In: Nonlinear Analysis Research Trends
Editor: Ins N. Roux, pp. 277-293
ISBN 978-1-60456-358-0

Chapter 9

STATIONARY SOLUTIONS OF PHASE TRANSITIONS IN A COUPLED VISCOELASTIC SYSTEM

***Ming Mei*[1,2,*]*, Yau Shu Wong*[3] *and Liping Liu*[4]**
[1]Department of Mathematics, Champlain College St.-Lambert
St.-Lambert, Quebec, J4P 3P2, Canada
[2]Department of Mathematics and Statistics, McGill University
Montreal, Quebec, H3A 2K6, Canada
[3]Department of Mathematical and Statistical Sciences, University of Alberta
Edmonton, Alberta, T6G 2G1, Canada
[4]Department of Mathematics, North Carolina Agricultural and Technical State University
Greensboro, NC 27411, USA

Abstract

This study focuses on a coupled 2×2 system of mixed type for viscosity-capillarity with periodic initial-boundary condition in viscoelastic material. The main concern in the present study is the stationary solutions. It is shown that the stationary system has one or multiple stationary solutions depending on the value of the viscosities ε_1 and ε_2. In particular, the non-trivial solutions always present phase transitions. The criteria for the type of system responses (one or multiple solutions) are specified. Furthermore, the calculation formula for the number of multiple solutions is provided. Finally, numerical simulations on the prototypical system are reported to verify the theoretical results.

Keywords: Phase transitions, system of viscosity-capillarity, mixed type, stationary solutions, periodic boundary conditions.

AMS: 35R10, 35B40, 34K30, 58D25

1. Introduction and Main Results

The viscous-capillarity system in the viscoelastic material dynamics (resp. the compressible van der Waals fluids) can be written as a system of 2×2 viscous conservation laws of mixed

[*]Corresponding author.

type:

$$\begin{cases} v_t - u_x = \varepsilon_1 v_{xx}, \\ u_t - \sigma(v)_x = \varepsilon_2 u_{xx}, \end{cases} \quad (x,t) \in R \times R_+. \tag{1.1}$$

In this paper, we study the coupled system with the initial condition

$$(v,u)|_{t=0} = (v_0, u_0)(x), \ \ x \in (-\infty, \infty) \tag{1.2}$$

and the $2L$-periodic boundary condition

$$(v,u)(x,t) = (v,u)(x+2L,t), \quad (x,t) \in (-\infty,\infty) \times (0,\infty) \tag{1.3}$$

where $L > 0$ is a given constant. Note that from the compatibility condition, we have

$$v_0(x) = v_0(x+2L), \ \ u_0(x) = u_0(x+2L). \tag{1.4}$$

Here $v(x,t)$ is the strain (resp. specific volume), $u(x,t)$ the velocity, $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ the viscous constants, $\sigma(v)$ the stress function (resp. pressure function), which is assumed to be sufficiently smooth and non-monotonic.

As a prototype, c.f. [15, 16], the simplest function:

$$\sigma(v) = v^3 - v \tag{1.5}$$

is considered in this study. This function captures the basic features for the phase transition models. For such a stress function $\sigma(v)$, it has only two critical points $\pm\frac{1}{\sqrt{3}}$ such that $\sigma'(\pm\frac{1}{\sqrt{3}}) = 0$, and $\sigma'(v) > 0$ for $v \in (-\infty, -\frac{1}{\sqrt{3}}) \cup (\frac{1}{\sqrt{3}}, \infty)$, $\sigma'(v) < 0$ for $v \in (-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$. Physically, this determines three phases, for example, water, vapor, and water-vapor mixture phases in van der Waals fluids. Mathematically, Eq.(1.1) with $\varepsilon_1 = \varepsilon_2 = 0$ is hyperbolic in $(-\infty, -\frac{1}{\sqrt{3}}) \cup (\frac{1}{\sqrt{3}}, \infty)$ and elliptic in $(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$. Here, $v = \pm\frac{1}{\sqrt{3}}$ are the two phase boundaries. In the case of the van der Waals fluids, the pressure is exactly given by $-\sigma(v) = \frac{R\theta}{v-b} - \frac{a}{v^2}$ with positive constants R, θ, a and b satisfying $R\theta b/a < (2/3)^3$ and $v > b > 0$. It is known that there are also two critical points of $\sigma(v)$, says v_1 and v_2, such that $\sigma'(v_1) = \sigma'(v_2) = 0$, $\sigma'(v) > 0$ for $v \in (b, v_1) \cup (v_2, \infty)$ and $\sigma'(v) < 0$ for $v \in (v_1, v_2)$. The region (b, v_1) is the water-region, (v_2, ∞) is the vapor-region, and (v_1, v_2) is the water-vapor mixture region, c.f. [1, 13].

Since the periodic solutions $(v,u)(x,t)$ of (1.1)-(1.3) in the entire space $(-\infty,\infty)$ can be regarded as $2L$-periodic extensions of that on $[0, 2L]$, let us focus the system (1.1) on the bounded interval $[0, 2L]$. Integrating (1.1) over $[0, 2L] \times [0, t]$ and using the periodic boundary condition (1.3), we obtain

$$\int_0^{2L} v(x,t)dx = \int_0^{2L} v_0(x)dx, \quad \int_0^{2L} u(x,t)dx = \int_0^{2L} u_0(x)dx. \tag{1.6}$$

Let

$$m_0 := \frac{1}{2L}\int_0^{2L} v_0(x)dx, \quad m_1 := \frac{1}{2L}\int_0^{2L} u_0(x)dx, \tag{1.7}$$

then

$$\int_0^{2L} [v(x,t) - m_0]dx = 0, \qquad \int_0^{2L} [u(x,t) - m_1]dx = 0. \tag{1.8}$$

Therefore, (m_0, m_1) is the average of the initial value $(v_0, u_0)(x)$ over $[0, 2L]$, and so the average of the solution $(v, u)(x, t)$ over $[0, 2L]$.

