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Best asymptotic profile for linear damped p-system with boundary effect

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ABSTRACT

This paper is devoted to the study of the 2×2 linear damped p-system with boundary effect. By a heuristic analysis, we realize that the best asymptotic profile for the original solution is the parabolic solution of the IBVP for the corresponding porous media equation with a specified initial data. In particular, we further show the convergence rates of the original solution to its best asymptotic profile, which are much better than the existing rates obtained in the previous works. The approach adopted in the paper is the elementary weighted energy method with Green function method together.

1. Introduction

In this work, we are interested in the best time-asymptotic behavior of the solution to the p-system with linear damping on the quarter plane $\mathbb{R}^+ \times \mathbb{R}^+$ ($\mathbb{R}^+ = [0, \infty)$), given by

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v)_x = -\alpha u, \end{cases} \qquad (x,t) \in \mathbb{R}^+ \times \mathbb{R}^+. \tag{1.1}$$

Eq. (1.1) models the compressible flow through porous media in Lagrangian coordinates. Here, $v > 0$ is the specific volume, u is the velocity, the pressure $p(v)$ is a smooth function of v such that $p(v) > 0$, $p'(v) < 0$, and $\alpha > 0$ is the damping constant.

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In this work, we mainly consider the IBVPs for the p-system with linear damping in two different boundary conditions, respectively:

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v)_x = -\alpha u, \quad (x,t) \in \mathbb{R}^+ \times \mathbb{R}^+, \\ (v,u)(x,0) = (v_0,u_0)(x) \to (v_+,u_+) \quad \text{as } x \to \infty, \\ v(0,t) = v_-, \end{cases} \tag{1.2}$$

and

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v)_x = -\alpha u, \quad (x,t) \in \mathbb{R}^+ \times \mathbb{R}^+, \\ (v,u)(x,0) = (v_0,u_0)(x) \to (v_+,u_+) \quad \text{as } x \to \infty, \\ u(0,t) = 0. \end{cases} \tag{1.3}$$

Here, for the system (1.2), we need $v_\pm > 0$ because v denotes the specific volume, but u can be any because it denotes the velocity. The compatibility condition is $v_0(0) = v_-$, and the corresponding boundary condition for u, from the first equation, is $u_x|_{x=0} = 0$. While, for the system (1.3), the compatibility condition is $u_0(0) = 0$, and the corresponding boundary condition for v, from the second equation, is $v_x|_{x=0} = 0$.

For the corresponding Cauchy problem (CP), the solution was first shown by Hsiao and Liu [4,5] to time-asymptotically behave like the so-called diffusion waves of Darcy's law,

$$\begin{cases} \bar{v}_t - \bar{u}_x = 0, \\ p(\bar{v})_x = -\alpha \bar{u}, \end{cases} \quad (x,t) \in \mathbb{R}^+ \times \mathbb{R}^+, \tag{1.4}$$

in the form $\bar{v}(x,t) = \bar{v}(x/\sqrt{1+t})$ (i.e., the self-similar solution, whose existence and properties had been studied by van Duyn and Peletier [3] in 1970), with the convergence rates $\|(v-\bar{v}, u-\bar{u})(t)\|_{L^\infty} = O(1)(t^{-1/2}, t^{-1/2})$. Then, by taking more detailed but elegant energy estimates, Nishihara [22] succeeded in improving the convergence rates as $\|(v-\bar{v}, u-\bar{u})(t)\|_{L^\infty} = O(1)(t^{-3/4}, t^{-5/4})$, which are optimal in the sense that the initial perturbation around the specified diffusion wave is in L^2, but still less sufficient for the initial perturbation in L^1. Furthermore, by constructing an approximate Green function with the energy method together, Nishihara, Wang and Yang [25] (see also the precise piecewise-rates later than by Wang and Yang [29]) completely improved the rates as $\|(v-\bar{v}, u-\bar{u})(t)\|_{L^\infty} = O(1)(t^{-1}, t^{-3/2})$, which are optimal when the initial perturbation around the diffusion wave is in L^1. Notice that, for the Darcy's law, with different initial data, the solution $(\bar{v}, \bar{u})(x,t)$ is different. In another word, the asymptotic profiles for the p-system with linear damping are not unique. Based on this observation, Nishihara [24] and Mei [21] found the best asymptotic profiles for the cases $v_- = v_+$ and $v_- \neq v_+$ respectively, and both of them obtained the convergence rates as $\|(v-\bar{v}, u-\bar{u})(t)\|_{L^\infty} = O(1)(t^{-3/2}\log t, t^{-2}\log t)$, where the slower decay t^{-1} of the diffusion wave $\bar{v}_t$ in L^1 causes the extra term $\log t$.

For the IBVP (1.2) replaced the boundary condition $v(0,t) = v_-$ by $v(0,t) = g(t)$, $g(t) \to v_+$, has been considered by Marcati and Mei [16]. However, the case $g(t) \equiv v_- (\neq v_+)$ or $g(t) \to v_- (\neq v_+)$ is not treated there. Later on, Nishihara and Yang [26] studied the IBVP (1.2). For the case $v_- \neq v_+$, they selected asymptotic profile of (1.2) as the diffusion wave $(\bar{v}, \bar{u})(x,t)$ in the form of $\bar{v}(x,t) = \phi(\frac{x+d(t)}{\sqrt{1+t}})$ with some shift function $d(t)$. And for $v_- = v_+$, they selected the asymptotic profile of (1.2) is, obviously, $(\bar{v}, \bar{u}) \equiv (v_+, 0)$. In both cases, they proved its stability with algebraic decay as $\|(v-\bar{v}, u-\bar{u})(t)\|_{L^\infty} = O(1)(t^{-3/4}, t^{-5/4})$. While for the IBVP (1.3), Nishihara and Yang [26] studied the convergence to diffusion wave, where the selected asymptotic profile of (1.3) is the linear diffusion wave:

$$\begin{cases} \bar{v} = v_+ + \dfrac{\bar{\delta}_0}{\sqrt{4\kappa\pi(t+1)}} \exp\left(-\dfrac{x^2}{4\kappa(t+1)}\right), \\ \bar{u}(x,t) = \kappa \bar{v}_x(x,t), \end{cases} \quad (x,t) \in \mathbb{R}^+ \times \mathbb{R}^+, \tag{1.5}$$

with

$$\kappa = -p'(v_+)/\alpha, \qquad \bar{\delta}_0 = 2\left(\int_0^\infty \big(v_0(x) - v_+\big)\,dx - \frac{u_+}{\alpha}\right).$$

They further showed the convergence rate of (1.3) to the linear diffusion wave $(\bar{v}, \bar{u})(x,t)$ as $\|(v - \bar{v}, u - \bar{u})(t)\|_{L^\infty} = O(1)(t^{-3/4}, t^{-5/4})$. Observing that the decay rates can be better by eliminate the slower decay term $(p'(\bar{v}) - p'(v_+))\bar{v}_x$ in [26], Marcati, Mei and Rubino [17] chose the asymptotic profile as the nonlinear diffusion wave

$$\begin{cases} \bar{\bar{v}}_t - \bar{\bar{u}}_x = 0, \\ p(\bar{\bar{v}})_x = -\alpha \bar{\bar{u}}, \\ \bar{\bar{v}}_x|_{x=0} = 0, \\ \bar{\bar{v}}|_{x=\infty} = v_+, \end{cases} \quad (x,t) \in \mathbb{R}^+ \times \mathbb{R}^+. \tag{1.6}$$

Then they improved the convergence rates to be optimal $\|(v - \bar{v}, u - \bar{u})(t)\|_{L^\infty} = O(1)(t^{-1}, t^{-3/2})$ when the initial perturbation belongs to L^1.

