|_4^2 \le N_0^2 e^{-2\theta t}. \tag{3.6}$$

Applying Sobolev's inequality to (3.6) yields our desired estimate (3.5). □

References

[1] Jiang, T., Nonconstant equilibrium solutions for a model of pattern formation on shells of mulluscs, *J. Xinjiang Univ.*, (*Natural Sci. Ed.*), **7** (1990), 1–5. (Chinese)

[2] Lindsay, D., A model for pattern formation of bivalved molluscs, *Differentsial'nye Uravneniya*, **2** (1982), 32–40. (Russia)

[3] Matsumura, A., Global existence and asymptotics of the solutions of the second-order quasilinear hyperbolic equation with the first-order dissipation, *Publ. RIMS, Kyoto Univ.*, **13** (1977), 349–379.

[4] Mei, M., Global smooth solutions of the Cauchy problem for generalized equations of pulse transmission type with high dimension, *Acta Appl. Math. Sinica*, **14** (1991), 450–461. (Chinese)

[5] Meinhardt, H. and Klingler, M., A model for pattern formation on the shells of molluscs, *J. Theory Biol.*, **126** (1987), 63–89.

[6] Minzahata, S., Theory of Partial Differential Equations, Cambridge Univ. Prin., 1973.

[7] Wadington, C.H. and Cowe, J., Tent-like pattern formation on the shells of molluscs, *J. Theory Biol.*, **25** (1969), 219–231.

Department of Mathematics, Faculty of Science, Kanazawa University, Kanazawa 920-11, Japan.
E-mail address: mei@kappa.s.kanazawa-u.ac.jp

Department of Mathematics, Jiangxi Normal University, Nanchang 330027.