Phase transitions are very common and interesting phenomena arising thermodynamics, and have been the hottest spots in mathematical and physical communities. A lot of great progress in, for example, the construction of the solutions and their stationary solutions, the behaviors of the solutions, as well as the asymptotic stabilities of the solutions, has been made by many mathematicians and physicians, see [1]-[25] and the references therein. In this paper, we are interested in the steady-state solutions of the system (1.1)-(1.3). The corresponding stationary problem of (1.1)-(1.3) is

$$\begin{cases} -U_x = \varepsilon_1 V_{xx}, \\ -\sigma(V)_x = \varepsilon_2 U_{xx}, \\ (V,U)(x) = (V,U)(x+2L), \\ \frac{1}{2L}\int_0^{2L} V(x)dx = m_0, \\ \frac{1}{2L}\int_0^{2L} U(x)dx = m_1, \end{cases} \tag{1.9}$$

where $(V, U) = (V, U)(x)$. It is easy to see that the average initial data (m_0, m_1) is a trivial stationary solution of (1.9). In the present paper, we prove that, if m_0 is in the hyperbolic region $(-\infty, -\frac{1}{\sqrt{3}})$ or $(\frac{1}{\sqrt{3}}, \infty)$ but not so close to the phase boundaries $v = \pm\frac{1}{\sqrt{3}}$, or when the viscosities ε_1 and ε_2 are suitably large (regardless of the value of m_0), then the stationary problem (1.9) has and only has the trivial solution (m_0, m_1). On the other hand, if m_0 is in the elliptic region $(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$ and the viscosities ε_1 and ε_2 are sufficiently small, then the stationary problem (1.9) has multiple non-trivial solutions, and these solutions always admit phase transitions. The number of those multiple solutions will be counted shortly. Here we provide a necessary preparation for our further study in [15, 16] on the asymptotic behaviors of the solution of (1.1), specifically, the convergence of the solution $(v, u)(x, t)$ to the stationary solution $(V, U)(x)$. We further present some numerical simulations in different cases which confirm also our theoretical results.

Our main results are stated as follows.

Theorem 1.1 (Stationary Solutions)

1. *If the viscosity is large, such that*

$$\varepsilon_1\varepsilon_2 > \frac{L^2}{\pi^2}\Big(1 - \frac{3}{4}m_0^2\Big), \tag{1.10}$$

then the stationary problem (1.9) *has a unique solution* $(V, U)(x) = (m_0, m_1)$. *No phase transition exhibits.*

2. *If the viscosity is sufficiently small such that*

$$\varepsilon_1\varepsilon_2 < \frac{L^2}{\pi^2}(1 - 3m_0^2), \tag{1.11}$$

then besides the trivial solution (m_0, m_1)*, the stationary problem* (1.9) *has at least one non-trivial solution, which is oscillating periodically, and must occur phase transitions.*

Theorem 1.2 (Number of Non-Trivial Stationary Solutions)

1. *When* $m_0 = 0$ *and* $\varepsilon_1\varepsilon_2 < \frac{L^2}{\pi^2}(1-3m_0^2)$*, then there exist* $(2N_0+1)$ *solutions for* (1.9)*, where* N_0 *is the integer of*

$$N_0 = \Big[\frac{2L}{\pi\sqrt{\varepsilon_1\varepsilon_2}}\Big]. \tag{1.12}$$

Here, $[x]$ *denotes the greatest integer* $\leq x$*.*

2. *When* $0 < m_0 < \frac{1}{\sqrt{5}}$ *and* $\varepsilon_1\varepsilon_2 < \frac{L^2}{\pi^2}(1-3m_0^2)$*, there is a unique non-trivial stationary solution for* (1.9)*.*

3. *When* $\frac{1}{\sqrt{5}} < m_0 < \frac{1}{\sqrt{3}}$*, there exists a number* $\varepsilon^* > \frac{L^2}{\pi^2}(1-3m_0^2)$ *such that, when* $\frac{L^2}{\pi^2}(1-3m_0^2) < \varepsilon_1\varepsilon_2 < \varepsilon^*$*, there are two non-trivial solutions. However, when* $\varepsilon_1\varepsilon_2 \leq \frac{L^2}{\pi^2}(1-3m_0^2)$*, there is only one non-trivial solution for* (1.9)*.*

Remark 1.3

1. *In the first part of Theorem 1.1, the condition* (1.10) *implies large viscosities if* m_0 *is in the elliptic region. However, when* $m_0 > \frac{2}{\sqrt{3}}$ *or* $m_0 < -\frac{2}{\sqrt{3}}$*, which implies that* m_0 *is in the hyperbolic regions, even if the viscosity disappears, i.e.,* $\varepsilon_1 = \varepsilon_2 = 0$*, the condition* (1.10) *still holds, and thus the stationary solution of* (1.9) *is unique and* $(V,U)(x) \equiv (m_0, m_1)$*. Note that, in the second part of Theorem 1.1, the condition (1.11) implies that the initial average* m_0 *is in the elliptic region. The theorem indicates that for the initial average* m_0 *in the elliptic region, when the viscosity* $\varepsilon_1\varepsilon_2$ *is small, multiple non-trivial stationary solutions* $(V,U)(x)$ *exist, and they exhibit phase transitions.*

2. *When* $m_0 = 0$*, the conditions* (1.10) *and* (1.11) *become*

$$\varepsilon_1\varepsilon_2 \gtrless \frac{L^2}{\pi^2},$$

which is the critical condition, and it will lead to one or multiple solutions.

3. *Theorem 1.2 gives the exact numbers of the non-trivial stationary solutions for different range of the initial average* m_0*. These results are the same to [23, 9] for the Cahn-Hilliard equation in Neumann boundary condition.*

2. Proof of Main Theorem

In this section, we are going to prove Theorem 1.1 and Theorem 1.2. Substituting the first equation of (1.9) into the second equation and integrating the resultant equation over $[0, x]$, we obtain the stationary equation for $V(x)$ as follows

$$\begin{cases} \varepsilon_1\varepsilon_2 V_{xx} = \sigma(V) - a \\ V(x) = V(x+2L) \\ \frac{1}{2L}\int_0^{2L} V(x)dx = m_0, \end{cases} \tag{2.1}$$

where $a = \sigma(V(0)) - \varepsilon_1\varepsilon_2 V_{xx}(0)$ is a constant determined automatically by Eq. (2.1). In fact, integrating Eq. (2.1) over $[0, 2L]$ and noticing the periodic boundary condition $V(x) = V(x+2L)$, one verifies

$$a = \frac{1}{2L}\int_0^{2L} \sigma(V(x))dx. \tag{2.2}$$

Eq. (2.1) is similar to the so-called Cahn-Hilliard equation which has been studied widely by researchers in the mathematical physics community, for example, see [8, 9, 18, 19, 20, 23, 24], and the references therein. With the Neumann boundary condition, Zheng [23] studied the existence of the trivial and non-trivial solutions (although he just gave a roughly sufficient conditions on the size of viscosity), in particular, the number of non-trivial solutions for the initial mean $m_0 = 0$ in the case of small viscosity was counted. While in [8, 9, 18, 19, 20] Grinfeld, Novick-Cohen, Peletier, Segel *et al*, specified the values of m_0 and the size of the viscosity for the existence of trivial or nontrivial solutions by the transversality arguments. All of these works will be a great help in dealing with (2.1) for the periodic boundary condition.