We also notice that, recently, Said-Houari [28] claimed that he further improved the convergence rates for the IBVP (1.3) to $\|(v - \bar{v}, u - \bar{u})(t)\|_{L^\infty} = O(1)(t^{-1-\frac{\gamma}{2}}, t^{-\frac{3}{2}-\frac{\gamma}{2}})$ for $\gamma \in [0,1]$, when the initial perturbation is in the weighted $L^{1,1}$-space. However, his result and proof both are incorrect, because he did not treat with the nonlinear term which also involves the diffusion wave $\bar{v}_t$. It is just this term causes a slower decay and we cannot expect to have the faster decay in the weighted space $L^{1,1}$ (for details, see Remark 1.3 in [14]). In fact, as Lin, Lin and Mei [14] showed, even for the linear damping case, the convergence rate is $\|(v - \bar{v}, u - \bar{u})(t)\|_{L^\infty} = O(1)(t^{-1-\frac{\gamma}{2}}, t^{-\frac{3}{2}})$ for $\gamma = \frac{1}{4}$, and $\gamma = \frac{1}{4}$ is the best selected number under consideration of the slow decay from $\bar{v}_t$ in the nonlinear term.

In this paper, motivated by Nishihara [24] and Mei [21], by a heuristic analysis, we realize that the convergence rates can be further improved by constructing the best asymptotic profile, which is the parabolic solution of the IBVP for the corresponding porous media equation with a specified initial data (see (2.8)), and we still denote it as $(\bar{v}, \bar{u})(x,t)$. For the IBVP (1.3), we can prove $\|(v - \bar{v}, u - \bar{u})(t)\|_{L^\infty} = O(1)(t^{-\frac{3}{2}} \log t, t^{-2} \log t)$, which is same to the rate showed in [24,21]. For the IBVP (1.2), we get the decay rate as $\|(v - \bar{v}, u - \bar{u})(t)\|_{L^\infty} = O(1)(t^{-\frac{3}{2}-\frac{\gamma}{4}}, t^{-2})$ for $\gamma \in [0, \frac{1}{4}]$ as $v_+ = v_-$. This rate is much better than all existing rates obtained in the previous works.

For the other interesting results in different cases, we refer to [1,2,6–15,18,20,23,27,30–33] and the references therein.

The rest of the paper is organized as follows. In Section 2, we study the initial boundary value problem (1.2). First, we give a heuristic analysis to find the best asymptotic profile for the system (1.2), and build up the working system. Then we state the properties of the asymptotic profile and our main results. In the rest subsections of this section, we will establish the existence and decay properties of our asymptotic profile and prove our main results. In Section 3, we make an odd extension to u and an even extension to v respectively to change the IBVP (1.3) to the corresponding Cauchy problem. Then we make use of the known results of the Cauchy problem to obtain the asymptotic behavior of the system (1.3).

Notations. Throughout this paper, $C > 0$ denotes a generic constant which may change its value from line to line or even in the same line. $\|\cdot\|_{L^p}$ and $\|\cdot\|_l$ stand for the $L^p(\mathbb{R}^+)$-norm $(1 \leqslant p \leqslant \infty)$ and $H^l(\mathbb{R}^+)$-norm and sometimes, without confusion, for $L^p(\mathbb{R})$-norm and $H^l(\mathbb{R})$-norm, respectively. The

L^2-norm on $\mathbb{R}^+$ or $\mathbb{R}$ is simply denoted by $\|\cdot\|$. Also, $\forall p \in [1,\infty)$, for $\gamma \in [0,\infty)$, we define the weighted function space $L^{p,\gamma}(\mathbb{R}^+)$ as follows: $f \in L^{p,\gamma}(\mathbb{R}^+)$ iff $f \in L^p(\mathbb{R}^+)$ and

$$\|f\|_{p,\gamma}^p = \int_0^\infty (1+x)^\gamma \left|f(x)\right|^p dx < \infty.$$

2. Initial boundary value problem (1.2)

2.1. Best asymptotic profile and main results

As analyzed below, the best asymptotic profile for system (1.2) should be the solution to the following equations

$$\begin{cases} \bar v_t - \bar u_x = 0, \\ p(\bar v)_x = -\alpha \bar u, \quad (x,t) \in \mathbb{R}^+ \times \mathbb{R}^+, \\ (\bar v, \bar u)(x,0) = (\bar v_0, \bar u_0)(x) \to (v_+, 0) \quad \text{as } x \to +\infty, \\ \bar v(0,t) = v_-, \quad t \in \mathbb{R}^+, \end{cases} \tag{2.1}$$

where the initial data $(\bar v_0, \bar u_0)(x) = (\bar v_0, -\frac{1}{\alpha} p(\bar v_0))(x)$ will be specified later.

First of all, let us technically construct the correction function $(\hat v, \hat u)(x,t)$ as follows

$$\begin{cases} \hat v(x,t) = -\dfrac{1}{\alpha}\left[u_+ m_0(x) + \delta_0 m_0'(x)\right] e^{-\alpha t}, \\ \hat u(x,t) = \left[u_+ m(x) + \delta_0 m_0(x)\right] e^{-\alpha t}, \end{cases}$$

which is different from what selected in the previous works for the IBVPs [16,26,17]. Here $m_0(x)$ is a smooth and compact supported function $m_0(x) \in C_0^\infty(\mathbb{R}^+)$ satisfying

$$m_0(0) = m_0(\infty) = 0, \qquad m_0'(0) = 0, \qquad \int_0^\infty m_0(y)\,dy = 1,$$

and $m(x)$ is defined as

$$m(x) = \int_0^x m_0(y)\,dy, \qquad m(\infty) = 1.$$

δ_0 is a constant given by

$$\delta_0 := \frac{1}{\alpha}\left[p(v_+) - p(v_-)\right] + \int_0^\infty \left[u_0(x) - u_+ m(x)\right] dx. \tag{2.2}$$

Thus, $(\hat v, \hat u)(x,t)$ satisfies

$$\begin{cases} \hat v_t - \hat u_x = 0, \\ \hat u_t = -\alpha \hat u, \quad (x,t) \in \mathbb{R}^+ \times \mathbb{R}^+, \\ (\hat v, \hat u)(x,t) \to (0, u_+ e^{-\alpha t}) \quad \text{as } x \to +\infty. \end{cases} \tag{2.3}$$

Now we are going to determine $\bar{v}_0(x)$ such that the corresponding solution $(\bar{v}, \bar{u})(x,t)$ to the system (2.1) is the best asymptotic profile for the original solution $(v,u)(x,t)$, and then we derive the perturbation equations. From (1.2), (2.1) and (2.3), we have

$$\begin{cases} (v-\bar{v}-\hat{v})_t - (u-\bar{u}-\hat{u})_x = 0, \\ (u-\bar{u}-\hat{u})_t + \big[p(v)-p(\bar{v})\big]_x = -\alpha(u-\bar{u}-\hat{u}) + \dfrac{1}{\alpha}p(\bar{v})_{xt}. \end{cases} \tag{2.4}$$

Integrating the second equation of (2.4) with respect to x over $\mathbb{R}^+$ and noting the boundary condition $\bar{v}(0,t) = v(0,t) = v_-$ and $\bar{v}(+\infty,t) = v(+\infty,t) = v_+$ yield

$$\frac{d}{dt}\int_0^\infty \big[u(x,t)-\bar{u}(x,t)-\hat{u}(x,t)\big]\,dx = -\alpha\int_0^\infty \big[u(x,t)-\bar{u}(x,t)-\hat{u}(x,t)\big]\,dx$$

which can be solved as

$$\begin{aligned}\int_0^\infty \big[u(x,t)-\bar{u}(x,t)-\hat{u}(x,t)\big]\,dx &= e^{-\alpha t}\int_0^\infty \big[u_0(x)-\bar{u}(x,0)-\hat{u}(x,0)\big]\,dx \\ &= e^{-\alpha t}\int_0^\infty \bigg[u_0(x)+\frac{1}{\alpha}p\big(\bar{v}_0(x)\big)_x - \hat{u}(x,0)\bigg]\,dx \\ &= e^{-\alpha t}\bigg\{\int_0^\infty \big[u_0(x)-u_+m(x)\big]\,dx + \frac{1}{\alpha}\big[p(v_+)-p(v_-)\big]-\delta_0\bigg\} \\ &= 0, \end{aligned} \tag{2.5}$$

where δ_0 selected in (2.2) comes from the last step. Now we turn to the first equation of (2.4) to determine $\bar{v}_0(x)$. Integrating $(2.4)_1$ with respect to x over $[x,\infty)$, we obtain

$$\begin{aligned}\frac{d}{dt}\int_x^\infty \big[v(z,t)-\bar{v}(z,t)-\hat{v}(z,t)\big]\,dz &= (u-\bar{u}-\hat{u})(z,t)|_{z=\infty} - (u-\bar{u}-\hat{u})(z,t)|_{z=x} \\ &= -\big[u(x,t)-\bar{u}(x,t)-\hat{u}(x,t)\big], \end{aligned} \tag{2.6}$$

then integrate the above equation with respect to x over $\mathbb{R}^+$ and use (2.5) to have

$$\frac{d}{dt}\int_0^\infty\int_x^\infty \big[v(z,t)-\bar{v}(z,t)-\hat{v}(z,t)\big]\,dz\,dx = -\int_0^\infty \big[u(x,t)-\bar{u}(x,t)-\hat{u}(x,t)\big]\,dx = 0,$$

which gives

$$\int_0^\infty\int_x^\infty \big[v(z,t)-\bar{v}(z,t)-\hat{v}(z,t)\big]\,dz\,dx = \int_0^\infty\int_x^\infty \big[v_0(z)-\bar{v}_0(z)-\hat{v}(z,0)\big]\,dz\,dx. \tag{2.7}$$

By selecting $\bar{v}_0(x)$ as

$$\bar{v}_0(x) = v_0(x) - \hat{v}(x,0), \tag{2.8}$$

then from (2.7) and (2.8), we obtain

$$\int_0^\infty \int_x^\infty \big[v(z,t) - \bar{v}(z,t) - \hat{v}(z,t)\big]\,dz\,dx = 0. \tag{2.9}$$

Thus, as explained in [21], the solution $(\bar{v},\bar{u})(x,t)$ for the system (1.2) with the specified initial data $\bar{v}_0$ in (2.8) is the best asymptotic profile for the original system (1.1). Therefore, let

$$\begin{aligned}
V(x,t) &:= \int_x^\infty \int_y^\infty (v - \bar{v} - \hat{v})(z,t)\,dz\,dy,\\
U(x,t) &:= \int_0^x (u - \bar{u} - \hat{u})(y,t)\,dy,\\
V_0(x) &:= \int_x^\infty \int_y^\infty \big(v_0(z) - \bar{v}_0(z) - \hat{v}(z,0)\big)\,dz\,dy = 0,\\
U_0(x) &:= \int_0^x \big(u_0(y) - \bar{u}(y,0) - \hat{u}(y,0)\big)\,dy,
\end{aligned} \tag{2.10}$$

namely

$$V_{xx} = v - \bar{v} - \hat{v}, \qquad U_x = u - \bar{u} - \hat{u}.$$

Then $U(\infty,t) = 0$, the original system can be reformulated as

$$\begin{cases}
V_t - U = 0,\\
U_t + p(\bar{v} + \hat{v} + V_{xx}) - p(\bar{v}) = -\alpha U + p(\bar{v})_t, \quad (x,t) \in \mathbb{R}^+ \times \mathbb{R}^+,\\
(V,U)|_{t=0} = \big(0, U_0(x)\big),\\
V(0,t) = 0,
\end{cases} \tag{2.11}$$

which can be rewritten as

$$\begin{cases}
V_t - U = 0,\\
U_t + \big(p'(\bar{v})V_x\big)_x = -\alpha U - F_1 - F_2, \quad (x,t) \in \mathbb{R}^+ \times \mathbb{R}^+,\\
(V,U)|_{t=0} = \big(0, U_0(x)\big),\\
V(0,t) = 0,
\end{cases} \tag{2.12}$$

or

$$\begin{cases}
V_t - U = 0,\\
U_t + p'(v_+)V_{xx} = -\alpha U - F_1 - F_3, \quad (x,t) \in \mathbb{R}^+ \times \mathbb{R}^+,\\
(V,U)|_{t=0} = \big(0, U_0(x)\big),\\
V(0,t) = 0,
\end{cases} \tag{2.13}$$

where

$$
\begin{aligned}
&F_1 := -p(\bar{v})_t, \\
&F_2 := p(\bar{v} + \hat{v} + V_{xx}) - p(\bar{v}) - p'(\bar{v})V_{xx} - p'(\bar{v})_x V_x, \\
&F_3 := p(\bar{v} + \hat{v} + V_{xx}) - p(\bar{v}) - p'(v_+)V_{xx},
\end{aligned}
$$

which implies, by Taylor's formula, that

$$
\begin{aligned}
&|F_1| = O\big(|\bar{v}_t|\big), \\
&|F_2| = O\big(|\hat{v}| + |V_{xx}|^2 + |\bar{v}_x V_x|\big). \\
&|F_3| = O\big(|\hat{v}| + |V_{xx}|^2 + |\bar{v} - v_+||V_{xx}|\big).
\end{aligned}
\tag{2.14}
$$

In this section, we mainly consider the case $v_- = v_+$, and for the case $v_- \neq v_+$ we will give a remark at the end of this section. Now we are going to state our convergence results. First of all, we have the following existence and stability of the solution $(\bar{v}, \bar{u})(x,t)$ (the best asymptotic profile) for the system (2.1) and (2.8).

Theorem 2.1 *(Decay rates of best asymptotic profile). Let $v_+ = v_-$ and $l \geqslant 3$. Suppose $\bar{v}_0 - v_+ \in L^1(\mathbb{R}^+) \cap H^l(\mathbb{R}^+) \cap W^{l-1,1}(\mathbb{R}^+)$, let $\delta_{0\bar{v}} = |\int_0^\infty (\bar{v}_0(x) - v_+)\,dx|$ be suitably small. Then there exists a unique solution $(\bar{v}, \bar{u})(x,t)$ to* (2.1) *and* (2.8) *satisfying the decay properties*

$$
\begin{aligned}
&\big\|\partial_t^j \partial_x^k (\bar{v} - v_+)(t)\big\|_{L^p} \leqslant C\delta_{0\bar{v}}(1+t)^{-(1-1/p)/2-(k+2j)/2}, \\
&\quad t \geqslant 0,\ 1 \leqslant p \leqslant \infty,\ j,k \geqslant 0,\ 2j+k \leqslant l-1, \\
&\quad \big\|\partial_x^k \bar{u}(t)\big\|_{L^p} \leqslant C\delta_{0\bar{v}}(1+t)^{-(1-1/p)/2-(1+k)/2}, \\
&\qquad t \geqslant 0,\ 1 \leqslant p \leqslant \infty,\ 0 \leqslant k \leqslant l-2.
\end{aligned}
\tag{2.15}
$$

Moreover, if $\bar{v}_0 - v_+ \in L^{1,\gamma}(\mathbb{R}^+)$, $0 < \gamma \leqslant 1$, then $\forall a \in (0,\gamma)$, the solution $\bar{v}(x,t)$ to the system (2.1) *and* (2.8) *satisfies*

$$
\begin{aligned}
&\big\|\partial_t^j \partial_x^k (\bar{v} - v_+)(t)\big\|_{1,a} \leqslant C(1+t)^{-(2j+k+\gamma-a)/2}, \\
&\quad t \geqslant 0,\ j,k \geqslant 0,\ 2j+k \leqslant 2, \\
&\big\|\partial_t^j \partial_x^k (\bar{v} - v_+)(t)\big\|_{L^p} \leqslant C(1+t)^{-(1-1/p)/2-(k+2j+\gamma)/2}, \\
&\quad t \geqslant 0,\ 1 \leqslant p \leqslant \infty,\ j,k \geqslant 0,\ 2j+k \leqslant 2.
\end{aligned}
\tag{2.16}
$$

Now we are going to state the convergence to the best asymptotic profile, the so-called stability of the nonlinear diffusion waves. Our first main result is as follows.