Inspiring by [23] and [9], we can testify the existence of the non-trivial solutions by specifying the size of viscosity, i.e., the criteria (1.10) and (1.11), and we further count the number of the non-trivial solutions by specifying the location of the initial average m_0.

Letting

$$\bar{V}(x) := V(x) - m_0, \tag{2.3}$$

and applying (1.8), then we reduce (2.1) into

$$\begin{cases} \varepsilon_1\varepsilon_2 \bar{V}_{xx} = \bar{\sigma}(\bar{V}) - \bar{a} \\ \bar{V}(x) = \bar{V}(x+2L) \\ \int_0^{2L} \bar{V}(x)dx = 0, \end{cases} \tag{2.4}$$

where

$$\bar{\sigma}(\bar{V}) := \sigma(\bar{V} + m_0) - \sigma(m_0), \quad \bar{a} := \frac{1}{2L}\int_0^{2L} \bar{\sigma}(\bar{V}(x))dx. \tag{2.5}$$

Define a periodic Sobolev space by

$$H^1_{per,0} = \left\{ v(y) | v(y) \in H^1_{per}(R), \int_0^{2L} v(y)dy = 0 \right\} \tag{2.6}$$

and a functional on $H^1_{per,0}$ by

$$G(v)=\int_0^{2L}\Big[\frac{\varepsilon_1\varepsilon_2}{2}v_x^2+H(v)\Big]dx, \tag{2.7}$$

where

$$H(v)=\int_0^v\bar\sigma(s)ds=\frac{1}{4}v^4+m_0v^3+\frac{3m_0^2-1}{2}v^2,$$

we now have the first lemma as follows.

Lemma 2.1 *The solutions of* (2.4) *are equivalent to the critical points of the functional* $G(v)$ *defined in* (2.7) *over* $H^1_{per,0}$.

Proof. If $\bar V(x)$ is a solution of (2.4), then for any $w(x)\in H^1_{per,0}$, multiplying (2.4) by $w(x)$ and integrating it over $[0,2L]$, and noting $\int_0^{2L}\bar a w(x)dx=\bar a\int_0^{2L}w(x)dx=0$, one gets

$$\int_0^{2L}\Big(\varepsilon_1\varepsilon_2\bar V_x w_x+\bar\sigma(\bar V)w\Big)dx=0,\quad \text{for any } w\in H^1_{per,0}. \tag{2.8}$$

This implies $\bar V(x)$ is a critical point of $G(v)$ over $H^1_{per,0}$.

On the other hand, let $\bar V(x)$ be a critical point of $G(v)$ over $H^1_{per,0}$, namely, (2.8) holds, we prove that $\bar V(x)$ is a solution of (2.4). For any $\bar w(x)\in H^1(0,2L)$ satisfying $\bar w(x)=\bar w(x+2L)$, it can be verified that

$$w(x):=\bar w(x)-\frac{1}{2L}\int_0^{2L}\bar w(x)dx\in H^1_{per,0} \tag{2.9}$$

and

$$\int_0^{2L}\bar\sigma(\bar V(x))\cdot\Big(\frac{1}{2L}\int_0^{2L}\bar w(x)dx\Big)dx=\int_0^{2L}\bar w(x)\cdot\Big(\frac{1}{2L}\int_0^{2L}\bar\sigma(\bar V(x))dx\Big)dx. \tag{2.10}$$

Thus, (2.8)–(2.10) gives

$$\begin{aligned}
0 &= \int_0^{2L}\Big(\varepsilon_1\varepsilon_2\bar V_x w_x+\bar\sigma(\bar V)w\Big)dx\\
&= \int_0^{2L}\Big[\varepsilon_1\varepsilon_2\bar V_x\Big(\bar w(x)-\frac{1}{2L}\int_0^{2L}\bar w(x)dx\Big)_x\\
&\qquad +\bar\sigma(\bar V(x))\Big(\bar w(x)-\frac{1}{2L}\int_0^{2L}\bar w(x)dx\Big)\Big]dx\\
&= \int_0^{2L}\Big[\varepsilon_1\varepsilon_2\bar V_x\bar w_x+\bar\sigma(\bar V(x))\bar w(x)\Big]dx\\
&\qquad -\int_0^{2L}\bar w(x)\Big(\frac{1}{2L}\int_0^{2L}\bar\sigma(\bar V(x))dx\Big)dx\\
&= \int_0^{2L}\Big[\varepsilon_1\varepsilon_2\bar V_x\bar w_x+\Big(\bar\sigma(\bar V(x))-\frac{1}{2L}\int_0^{2L}\bar\sigma(\bar V(x))dx\Big)\bar w(x)\Big]dx\\
&= \int_0^{2L}\Big[\varepsilon_1\varepsilon_2\bar V_x\bar w_x+(\bar\sigma(\bar V(x))-\bar a)\bar w(x)\Big]dx,
\end{aligned} \tag{2.11}$$

where we used (2.5). Thus, we prove in (2.11) that $\bar{V}(x)$ is a weak solution of the stationary problem (2.4). Furthermore, by the usual bootstrap argument, we conclude that $\bar{V}(x)$ is the classical solution of (2.4). The detail is omitted. □

We now prove a useful inequality which is somewhat similar to the so-called Poincaré inequality.

Lemma 2.2 *Let $\bar{V}(x) \in H^1_{per,0}$ be the solution of Eq.* (2.4). *Then there exists at least one point x_* such that $\bar{V}(x_*) = 0$, and*

$$\|\bar{V}\|_{L^2} \leq \frac{L}{\pi}\|\bar{V}_x\|_{L^2}. \tag{2.12}$$

Proof. It is easy to verify that 0 is a solution of (2.4). If $\bar{V}(x) \equiv 0$, then (2.12) automatically holds. If $\bar{V}(x) \not\equiv 0$ on $[0, 2L]$, then there must exist at least one point, says $x_* \in [0, 2L]$, such that $\bar{V}(x_*) = 0$. In fact, notice that $\int_0^{2L} \bar{V}(s)ds = 0$ and $\bar{V}(x) \in C^0(0, 2L)$ (because $\bar{V}(x) \in H^1(0, 2L)$), then $\bar{V}(x)$ must change signs on $[0, 2L]$, which implies that $\bar{V}(x_*) = 0$ for some $x_* \in [0, 2L]$. Otherwise, either $\bar{V}(x) > 0$ or $\bar{V}(x) < 0$ on $[0, 2L]$ leads to $\int_0^{2L} \bar{V}(s)ds > 0$ or $\int_0^{2L} \bar{V}(s)ds < 0$, but this is a contradiction with the condition $\int_0^{2L} \bar{V}(s)ds = 0$.