Theorem 2.2 *(Convergence). Let $v_+ = v_-$ and $l \geqslant 3$, $\delta_{0v} := |\int_0^\infty (v_0(x) - v_+)\,dx|$ and δ_0 be defined as before. Suppose that $v_0 - v_+ \in H^l(\mathbb{R}^+) \cap W^{l-1,1}(\mathbb{R}^+)$ and $U_0(x) \in H^{l-1}(\mathbb{R}^+)$. If $\lambda_l = \|U_0\|_{l-1}^2 + \delta_0 + \delta_{0v}$ is suitably small, then there exists a unique time-global solution $(V, U)(x,t)$ of* (2.11)

$$
\begin{aligned}
&V(x,t) \in C^k\big([0,\infty); H^{l-k}\big), \quad k = 0,1,\ldots,l, \\
&U(x,t) \in C^k\big([0,\infty); H^{l-1-k}\big), \quad k = 0,1,\ldots,l-1,
\end{aligned}
$$

satisfying

$$(1+t)^{k/2+j}\left\|\partial_x^k\partial_t^j V(t)\right\| \leqslant C\lambda_l, \tag{2.17}$$

for $j=0,1,\ldots,l-2$ *and* $k=0,1,\ldots,l-j$,

$$(1+t)^{(k+1)/2+l-2}\left\|\partial_x^k\partial_t^{l-1} V(t)\right\| \leqslant C\lambda_l, \tag{2.18}$$

for $k=0,1$, *and*

$$(1+t)^{l-1}\left\|\partial_t^l V(t)\right\| \leqslant C\lambda_l. \tag{2.19}$$

Furthermore, let $l=4$, *if* $U_0(x)\in L^1(\mathbb{R}^+)$ *and* $v_0-v_+\in L^{1,\gamma}(\mathbb{R}^+)$ $(0<\gamma\leqslant 1)$, *then the convergence rates can be further improved as*

$$\begin{aligned}
&\left\|\partial_x^k V(t)\right\|_{L^p} \leqslant C\big(\lambda_4+\|U_0\|_{L^1}\big)(1+t)^{-(\frac{1}{2}-\frac{1}{2p})-\frac{k}{2}}, \quad k=0,1,2,\\
&\left\|\partial_x^k U(t)\right\|_{L^p} \leqslant C\big(\lambda_4+\|U_0\|_{L^1}\big)(1+t)^{-(\frac{3}{2}-\frac{1}{2p})-\frac{k}{2}}, \quad k=0,1,
\end{aligned} \tag{2.20}$$

for $t\geqslant 0$, $2\leqslant p\leqslant\infty$.

Based on Theorem 2.2, we have the following decay properties of the solution $(V,U)(x,t)$ to the system (2.11).

Theorem 2.3. *Let* $v_+=v_-$ *and* $a\in[0,\frac{1}{2})$. *Suppose the conditions in Theorem* 2.2 *hold, and in addition,* $\sum_{k=0}^2\|\partial_x^k U_0\|_{2,a}^2<\infty$. *Then the unique time-global solution* $(V,U)(x,t)$ *of* (2.11) *satisfies*

$$\begin{aligned}
&\sum_{k=0}^{2}(1+t)^k\left\|\partial_x^k V(t)\right\|_{2,a}^2+\sum_{k=0}^{1}(1+t)^{k+1}\left\|\partial_x^k U(t)\right\|_{2,a}^2\\
&\quad+\int_0^t\left\{\sum_{k=1}^{2}(1+s)^{k-1}\left\|\partial_x^k V(s)\right\|_{2,a}^2+\sum_{k=0}^{1}(1+s)^{k+1}\left\|\partial_x^k U(s)\right\|_{2,a}^2\right\}ds\\
&\leqslant C.
\end{aligned} \tag{2.21}$$

Finally, we obtain much better decay rates as follows.

Theorem 2.4 *(Improved convergence). Let* $a\in(0,\frac{1}{4}]$. *Suppose the conditions in Theorems* 2.2 *and* 2.3 *hold. In addition, we assume that* $v_0-v_+\in L^{1,1}(\mathbb{R}^+)$ *and* $U_0\in L^{1,a}(\mathbb{R}^+)$. *Then the decay rates of the solution V to* (2.11) *can be further improved to be optimal as follows*

$$\left\|\partial_x^k V(t)\right\| \leqslant C(1+t)^{-\frac{2k+1}{4}-\frac{a}{2}}, \quad k=0,1,2. \tag{2.22}$$

From Theorems 2.2 and 2.4, noticing that $\|\partial_x^k(\hat v,\hat u)\|_{L^\infty}\leqslant Ce^{-\alpha t}$, we can easily obtain the following decay properties for the solution $(v,u)(x,t)$ of (1.2) to the solution $(\bar v,\bar u)(x,t)$ of (2.1).

Corollary 2.1. *Under the conditions in Theorem* 2.4*, the system* (1.2) *possesses a unique time-global solution* $(v,u)(x,t)$*, which converges to its best asymptotic profile* $(\bar v,\bar u)(x,t)$ *defined in* (2.1) *and* (2.8) *in the form of*

$$\begin{aligned}
\big\|(v-\bar v-\hat v)(t)\big\| &\leqslant C(1+t)^{-\frac{5}{4}-\frac{a}{2}},\\
\big\|(v-\bar v)(t)\big\|_{L^\infty} &\leqslant C(1+t)^{-\frac{3}{2}-\frac{a}{4}},\\
\big\|(u-\bar u-\hat u)(t)\big\| &\leqslant C(1+t)^{-\frac{7}{4}},\\
\big\|(u-\bar u)(t)\big\|_{L^\infty} &\leqslant C(1+t)^{-2}.
\end{aligned}\tag{2.23}$$

Remark 2.1. It is easy to see that our new convergence rates obtained are much better than the existing rates showed in the previous works for the special case $v_+=v_-$.

Finally, we give a remark on the case $v_-\neq v_+$.

Remark 2.2. For the case $v_-\neq v_+$, $\bar v(x,t)$ decays as

$$\big\|\partial_t^j\partial_x^k(\bar v-v_+)(t)\big\|_{L^p}=O(1)(1+t)^{-(1-1/p)/2-(2j+k-1)/2},$$

even if $\bar v_0-v_+\in L^{1,1}(\mathbb{R}^+)$. As a result, we can only obtain the following decay properties for the solution $(v,u)(x,t)$ of (1.2),

$$\begin{aligned}
\big\|(v-\bar v-\hat v)\big\| &= O(1)(1+t)^{-\frac{3}{4}},\\
\big\|(v-\bar v)\big\|_{L^\infty} &= O(1)(1+t)^{-1},\\
\big\|(u-\bar u-\hat u)\big\| &= O(1)(1+t)^{-\frac{5}{4}},\\
\big\|(u-\bar u)\big\|_{L^\infty} &= O(1)(1+t)^{-\frac{3}{2}}.
\end{aligned}$$

These rates are exactly same to those obtained by Marcati, Mei and Rubino [17] for the IBVPs.

2.2. Property of the best asymptotic profile

This section is devoted to the proof of Theorem 2.1, that is, we are going to prove the unique existence of the particular solution $(\bar v,\bar u)(x,t)$ to (2.1) and (2.8), as well as its optimal decay rates.

From the system (2.1), $\bar v$ satisfies the following IBVP

$$\begin{cases}\alpha\bar v_t+p(\bar v)_{xx}=0, & (x,t)\in\mathbb{R}^+\times\mathbb{R}^+,\\ \bar v(x,0)=\bar v_0(x)\to v_+ & \text{as } x\to+\infty,\\ \bar v(0,t)=v_+.\end{cases}\tag{2.24}$$

Let

$$h(x,t)=\bar v(x,t)-v_+,\qquad \beta=\frac{-p'(v_+)}{\alpha},\qquad H(h)=-\frac{p(h+v_+)-p(v_+)-p'(v_+)h}{\alpha}.$$

Then $h(x,t)$ satisfies

$$\begin{cases}h_t-\beta h_{xx}=H(h)_{xx}, & (x,t)\in\mathbb{R}^+\times\mathbb{R}^+,\\ h(x,0)=\bar v_0(x)-v_+:=h_0(x)\to 0 & \text{as } x\to+\infty,\\ h(0,t)=0.\end{cases}\tag{2.25}$$

Denote

$$K(x,y;t)=G(x-y,t)-G(x+y,t),$$

where $G(x,t)=\frac{1}{\sqrt{4\beta\pi t}}e^{-x^2/4\beta t}$ is the heat kernel. Then (2.25) can be expressed in the following integral form

$$h(x,t)=\int_0^\infty K(x,y;t)h_0(y)\,dy+\int_0^t\int_0^\infty K(x,y;t-s)H\big(h(y,s)\big)_{yy}\,dy\,ds. \tag{2.26}$$

Let

$$\phi(x,t)=\int_0^\infty K(x,y;t)h_0(y)\,dy.$$

We define the iteration $\{h^n(x,t)\}$ by

$$h^0(x,t)=\phi(x,t)+\int_0^t\int_0^\infty K(x,y;t-s)H(\phi)_{yy}\,dy\,ds,$$

and

$$h^{n+1}(x,t)=\phi(x,t)+\int_0^t\int_0^\infty K(x,y;t-s)H\big(h^n\big)_{yy}\,dy\,ds.$$

Similarly, as in [17,24], we can prove that $\{h^n(x,t)\}$ is a Cauchy sequence and converges to a limit, say h, which is the unique global solution of the IBVP (2.25). Furthermore, by using the Green function method and the standard energy estimates with the a priori estimates, when the initial perturbation is small enough, we can have the following decay rates

$$\big\|\partial_t^j\partial_x^k h\big\|_{L^p}\leqslant C\delta_{0\bar v}(1+t)^{-(1-1/p)/2-(2j+k)/2}, \tag{2.27}$$

for $k,j\geqslant 0$, $0\leqslant k+2j\leqslant l-3$, $1\leqslant p\leqslant\infty$. Here, the details are omitted. Hence we get $(2.15)_1$. Note that $\bar u=-\frac{1}{\alpha}p(\bar v)_x$, then (2.27) gives $(2.15)_2$.

In order to prove (2.16), we first prove the following two lemmas.

Lemma 2.1. *Let $\gamma\in[0,1]$. If $f\in L^{1,\gamma}(\mathbb{R})\cap H^l(\mathbb{R})$ $(l\geqslant 1)$ and f is an odd function. Then it holds, for $2\leqslant p\leqslant\infty$,*

$$\left\|\partial_x^k\int_{-\infty}^\infty G(x-y,t)f(y)\,dy\right\|_{L^p}\leqslant C(1+t)^{-(1-1/p)/2-\frac{k+\gamma}{2}}\big(\|f\|_{1,\gamma}+\|f\|_{k+1}\big),$$
$$t\geqslant 0,\ k=0,1,2,\dots,l-1, \tag{2.28}$$

where $G(x,t)$ is the heat kernel defined as before. In addition, if $f\in W^{l-1,1}$, then it holds, for $1\leqslant p\leqslant\infty$,

$$\left\| \partial_x^k \int_{-\infty}^{\infty} G(x-y,t) f(y)\,dy \right\|_{L^p} \leqslant C(1+t)^{-(1-1/p)/2-\frac{k+\gamma}{2}} \left(\|f\|_{1,\gamma} + \|f\|_{H^l \cap W^{l-1,1}} \right),$$
$$t \geqslant 0,\ k = 0,1,2,\ldots,l-1. \tag{2.29}$$

Proof. For the inequality (2.28), we only show the case $p=\infty$. The case $p=2$ can be proved in the same way, and the other cases can be obtained by using the interpolation inequality based on the L^∞ and L^2-estimates.

Let $u = \int_{-\infty}^{\infty} G(x-y,t) f(y)\,dy$. Then u is the solution of the following Cauchy problem:

$$\begin{cases} u_t - \beta u_{xx} = 0, \\ u(x,0) = f(x), \end{cases} \quad (x,t) \in \mathbb{R} \times \mathbb{R}^+, \tag{2.30}$$

where $\beta = \frac{-p'(v_+)}{\alpha}$. Using Fourier transform, we have

$$\begin{cases} \hat{u}_t + \beta \xi^2 u = 0, \\ \hat{u}(\xi,0) = \hat{f}(\xi), \end{cases} \quad (\xi,t) \in \mathbb{R} \times \mathbb{R}^+, \tag{2.31}$$

which solves $\hat{u} = e^{-\beta\xi^2 t} \hat{f}$.

It is well known that

$$\|f\|_{L^p} \leqslant \|\hat{f}\|_{L^q}, \qquad \frac{1}{p} + \frac{1}{q} = 1, \qquad 1 \leqslant q \leqslant 2.$$

Then we have

$$\left\| \partial_x^k u \right\|_{L^\infty} \leqslant \left\| \widehat{\partial_x^k u} \right\|_{L^1}.$$

Since $\widehat{\partial_x^k u} = (i\xi)^k \hat{u}$, we can get

$$\left\| \widehat{\partial_x^k u} \right\|_{L^1} = \left\| (i\xi)^k \hat{u} \right\|_{L^1} \leqslant C \left\| |\xi|^k e^{-\beta\xi^2 t} \hat{f} \right\|_{L^1} = C \int_{\mathbb{R}} |\xi|^k e^{-\beta\xi^2 t} |\hat{f}|\,d\xi.$$

Notice that f is odd, it holds that

$$\begin{aligned} |\hat{f}| &= \frac{1}{\sqrt{2\pi}} \left| \int_{\mathbb{R}} e^{-ix\xi} f(x)\,dx \right| \\ &= \frac{1}{\sqrt{2\pi}} \left| \int_{\mathbb{R}} f(x) \sin(x\xi)\,dx \right| \\ &\leqslant C \int_{\mathbb{R}} |x\xi|^\gamma \left| f(x) \right| \left| \frac{\sin(x\xi)}{(x\xi)^\gamma} \right| dx. \end{aligned}$$

Since

$$\left| \frac{\sin(x\xi)}{(x\xi)^\gamma} \right| \leqslant C \quad \text{for } 0 \leqslant \gamma \leqslant 1,$$

we have $|\hat{f}| \leqslant C|\xi|^{\gamma}\|f\|_{1,\gamma}$. Thus, for $t \geqslant 2$, we have

$$\left\||\xi|^k e^{-\beta\xi^2 t}\hat{f}\right\|_{L^1} \leqslant C\left(\int_{\mathbb{R}} |\xi|^{(k+\gamma)} e^{-\beta\xi^2 t}\,d\xi\right)\|f\|_{1,\gamma} \leqslant Ct^{-\frac{(k+\gamma)+1}{2}}\|f\|_{1,\gamma} \tag{2.32}$$

and for $0 \leqslant t < 2$, we have

$$\left\||\xi|^k e^{-\beta\xi^2 t}\hat{f}\right\|_{L^1} \leqslant \left\||\xi|^{-1}\right\|\left\||\xi|^{k+1}\hat{f}\right\| \leqslant C\|f\|_{k+1} \leqslant C(1+t)^{-\frac{(k+\gamma)+1}{2}}\|f\|_{k+1}. \tag{2.33}$$

Thus, we obtain

$$\left\|\widehat{\partial_x^k u}\right\|_{L^1} \leqslant C(1+t)^{-\frac{(k+\gamma)+1}{2}}\left(\|f\|_{1,\gamma} + \|f\|_{k+1}\right), \quad t \geqslant 0, \tag{2.34}$$

hence,

$$\left\|\partial_x^k u\right\|_{L^\infty} \leqslant C(1+t)^{-\frac{(k+\gamma)+1}{2}}\left(\|f\|_{1,\gamma} + \|f\|_{k+1}\right), \quad t \geqslant 0. \tag{2.35}$$

Based on (2.34) and (2.35), the interpolation inequality gives (2.28).