Consider the following eigenvalue problem

$$\begin{cases} -\tilde{v}_{xx} = \beta^2 \tilde{v}, \\ \tilde{v}(x) = \tilde{v}(x + 2L), \\ \int_0^{2L} \tilde{v}(x)dx = 0. \end{cases} \tag{2.13}$$

The eigenvalues are given by

$$\beta_k = \frac{kL}{\pi}, \quad k = 1, 2, 3, \cdots, \tag{2.14}$$

and the corresponding eigenfunctions are

$$\tilde{v}_{1,k}(x) = \frac{1}{\sqrt{L}} \sin \beta_k x, \quad \tilde{v}_{2,k}(x) = \frac{1}{\sqrt{L}} \cos \beta_k x, \quad k = 1, 2, 3, \cdots \tag{2.15}$$

satisfying

$$< \tilde{v}_{i,k}, \tilde{v}_{j,l} >= \begin{cases} 1, & i = j, k = l \\ 0, & \text{otherwise}, \end{cases} \tag{2.16}$$

where

$$< \tilde{v}_{i,k}, \tilde{v}_{j,l} >= \int_0^{2L} \tilde{v}_{i,k}(x)\tilde{v}_{j,l}(x)dx$$

is the inner product of L^2_{per}. It is known that the sequence $\{\tilde{v}_{i,k}(x)\}$ ($i = 1, 2$ and $k = 1, 2, 3, \cdots$) forms an orthonormal basis for the space

$$L^2_{per,0} = \Big\{\tilde{v}(x) | \tilde{v}(x) \in L^2_{per} \text{ and } \int_0^{2L} \tilde{v}(x)dx = 0\Big\}.$$

Therefore, as a periodic function satisfying $\bar{V}(x) = \bar{V}(x+2L)$ and $\int_0^{2L} \bar{V}(x)dx = 0$, $\bar{V}(x)$ is in the space $L^2_{per,0}$, and can be expressed in the Fourier form

$$\bar{V}(x) = \sum_{k=1}^{\infty}(A_k\tilde{v}_{1,k}(x) + B_k\tilde{v}_{2,k}(x)),$$

where the coefficients A_k and B_k are determined by

$$A_k =< \bar{V}, \tilde{v}_{1,k} >, \quad B_k =< \bar{V}, \tilde{v}_{2,k} > .$$

Its derivative is given by

$$\bar{V}_x(x) = \sum_{k=1}^{\infty}\beta_k(A_k\tilde{v}_{2,k}(x) - B_k\tilde{v}_{1,k}(x)).$$

Making the inner products, we have

$$\begin{aligned}
\|\bar{V}\|^2 &= \int_0^{2L}\bar{V}^2(x)dx \\
&= \int_0^{2L}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}(A_k\tilde{v}_{1,k}(x) + B_k\tilde{v}_{2,k}(x))(A_l\tilde{v}_{1,l}(x) + B_l\tilde{v}_{2,l}(x))\, dx \\
&= \sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\Big(A_kA_l\int_0^{2L}\tilde{v}_{1,k}(x)\tilde{v}_{1,l}(x)dx + A_kB_l\int_0^{2L}\tilde{v}_{1,k}(x)\tilde{v}_{2,l}(x)dx \\
&\quad + B_kA_l\int_0^{2L}\tilde{v}_{2,k}(x)\tilde{v}_{1,l}(x)dx + B_kB_l\int_0^{2L}\tilde{v}_{2,k}(x)\tilde{v}_{2,l}(x)dx\Big) \\
&= \sum_{k=1}^{\infty}(A_k^2 + B_k^2) \qquad (2.17)
\end{aligned}$$

and

$$\begin{aligned}
\|\bar{V}_x\|^2 &= \int_0^{2L}\bar{V}_x^2(x)dx \\
&= \int_0^{2L}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\beta_k\beta_l(A_k\tilde{v}_{2,k}(x) - B_k\tilde{v}_{1,k}(x))(A_l\tilde{v}_{2,l}(x) - B_l\tilde{v}_{1,l}(x))\, dx \\
&= \sum_{k=1}^{\infty}\beta_k^2(A_k^2 + B_k^2). \qquad (2.18)
\end{aligned}$$

Since $\beta_k \geq \beta_1 = \frac{\pi}{L}$ for $k = 1, 2, \cdots$, then (2.17) and (2.18) give

$$\|\bar{V}_x\|^2 = \sum_{k=1}^{\infty}\beta_k^2(A_k^2 + B_k^2) \geq \beta_1^2\sum_{k=1}^{\infty}(A_k^2 + B_k^2) = \frac{\pi^2}{L^2}\|\bar{V}\|^2,$$

which implies (2.12). □

Now we are ready to discuss the existence of the trivial and non-trivial stationary solutions, as well as their phase transitions.

Lemma 2.3 *If* (1.10) *holds, then the stationary problem* (2.4) *has and only has the trivial solution* $\bar V(x) \equiv 0$*.*

Proof. Multiplying (2.4) by $\bar V(x)$ and integrating by parts with respect to x over $[0, 2L]$, as well as using the periodic boundary condition, we obtain

$$0 = \int_0^{2L} (-\varepsilon_1\varepsilon_2 \bar V_{xx} + \bar\sigma(\bar V) - \bar a)\bar V(x)dx = \int_0^{2L} (\varepsilon_1\varepsilon_2 \bar V_x^2 + \bar\sigma(\bar V)\bar V)dx. \tag{2.19}$$

Since

$$\begin{aligned} \bar\sigma(\bar V)\bar V &= \bar V^4 + 3m_0\bar V^3 + (3m_0^2 - 1)\bar V^2 \\ &= \bar V^2\Big(\bar V + \frac{3}{2}m_0\Big)^2 + \Big(\frac{3}{4}m_0^2 - 1\Big)\bar V^2 \\ &\geq \Big(\frac{3}{4}m_0^2 - 1\Big)\bar V^2, \end{aligned} \tag{2.20}$$

applying (2.12) and (2.20) into (2.19), as well as noting (1.10), we have

$$\begin{aligned} 0 &= \int_0^{2L} (\varepsilon_1\varepsilon_2 \bar V_x^2 + \bar\sigma(\bar V)\bar V)dx \\ &\geq \int_0^{2L} \Big[\frac{\varepsilon_1\varepsilon_2\pi^2}{L^2}\bar V^2 + \Big(\frac{3}{4}m_0^2 - 1\Big)\bar V^2\Big]dx \\ &= \Big[\frac{\varepsilon_1\varepsilon_2\pi^2}{L^2} - \Big(1 - \frac{3}{4}m_0^2\Big)\Big]\int_0^{2L} \bar V^2(x)dx \geq 0, \end{aligned} \tag{2.21}$$

which implies $\int_0^{2L} \bar V^2(x)dx \equiv 0$, i.e., $\bar V(x) \equiv 0$. Here, in order to obtain the non-negativity of the last step of (2.21), we need

$$\frac{\varepsilon_1\varepsilon_2\pi^2}{L^2} - \Big(1 - \frac{3}{4}m_0^2\Big) > 0,$$

which leads to the sufficient condition (1.10). □

Lemma 2.4 *If* (1.11) *holds, then, except the trivial solution* 0*, the stationary problem* (2.4) *has at least one non-trivial solution.*

Proof. Let $\bar V_n(x)$ be a minimizing sequence:

$$G(\bar V_n) \to g = \inf_{v\in H^1_{per,0}} G(v), \tag{2.22}$$

where $G(V)$ is defined in (2.7). Notice that $\bar V_n$ is bounded in $H^1_{per,0}$, hence there exists a subsequence, still denoting by $\bar V_n$, such that

$$\bar V_n \to \bar V \quad \text{weakly in} \;\; H^1_{per,0}, \tag{2.23}$$

and

$$\bar V_n \to \bar V \quad \text{strongly in} \;\; L^p(0, 2L), 1 < p < 6. \tag{2.24}$$

This implies

$$\int_0^{2L} \bar{V}_x^2 dx \leq \underline{\lim}_{n\to\infty} \int_0^{2L} |(\bar{V}_n)_x|^2 dx, \tag{2.25}$$

and

$$\int_0^{2L} H(\bar{V}_n)dx \to \int_0^{2L} H(\bar{V})dx. \tag{2.26}$$

From (2.22), (2.25) and (2.26), we see that $\bar{V}(x)$ is a minimum element of functional G, and by Lemma 2.1, it is also a solution of stationary problem (2.4).

It is known that 0 satisfying $G(0) = 0$ is the trivial solution of (2.4). We are now going to prove that the minimum element $\bar{V}(x)$ is a non-trivial solution. In fact, we can prove $g = G(\bar{V}) < 0$, which implies that $\bar{V}(x) \not\equiv 0$ is non-trivial.

Choosing a test function as $\eta\tilde{v}_{1,1}(x)$, where $\tilde{v}_{1,1}(x) = \frac{1}{\sqrt{L}} \sin \frac{\pi x}{L}$, and $\eta > 0$ is a constant such that

$$\eta^2 < \frac{8L}{3}\Big(1 - 3m_0^2 - \frac{\varepsilon_1\varepsilon_2\pi^2}{L^2}\Big). \tag{2.27}$$

It is known that $\eta\tilde{v}_{1,1}(x) \in H^1_{per,0}$. A straightforward calculation gives

$$\int_0^{2L} |(\tilde{v}_{1,1})_x|^2 dx = \frac{\pi^2}{L^2}\int_0^{2L} \tilde{v}_{1,1}^2 dx.$$

and

$$\int_0^{2L} \tilde{v}_{1,1}^2(x)dx = 1, \ \int_0^{2L} \tilde{v}_{1,1}^3(x)dx = 0, \ \int_0^{2L} \tilde{v}_{1,1}^4(x)dx = \frac{3}{4L}.$$

Hence,

$$\begin{aligned}
G(\eta\tilde{v}_{1,1}) &= \int_0^{2L} \Big[\frac{\varepsilon_1\varepsilon_2\eta^2}{2}|(\tilde{v}_{1,1})_x|^2 + H(\eta\tilde{v}_{1,1})\Big]dx \\
&= \int_0^{2L} \Big[\frac{\varepsilon_1\varepsilon_2\eta^2\pi^2}{2L^2}\tilde{v}_{1,1}^2 + \frac{\eta^4}{4}\tilde{v}_{1,1}^4 + m_0\eta^3\tilde{v}_{1,1}^3 + \frac{(3m_0^2-1)\eta^2}{2}\tilde{v}_{1,1}^2\Big]dx \\
&= \Big(\frac{\varepsilon_1\varepsilon_2\pi^2}{2L^2} - \frac{1-3m_0^2}{2}\Big)\eta^2 \int_0^{2L} \tilde{v}_{1,1}^2 dx + \frac{\eta^4}{4}\int_0^{2L} \tilde{v}_{1,1}^4 dx + m_0\eta^3 \int_0^{2L} \tilde{v}_{1,1}^3 dx \\
&= -\Big(\frac{1-3m_0^2}{2} - \frac{\varepsilon_1\varepsilon_2\pi^2}{2L^2} - \frac{3}{16L}\eta^2\Big)\eta^2 \\
&< 0,
\end{aligned} \tag{2.28}$$

because of (1.11) and (2.27). Since $\bar{V}(x)$ is the minimum element of $G(v)$ on H^1_{per}, (2.28) implies $G(\bar{V}) = g < 0$. □

Lemma 2.5 *The non-trivial solutions are oscillatory, and always exhibit phase transitions.*

Proof. According to the periodicity, the non-trivial solution is always oscillatory.

Now we are going to prove that the non-trivial solutions always exhibit phase transitions. In fact, differentiating (2.1) with respect to x gives

$$\sigma'(V)V_x = \varepsilon_1\varepsilon_2 V_{xxx}. \tag{2.29}$$

Notice that $V_{xxx}(x)$ is non-monotone, namely, $V_{xxx}(x)$ changes signs on $[0, 2L]$. Let $V_{xxx}(x) \neq 0$ at some points x, we have four possible cases:

Case 1. $V_{xxx}(x) > 0$ and $V_x(x) > 0$;

Case 2. $V_{xxx}(x) > 0$ and $V_x(x) < 0$;

Case 3. $V_{xxx}(x) < 0$ and $V_x(x) > 0$;

Case 4. $V_{xxx}(x) < 0$ and $V_x(x) < 0$.

For Case 1, when $V_{xxx}(x) > 0$ and $V_x(x) > 0$, then from (2.29) we have $\sigma'(V(x)) > 0$, which is equivalent to $|V(x)| > \frac{1}{\sqrt{3}}$, i.e., $V(x) > \frac{1}{\sqrt{3}}$ or $V(x) < -\frac{1}{\sqrt{3}}$. This indicates that the phase transitions exhibit. Similarly, we can verify that, for Case 2, the solution V at x doesn't exhibit a phase transition, i.e., we have $-\frac{1}{\sqrt{3}} < V(x) < \frac{1}{\sqrt{3}}$, while Case 3 like Case 1 exhibits a phase transition, and Case 4 like Case 2 doesn't exhibit a phase transition.