For the inequality (2.29), based on (2.28), it suffices to show that

$$\left\|\partial_x^k \int_{-\infty}^{\infty} G(x-y,t)f(y)\,dy\right\|_{L^1} \leqslant C(1+t)^{-\frac{k+\gamma}{2}}\left(\|f\|_{1,\gamma} + \|f\|_{W^{k,1}}\right), \tag{2.36}$$

for $t \geqslant 0$, $k = 0, 1, 2, \dots, l-1$.

We only show (2.36) for the case $k = 1$, the other cases can be proved similarly. If $t \geqslant 2$, since f is odd,

$$\begin{aligned}
\left\|\partial_x \int_{-\infty}^{\infty} G(x-y,t)f(y)\,dy\right\|_{L^1} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left|\partial_x G(x-y,t)f(y)\right|dy\,dx \\
&= 2\int_0^{\infty}\int_0^{\infty} \left|\partial_x K(x,y;t)\right|\left|f(y)\right|dy\,dx \\
&= 2(\sqrt{4\beta t})^{-\gamma}\int_0^{\infty} y^{\gamma}\left|f(y)\right|dy \int_0^{\infty} \frac{|\partial_x K(x,y;t)|}{(\frac{y}{\sqrt{4\beta t}})^{\gamma}}\,dx.
\end{aligned} \tag{2.37}$$

In order to show that

$$\left\|\int_{-\infty}^{\infty} \partial_x G(x-y,t)f(y)\,dy\right\|_{L^1} \leqslant Ct^{-(1+\gamma)/2}\|f\|_{1,\gamma}, \tag{2.38}$$

it is enough to show that $\int_0^\infty \frac{|\partial_x K(x,y;t)|}{(\frac{y}{\sqrt{4\beta t}})^{\gamma}}\,dx \leqslant Ct^{-1/2}$. Since $\int_0^\infty |\partial_x K(x,y;t)|\,dx = O(1)t^{-1/2}$, it is easy to see that if $\frac{y}{\sqrt{4\beta t}} \gg 1$, then $\int_0^\infty \frac{|\partial_x K(x,y;t)|}{(\frac{|y|}{\sqrt{4\beta t}})^{\gamma}}\,dx \leqslant Ct^{-1/2}$. Thus we only need to show that $\int_0^\infty \frac{|\partial_x K(x,y;t)|}{(\frac{y}{\sqrt{4\beta t}})^{\gamma}}\,dx \leqslant Ct^{-1/2}$ for the case $\frac{y}{\sqrt{4\beta t}} \to 0$. Note that,

$$\int_0^\infty \frac{|\partial_x K(x,y;t)|}{(\frac{y}{\sqrt{4\beta t}})^\gamma}\,dx = \frac{1}{\sqrt{4\beta\pi t}}\int_0^\infty \frac{|-\frac{(x-y)}{2\beta t}e^{-\frac{(x-y)^2}{4\beta t}}+\frac{(x+y)}{2\beta t}e^{-\frac{(x+y)^2}{4\beta t}}|}{(\frac{y}{\sqrt{4\beta t}})^\gamma}\,dx$$
$$= \frac{1}{\sqrt{4\beta\pi t}}\int_0^\infty e^{-\frac{x^2+y^2}{4\beta t}}\frac{|\frac{x-y}{2\beta t}e^{\frac{xy}{2\beta t}}-\frac{x+y}{2\beta t}e^{-\frac{xy}{2\beta t}}|}{(\frac{y}{\sqrt{4\beta t}})^\gamma}\,dx$$
$$= \frac{1}{\sqrt{\beta t}}\frac{1}{\sqrt{4\beta\pi t}}\int_0^\infty e^{-\frac{x^2+y^2}{4\beta t}}\frac{|\frac{x-y}{\sqrt{4\beta t}}e^{\frac{x}{\sqrt{\beta t}}\cdot\frac{y}{\sqrt{4\beta t}}}-\frac{x+y}{\sqrt{4\beta t}}e^{-\frac{x}{\sqrt{\beta t}}\cdot\frac{y}{\sqrt{4\beta t}}}|}{(\frac{y}{\sqrt{4\beta t}})^\gamma}\,dx. \tag{2.39}$$

Let $z=\frac{y}{\sqrt{4\beta t}}$, since $0<\gamma\leqslant 1$, we have

$$\lim_{z\to 0}\frac{|(\frac{x}{\sqrt{4\beta t}}-z)e^{\frac{x}{\sqrt{\beta t}}\cdot z}-(\frac{x}{\sqrt{4\beta t}}+z)e^{-\frac{x}{\sqrt{\beta t}}\cdot z}|}{z^\gamma}=\begin{cases}0, & \text{if } 0<\gamma<1,\\ \frac{x^2}{\beta t}, & \text{if } \gamma=1.\end{cases} \tag{2.40}$$

Note also that $|\frac{1}{\sqrt{4\beta\pi t}}\int_0^\infty e^{-\frac{x^2+y^2}{4\beta t}}(1+\frac{x^2}{\beta t})\,dx|\leqslant C$, from (2.39) and (2.40), we have proved (2.38) for $t\geqslant 2$. For the case $0<t<2$, it obviously holds that

$$\left\|\int_{-\infty}^\infty \partial_x G(x-y,t)f(y)\,dy\right\|_{L^1}$$
$$=\left\|\int_{-\infty}^\infty G(x-y,t)\partial_y f(y)\,dy\right\|_{L^1} \tag{2.41}$$
$$\leqslant \|\partial_x f\|_{L^1}\leqslant C(1+t)^{-(\gamma+1)/2}\|\partial_x f\|_{L^1}. \tag{2.42}$$

From the above analysis, we have

$$\left\|\partial_x\int_{-\infty}^\infty G(x-y,t)f(y)\,dy\right\|_{L^1}\leqslant C(1+t)^{-(\gamma+1)/2}\big\{\|f\|_{1,\gamma}+\|\partial_x f\|_{L^1}\big\},\quad t\geqslant 0. \tag{2.43}$$

This completes the proof of Lemma 2.1. □

Based on Lemma 2.1, we have the following result.

Lemma 2.2. *Let $\gamma\in(0,1]$. If $f\in L^{1,\gamma}\cap H^l\cap W^{l-1,1}(\mathbb{R}^+)$ $(1\leqslant l\leqslant 3)$, then it holds that*

$$\left\|\partial_x^k\int_0^\infty K(x,y;t)f(y)\,dy\right\|_{L^p(\mathbb{R}^+)}\leqslant C(1+t)^{-(1-1/p)/2-\frac{k+\gamma}{2}}\big(\|f\|_{1,\gamma}+\|f\|_{H^l\cap W^{l-1,1}}\big), \tag{2.44}$$

for $t\geqslant 0$, $k=0,1,\dots,l-1$, $1\leqslant p\leqslant\infty$.