If there are some points such that Case 1 or Case 3 occurs, we immediately prove that the non-trivial solutions exhibit phase transitions. This proves our lemma. If there are only Case 2 and Case 4, but no Case 1 and Case 3, we see that, whatever $V_{xxx} > 0$ or < 0, we always have $V_x < 0$. This is a contradiction to the non-monotonicity of $V(x)$. Therefore, the non-trivial stationary solution $V(x)$ always occurs phase transitions. $\square$

Proof of Theorem 1.1. Thanks to Lemma 2.3, and notice that $V(x) = \bar{V}(x) + m_0$, we proved Part 1 of Theorem 1.1. Furthermore, Lemma 2.4 implies the existence of multiple non-trivial stationary solutions of (2.1) in Part 2 of Theorem 1.1, and Lemma 2.5 proves that all non-trivial stationary solutions are oscillatory and exhibit the phase transitions. Thus, Theorem 1.1 is proved. $\square$

Now we are going to prove Theorem 1.2, i.e., we count the number of the non-trivial solutions for $m_0 \in [0, \frac{1}{\sqrt{3}})$.

Firstly, when $m_0 = 0$, for the stationary equation (2.1), if we change the periodic interval from $[0, 2L]$ to $[-L, L]$, then it is equivalent to

$$\begin{cases} \varepsilon_1\varepsilon_2 V_{xx} = \sigma(V) - a \\ V(x-L) = V(x+L) \\ \frac{1}{2L}\int_{-L}^{L} V(x)dx = m_0. \end{cases} \tag{2.30}$$

Let $y = \frac{x}{L}$, (2.30) can be rewritten as

$$\begin{cases} \frac{\varepsilon_1\varepsilon_2}{L^2} V_{yy} = \sigma(V) - a \\ V(y-1) = V(y+1) \\ \frac{1}{2}\int_{-1}^{1} V(y)dy = m_0. \end{cases} \tag{2.31}$$

By a very similar method as shown in [2, 23] (see [23]: Eqs. (3.16) and (3.17) on page 178 and Theorem 3.4 on page 179), we can prove that the modified stationary equation (2.31) has $2N_0 + 1$ solutions for N_0 satisfying (1.12). Here the detail of the proof is omitted.

Lemma 2.6 *For $m_0 = 0$ and $\varepsilon_1\varepsilon_2 < \frac{L^2}{\pi^2}(1 - 3m_0^2)$, there are $(2N_0 + 1)$ solutions for the stationary equation* (2.31)*, where N_0 is the integer satisfying* (1.12)*, i.e.,*

$$\frac{2L}{\pi\sqrt{\varepsilon_1\varepsilon_2}} - 1 < N_0 \le \frac{2L}{\pi\sqrt{\varepsilon_1\varepsilon_2}}. \tag{2.32}$$

Furthermore, when $m_0 \neq 0$, by the change of variable $y = \frac{x}{2L} + \frac{1}{2}$, then we deduce (2.30) to

$$\begin{cases} \frac{\varepsilon_1\varepsilon_2}{4L^2} V_{yy} = \sigma(V) - a \\ V(y) = V(y+1) \\ \int_0^1 V(y)dy = m_0. \end{cases} \tag{2.33}$$

By the transversality arguments as presented in [9], based on the size of viscosity $\varepsilon_1\varepsilon_2$ and the location of the initial average m_0, we can very similarly count the number of stationary solutions as follows. Here, the proof is also omitted.

Lemma 2.7

1. *When* $m_0 \in (0, \frac{1}{\sqrt{5}})$, *if* $\varepsilon_1\varepsilon_2 \geq \frac{L^2}{\pi^2}(1 - 3m_0^2)$, *then there is no non-trivial stationary solution to Eq.* (2.33). *However, if* $\varepsilon_1\varepsilon_2 < \frac{L^2}{\pi^2}(1 - 3m_0^2)$, *then there is only one non-trivial solution to Eq.* (2.33).

2. *When* $m_0 \in (\frac{1}{\sqrt{5}}, \frac{1}{\sqrt{3}})$, *there exists a positive number* $\varepsilon^* > \frac{L^2}{\pi^2}(1 - 3m_0^2)$ *such that, if* $\varepsilon_1\varepsilon_2 > \varepsilon^*$, *then there is no non-trivial solution to Eq.* (2.33). *However, if* $\frac{L^2}{\pi^2}(1 - 3m_0^2) < \varepsilon_1\varepsilon_2 < \varepsilon^*$, *then there are only two non-trivial solutions to Eq.* (2.33). *While, if* $\varepsilon_1\varepsilon_2 \leq \frac{L^2}{\pi^2}(1 - 3m_0^2)$, *then there is only one non-trivial solution to Eq.* (2.33).

Proof of Theorem 1.2. Combining Lemma 2.6 and Lemma 2.7, we immediately prove Theorem 1.2. □

3. Numerical Simulations

With the central finite difference for the second order derivative and the Simpson's rule for the integral, system (2.1) is discretized as follows:

$$\begin{cases} \varepsilon_1\varepsilon_2(v_{i-1} - 2v_i + v_{i+1}) = h^2(v_i^3 - v_i) - a, \quad i = 1, 2, \cdots, n-1 \\ v_n = v_0 \\ h\sum_{i=1}^{n/2}(2v_{2i-2} + 4v_{2i-1}) = 6Lm_0 \\ a = h^2(v_0^3 - v_0) - \varepsilon_1\varepsilon_2(v_1 - 2v_0 + v_{n-1}) \end{cases} \tag{3.1}$$

where n is even, and $h = \frac{2L}{n}$.

The above scheme is a system of $n+1$ nonlinear algebraic equations with $n+1$ variables. The Newton-Raphson's nonlinear solver is used with various initial guessing points. The computer program uses a random generator for the initial guess. For each case, a reasonable number of sets of guessing values has been implemented. The convergent solution could be unique or multiple depending on the system configuration, i.e. the values of the initial average m_0 and the viscosity ε_1 and ε_2. The smaller the space step h is, i.e. the larger the number of grids n is, the more accurate the numerical solutions are. However, when n is large, the computational cost for the nonlinear solver increases dramatically. For the chosen $L = \pi$ for the cases reported in the following, n near 60 (i.e. h near 0.1) provides convergent solutions in a short time (within couple of minutes) and with good accuracy. Further increasing the value n does not change the solutions significantly.

To verify Theorem 1.1, first of all, by plugging $m_0 = 0$ and $\varepsilon_1\varepsilon_2 = 4$ in the numerical program, we obtain only the trivial zero solution $V \equiv 0$. For small viscosities, the case with $m_0 = 0$ and $\varepsilon_1\varepsilon_2 = 0.49$ is examined, and one non-trivial solution is found as shown in Figure 3.1, which also verifies the first item of Theorem 1.2 with $N_0 = 2$.