Proof. Let $\tilde{f}$ be the odd extension of f. Then

$$\int_0^\infty K(x,y;t)f(y)\,dy = \int_{-\infty}^\infty G(x-y,t)\tilde{f}(y)\,dy.$$

Hence

$$\begin{aligned}\left\|\partial_x^k \int_0^\infty K(x,y;t)f(y)\,dy\right\|_{L^p(\mathbb{R}^+)} &= \left\|\partial_x^k \int_{-\infty}^\infty G(x-y,t)\tilde{f}(y)\,dy\right\|_{L^p(\mathbb{R}^+)}\\ &\leqslant \left\|\partial_x^k \int_{-\infty}^\infty G(x-y,t)\tilde{f}(y)\,dy\right\|_{L^p(\mathbb{R})}\\ &\leqslant C(1+t)^{-(1-1/p)/2-\frac{k+\gamma}{2}}\big(\|\tilde{f}\|_{1,\gamma}+\|\tilde{f}\|_{H^l\cap W^{l-1,1}}\big). \qquad (2.45)\end{aligned}$$

Notice that

$$\|\tilde{f}\|_{L^{1,\gamma}} = \|f\|_{L^{1,\gamma}}, \qquad \|\tilde{f}\|_{H^l(\mathbb{R})\cap W^{l-1,1}(\mathbb{R})} \leqslant C\|f\|_{H^l(\mathbb{R}^+)\cap W^{l-1,1}(\mathbb{R}^+)}.$$

We can easily get (2.44). Hence we finish the proof of Lemma 2.2. □

Based on (2.27) and Lemma 2.2, we have the following property on $\bar{v}(x,t)$.

Lemma 2.3. *If $\bar{v}_0 - v_+ \in L^{1,\gamma}\cap H^l\cap W^{l-1,1}(\mathbb{R}^+)$ $(l \geqslant 3)$, then for any $1 \leqslant p \leqslant \infty$, the solution $\bar{v}$ to the system* (2.24) *satisfies*

$$\left\|\partial_t^j\partial_x^k(\bar{v}-v_+)\right\|_{L^p} \leqslant C(1+t)^{-(1-1/p)/2-(2j+k+\gamma)/2}, \quad k+2j\leqslant 2. \qquad (2.46)$$

Proof. From (2.25), we have

$$\bar{v}-v_+ = \int_0^\infty K(x,y;t)\big(\bar{v}_0(y)-v_+\big)\,dy + \int_0^t\int_0^\infty K(x,y;t-s)H(\bar{v}-v_+)_{yy}\,dy\,ds. \qquad (2.47)$$

Since $\bar{v}_0 - v_+ \in L^{1,\gamma}\cap H^l\cap W^{l-1,1}(\mathbb{R}^+)$ $(l\geqslant 3)$ and $(\bar{v}_0-v_+)|_{x=0}=0$, using Lemma 2.2, we have

$$\left\|\partial_x^k\int_0^\infty K(x,y;t)\big(\bar{v}_0(y)-v_+\big)\,dy\right\|_{L^p} \leqslant C(1+t)^{-(1-1/p)/2-(k+\gamma)/2}, \qquad (2.48)$$

for $t\geqslant 0$, $k=0,1,\dots,l-1$, $1\leqslant p\leqslant\infty$.

From the definition of H, by Taylor expansion, we have $H(\bar{v}-v_+)\sim(\bar{v}-v_+)^2$. Now we are going to show that

$$\left\|\partial_x^k\int_0^t\int_0^\infty K(x,y;t-s)H(\bar{v}-v_+)_{yy}\,dy\,ds\right\|_{L^p} \leqslant C(1+t)^{-(1-1/p)/2-(k+1)/2}, \qquad (2.49)$$

for $t\geqslant 0$, $k=0,1,\dots,l-1$, $1\leqslant p\leqslant\infty$.

From (2.27), for $t \geqslant 2$, we have

$$\begin{aligned}
&\left\| \partial_x^k \int_0^t \int_0^\infty K(x,y;t-s) H(\bar v - v_+)_{yy}\,dy\,ds \right\|_{L^\infty} \\
&\leqslant C \int_0^{t/2} \left\| \partial_x^{k+2} K(x,\cdot;t-s) \right\|_{L^\infty} \left\| (\bar v - v_+)^2(s) \right\|_{L^1} ds \\
&\quad + C \int_{t/2}^t \left\| \partial_x K(x,\cdot;t-s) \right\|_{L^1} \left\| \partial_y^{k+1} (\bar v - v_+)^2(s) \right\|_{L^\infty} ds \\
&\leqslant C t^{-\frac{k+3}{2}} \int_0^{t/2} (1+s)^{-1/2}\,ds + C(1+t)^{-\frac{k+3}{2}} \int_{t/2}^t (t-s)^{-1/2}\,ds \\
&\leqslant C(1+t)^{-\frac{k+2}{2}},
\end{aligned} \tag{2.50}$$

and

$$\begin{aligned}
&\left\| \partial_x^k \int_0^t \int_0^\infty K(x,y;t-s) H(\bar v - v_+)_{yy}\,dy\,ds \right\|_{L^1} \\
&\leqslant C \int_0^{t/2} \left\| \partial_x^{k+2} K(x,\cdot;t-s) \right\|_{L^1} \left\| (\bar v - v_+)^2(s) \right\|_{L^1} ds \\
&\quad + C \int_{t/2}^t \left\| \partial_x K(x,\cdot;t-s) \right\|_{L^1} \left\| \partial_y^{k+1} \big((\bar v - v_+)^2 \big)(s) \right\|_{L^1} ds \\
&\leqslant C t^{-\frac{k+2}{2}} \int_0^{t/2} (1+s)^{-1/2}\,ds + C(1+t)^{-\frac{k+2}{2}} \int_{t/2}^t (t-s)^{-1/2}\,ds \\
&\leqslant C(1+t)^{-\frac{k+1}{2}}.
\end{aligned} \tag{2.51}$$

Combining these with $\|\partial_x^k \int_0^t \int_0^\infty K(x,y;t-s)H(\bar v - v_+)_{yy}\,dy\,ds\|_{L^1\cap L^\infty} \leqslant C$ for $t \leqslant 2$, the desired estimates (2.49) are obtained. Thus, we have proved (2.46) for the case $j=0$. When $j=1$, note that $\partial_t \bar v = -\frac{1}{\alpha} p(\bar v)_{xx}$, we have

$$\left\| \partial_t (\bar v - v_+) \right\|_{L^p} = O(1) \left\| \partial_x^2 p(\bar v) \right\|_{L^p} \leqslant C(1+t)^{-(1-1/p)/2-3/2}. \tag{2.52}$$

This completes the proof of Lemma 2.3. □

Lemma 2.4. *If $\bar v_0 - v_+ \in L^{1,1} \cap H^l \cap W^{l-1,1}(\mathbb{R}^+)$ ($l \geqslant 3$), then $\forall a \in (0,1)$, the solution $\bar v$ to Eq. (2.24) satisfies*

$$\left\| \partial_t^j \partial_x^k (\bar v - v_+) \right\|_{1,a} \leqslant C(1+t)^{-1/2-(2j+k-a)/2}, \quad k + 2j \leqslant 2. \tag{2.53}$$

Proof. First, we prove that (2.53) holds for the case $j=0$, $k=0$. It is verified

$$\begin{aligned}\|\bar v - v_+\|_{1,a} &= \int_0^\infty (1+x)^a \left| \int_0^\infty K(x,y;t)\big(\bar v_0(y)-v_+\big)\,dy \right| dx \\ &\quad + \int_0^t \int_0^\infty (1+x)^a \left| \int_0^\infty K(x,y;t-s) H\big(\bar v(y,s)-v_+\big)_{yy}\,dy \right| dx\,ds \\ &=: L_1(t)+L_2(t). \end{aligned} \tag{2.54}$$

Now, we estimates $L_1(t)$ for $t \geqslant 2$. In this case, we write $L_1(t)$ in the form of

$$L_1(t) = (\sqrt{4\beta t}\,)^{-1} \int_0^\infty y\big|\bar v_0(y)-v_+\big|\,dy \int_0^\infty (1+x)^a \frac{|K(x,y;t)|}{\frac{y}{\sqrt{4\beta t}}}\,dx. \tag{2.55}$$