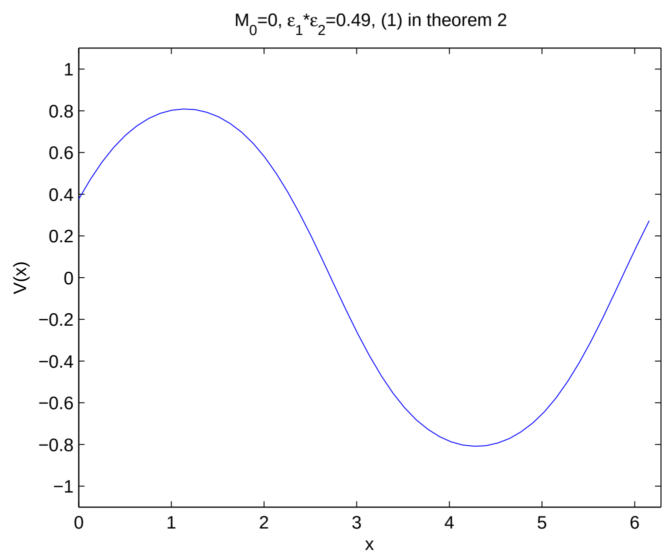

Figure 3.1. Case 1: $m_0 = 0$, $\varepsilon_1\varepsilon_2 = 0.49$, Item 1 in Theorem 1.2

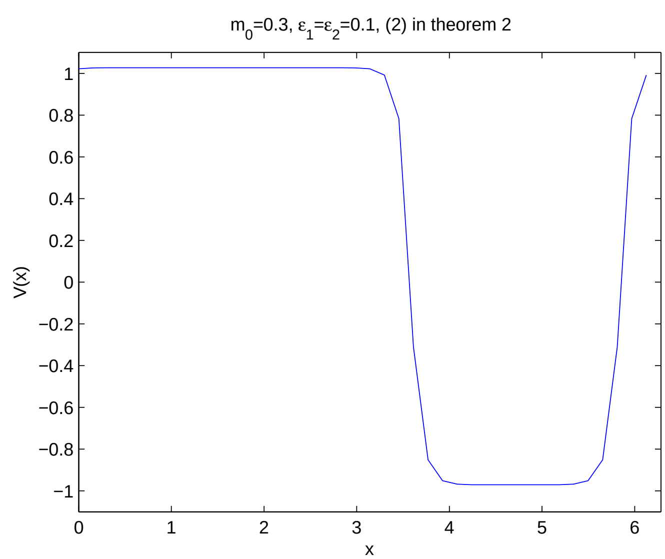

Figure 3.2. Case 2: $m_0 = 0.3$, $\varepsilon_1 = \varepsilon_2 = 0.1$, Item 2 in Theorem 1.2 with a unique non-trivial solution

For Item 2 in Theorem 1.2, the chosen parameters are $m_0 = 0.3$ and $\varepsilon_1 = \varepsilon_2 = 0.1$, and the unique non-trivial solution is found. See Figure 3.2.

Select $m_0 = 0.5$ and $\varepsilon_1\varepsilon_2 = 0.28$ that satisfies $\frac{L^2}{\pi^2}(1 - 3m_0^2) < \varepsilon_1\varepsilon_2 < \varepsilon^*$, we find two non-trivial solutions as shown in Figure 3.3.

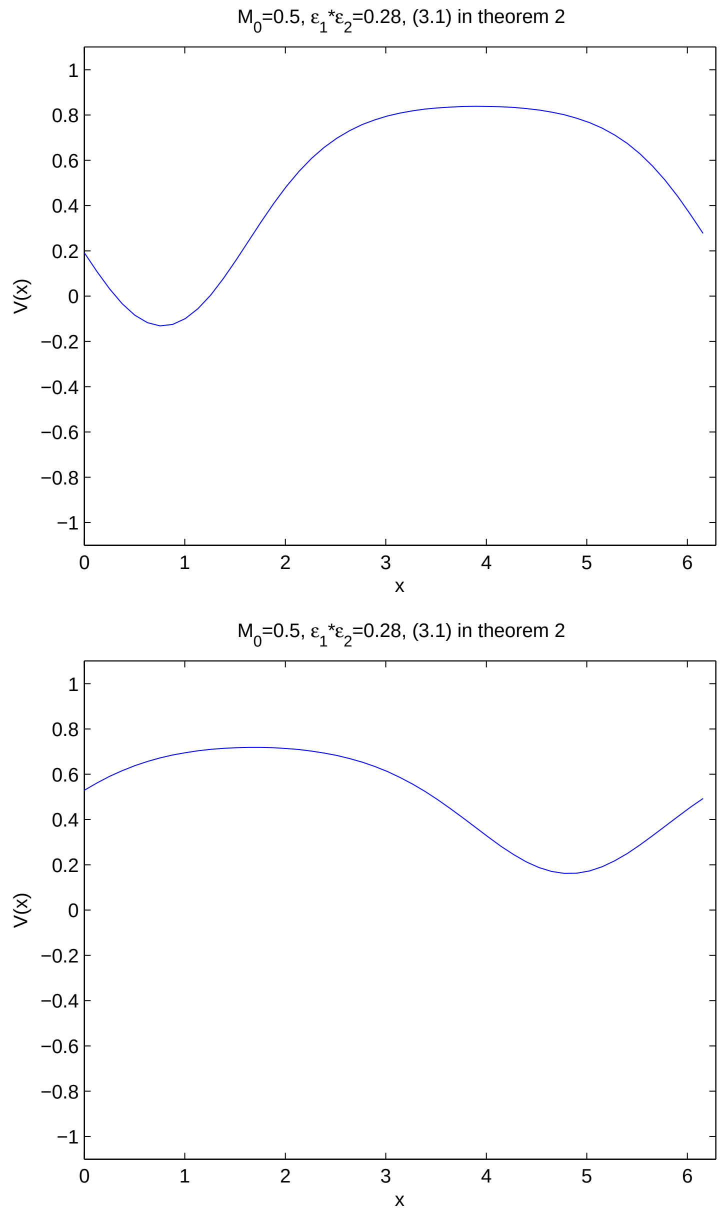

Figure 3.3. Case 3: $m_0 = 0.5$, $\varepsilon_1\varepsilon_2 = 0.28$, the first part of Item 3 in Theorem 1.2 with two non-trivial solutions.

For the same value of the initial average $m_0 = 0.5$, when the viscosity is small as $\varepsilon_1\varepsilon_2 = 0.02$ such that $\varepsilon_1\varepsilon_2 < \frac{L^2}{\pi^2}(1-3m_0^2)$, the program only finds one non-trivial solution as shown Figure 3.4.

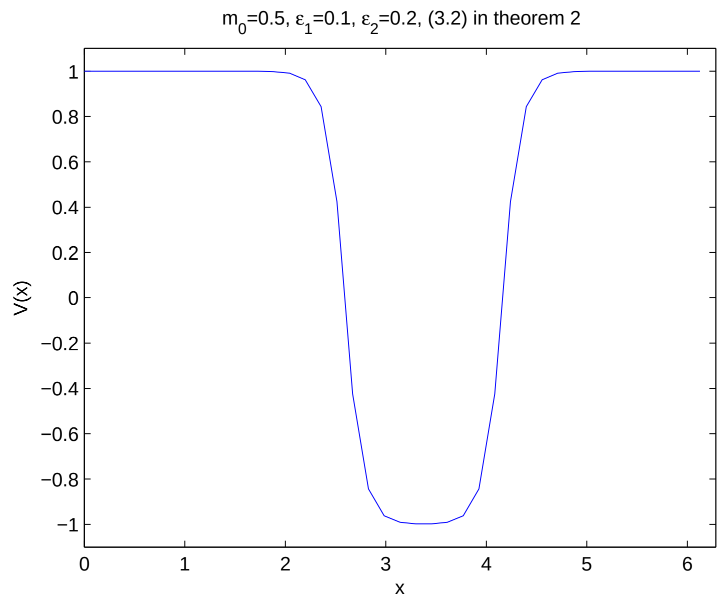

Figure 3.4. Case 4: $m_0 = 0.5$, $\varepsilon_1\varepsilon_2 = 0.02$, the second part of Item 3 in Theorem 1.2 with a unique non-trivial solutions.

References

[1] M. Affouf and R. E. Caflisch, A numerical study of Riemann problem solutions and stability for a system of viscous conservation laws of mixed type, *SIAM J. Appl. Math.* **51** (1991) 605-634.

[2] J. Carr, M. Gurtin and M. Slemrod, Structured phase transitions on a finite interval, *Arch. Rational Mech. Anal.*, **86** (1984) 317–351.

[3] J. Carr, M. Gurtin and M. Slemrod, One-dimensional structured phase transformations under prescribed loads, *J. Elasticity*, **15** (1985) 133–142.

[4] A. Eden, A. Milani and B. Nicolaenko, Local expential attractors for model of phase change for compressible gas dynamics, *Nonlinearity*, **6** (1993) 93–117.

[5] C. M. Elliott and S. Zheng, On the Cahn-Hilliard equation, *Arch. Rational Mech. Anal.*, **96** (1986) 339–357.

[6] H.-T. Fan, A limiting “viscosity” approach to the Riemann problem for materials exhibiting a change of phase. II, *Arch. Rational Mech. Anal.*, **116** (1992) 317–337.

[7] P. C. Fife and X.-P. Wang, Periodic structures in a van der Waals fluid, *Proc. Royal Soc. Edinburgh, Sect. A*, **128** (1998) 235–250.

[8] M. Grinfeld, J.E. Furter and J.C. Eilbeck, A monotonicity theorem and its application to stationary solutions of phase field model, *IMA J. Appl. Math.* **49** (1992) 61–72; **50** (1993) 203–204.

[9] M. Grinfeld and A. Novick-Cohen, Counting stationary solutions of the Cahn-Hilliard equation by transversality arguments, *Proc. Royal Soc. Edinburgh, Sect. A*, **125** (1995) 351–370.

[10] M. E. Gurtin and H. Matano, On the structure of equilibrium phase transitions within the gradient theory of fluids, *Quart. Appl. Math.*, **46** (1988) 301–317.

[11] M. E. Gurtin, On a theory of phase transitions with interfacial energy, *Arch. Rational Mech. Anal.*, **87** (1985) 187–212.

[12] R. Hagan and M. Slemrod, The viscous-capillarity criterion for shocks and phase transitions, *Arch. Rational Mech. Anal.* **83** (1983) 333–361.

[13] D.-Y. Hsieh and X.-P. Wang, Phase transition in van der Waals fluid, *SIAM J. Appl. Math.* **57** (1997) 871–892.

[14] R. D. James, The propagation of phase boundaries in elastic bars, *Arch. Rational Mech. Anal.* **73** (1980) 125–158.

[15] M. Mei, Y.-S. Wong and L. Liu, Phase transitions in a coupled viscoelastic system with periodic initial-boundary condition: (I) existence and uniform boundedness, *Discrete Continu. Dynam. Systems, Series B*, **7** (2007) 825–837.

[16] M. Mei, Y.-S. Wong and L. Liu, Phase transitions in a coupled viscoelastic system with periodic initial-boundary condition: (II) convergence, *Discrete Continu. Dynam. Systems, Series B*, **7** (2007) 839–857.

[17] B. Nicolaenko and W. Qian, Inertial manifolds for nonlinear viscoelasticity equations, *Nonlinearity*, **11** (1998) 1075–1093.

[18] A. Novick-Cohen, On Cahn-Hilliard type equations, *Nonlinear Analysis*, **15** (1990) 797–814.

[19] A. Novick-Cohen and L.A. Peletier, Steady states of the one-dimensional Cahn-Hilliard equation, *Proc. Royal Soc. Edinburgh, Sect. A*, **123** (1993) 1071–1098.

[20] A. Novick-Cohen and L.A. Segel, Nonlinear aspects of the Cahn-Hilliard equation, *Phsica D* **10** (1984) 277–298.

[21] M. Shearer, Dynamic phase transitions in a van der Waals gas, *Quart. Appl. Math.*, **46** (1988) 631–636.

[22] M. Slemrod, Admissibility criteria for propagating phase boundaries in a van der Waals fluid, *Arch. Rational Mech. Anal.*, **81** (1983) 301–315.

[23] S. Zheng, Asymptotic behavior of solution to the Cahn-Hillard equation, *Applicable Analysis*, **23** (1986) 164–184.

[24] S. Zheng, Nonlinear Parabolic Equations and Hyperbolic-Parabolic Coupled Systems, Pitman Monographs and Surverys in Pure and Applied Mathematics **76**, Longman Scientific & Technical, Harlow; copublished in the United States with John Wiley & Sons, Inc., New York, 1995.

[25] K. Zumbrun, Dynamical stability of phase transitions in the p-system with viscosity-capillarity, *SIAM J. Appl. Math.*, **60** (2000) 1913–1924.