If $\frac{y}{\sqrt{4\beta t}} \to 0$, similarly as above, we have

$$\int_0^\infty (1+x)^a \frac{|K(x,y;t)|}{\frac{y}{\sqrt{4\beta t}}}\,dx \leqslant C \int_0^\infty (1+x)^a e^{-\frac{x^2+y^2}{4\beta t}} \left(1+\frac{2x}{\sqrt{\beta t}}\right) dx \leqslant C(1+t)^{a/2}. \tag{2.56}$$

For the case that $\frac{y}{\sqrt{4\beta t}} \gg 1$, we have

$$\begin{aligned} &\int_0^\infty (1+x)^a \frac{|K(x,y;t)|}{\frac{y}{\sqrt{4\beta t}}}\,dx \\ &= \frac{1}{\sqrt{4\beta\pi t}} \left\{ \int_0^\infty (1+x)^a \frac{e^{-\frac{(x-y)^2}{4\beta t}}}{\frac{y}{\sqrt{4\beta t}}}\,dx - \int_0^\infty (1+x)^a \frac{e^{-\frac{(x+y)^2}{4\beta t}}}{\frac{y}{\sqrt{4\beta t}}}\,dx \right\} \\ &\left[\text{by change of variables: } z=\frac{x-y}{4\beta t} \text{ for the first integral,}\right. \\ &\left.\text{and } z=\frac{x+y}{4\beta t} \text{ for the second integral}\right] \\ &= \frac{1}{\sqrt{\pi}} \left\{ \int_{-\frac{y}{\sqrt{4\beta t}}}^\infty (1+\sqrt{4\beta t}z+y)^a \frac{e^{-z^2}}{\frac{y}{\sqrt{4\beta t}}}\,dz - \int_{\frac{y}{\sqrt{4\beta t}}}^\infty (1+\sqrt{4\beta t}z-y)^a \frac{e^{-z^2}}{\frac{y}{\sqrt{4\beta t}}}\,dz \right\} \\ &= \frac{1}{\sqrt{\pi}} \left\{ \int_{-\frac{y}{\sqrt{4\beta t}}}^{\frac{y}{\sqrt{4\beta t}}} \frac{(1+\sqrt{4\beta t}z+y)^a}{\frac{y}{\sqrt{4\beta t}}} e^{-z^2}\,dz \right. \\ &\quad \left. + \int_{\frac{y}{\sqrt{4\beta t}}}^\infty \frac{(1+\sqrt{4\beta t}z+y)^a-(1+\sqrt{4\beta t}z-y)^a}{\frac{y}{\sqrt{4\beta t}}} e^{-z^2}\,dz \right\} \end{aligned}$$

$$\left[\text{let } w = \frac{y}{\sqrt{4\beta t}}\right]$$

$$= \frac{1}{\sqrt{\pi}}\left\{\int_{-w}^{w} \frac{(1+\sqrt{4\beta t}w+\sqrt{4\beta t}z)^a}{w} e^{-z^2}\,dz\right.$$

$$\left.+\int_{w}^{\infty} \frac{(1+\sqrt{4\beta t}z+\sqrt{4\beta t}w)^a-(1+\sqrt{4\beta t}z-\sqrt{4\beta t}w)^a}{w} e^{-z^2}\,dz\right\}$$

$$\leqslant \frac{1}{\sqrt{\pi}}\left\{\int_{-w}^{w} \frac{(1+2\sqrt{4\beta t}w)^a}{w} e^{-z^2}\,dz \quad \bigl[\text{because } |z| \leqslant w\bigr]\right.$$

$$\left.+\int_{w}^{\infty} \frac{(1+\sqrt{4\beta t}z+\sqrt{4\beta t}w)^a-(1+\sqrt{4\beta t}z-\sqrt{4\beta t}w)^a}{w} e^{-z^2}\,dz\right\}.$$

Since $w \gg 1$, we have

$$\frac{(1+2\sqrt{4\beta t}w)^a}{w} \leqslant \frac{(1+2\sqrt{4\beta t}\,)^a(1+w)^a}{w} \leqslant C(1+t)^{a/2} \quad \text{for } 0 < a < 1.$$

And for $0 < a < 1$, $\forall z \geqslant w$, we can easily check that

$$(1+\sqrt{4\beta t}z+\sqrt{4\beta t}w)^a-(1+\sqrt{4\beta t}z-\sqrt{4\beta t}w)^a \leqslant (1+\sqrt{4\beta t}w)^a \leqslant C(1+t)^{a/2}(1+w)^a,$$

which implies that

$$\frac{(1+\sqrt{4\beta t}z+\sqrt{4\beta t}w)^a-(1+\sqrt{4\beta t}z-\sqrt{4\beta t}w)^a}{w} \leqslant C(1+t)^{a/2}.$$

Thus, we have

$$\int_0^\infty (1+x)^a \frac{|K(x,y;t)|}{\frac{y}{\sqrt{4\beta t}}}\,dx \leqslant C(1+t)^{a/2} \tag{2.57}$$

for the case $\frac{y}{\sqrt{4\beta t}} \gg 1$. Combining (2.56) and (2.57) gives

$$\int_0^\infty (1+x)^a \frac{|K(x,t;y,0)|}{\frac{y}{\sqrt{4\beta t}}}\,dx \leqslant C(1+t)^{a/2}, \quad \forall (y,t) \in \mathbb{R}^+ \times \mathbb{R}^+. \tag{2.58}$$

Thus, (2.58) and (2.55) imply that

$$L_1(t) \leqslant C(1+t)^{-(1-a)/2} \|\bar{v}_0 - v_+\|_{1,1}, \quad t \geqslant 2. \tag{2.59}$$

On the other hand, for $t < 2$, $L_1(t)$ is written as

$$
\begin{aligned}
L_1(t) &= \int_0^\infty \left\{ \int_0^\infty (1+x)^a K(x,y;t)\,dx \right\} \left| \bar{v}_0(y) - v_+ \right| dy \\
&\leqslant C \int_0^\infty (1+y)^a \left| \bar{v}_0(y) - v_+ \right| dy \\
&\leqslant C(1+t)^{-(1-a)/2} \|\bar{v}_0 - v_+\|_{1,a} \\
&\leqslant C(1+t)^{-(1-a)/2} \|\bar{v}_0 - v_+\|_{1,1} \quad \text{for } 0 < a < 1.
\end{aligned} \tag{2.60}
$$

From (2.59) and (2.60), we can get

$$
L_1(t) \leqslant C(1+t)^{-(1-a)/2} \|\bar{v}_0 - v_+\|_{1,1}, \quad t \geqslant 0. \tag{2.61}
$$

Now we estimate $L_2(t)$. When $t \geqslant 2$, we write $L_2(t)$ in the form of

$$
\begin{aligned}
L_2(t) &= \int_0^{t/2} \left(\sqrt{4\beta(t-s)}\right)^{-a} \left\{ \int_0^\infty y^a \left| H\left(\bar{v}(y,s) - v_+\right) \right| dy \right. \\
&\quad \left. \times \int_0^\infty (1+x)^a \frac{|\partial_x^2 K(x,y;t-s)|}{(\frac{y}{\sqrt{4\beta(t-s)}})^a}\,dx \right\} ds \\
&\quad + \int_{t/2}^{t} \left(\sqrt{4\beta(t-s)}\right)^{-a} \left\{ \int_0^\infty y^a \left| \partial_y H\left(\bar{v}(y,s) - v_+\right) \right| dy \right. \\
&\quad \left. \times \int_0^\infty (1+x)^a \frac{|\partial_x K(x,y;t-s)|}{(\frac{y}{\sqrt{4\beta(t-s)}})^a}\,dx \right\} ds.
\end{aligned} \tag{2.62}
$$

Similarly as above, we can get

$$
\int_0^\infty (1+x)^a \frac{|\partial_x^k K(x,y;t-s)|}{(\frac{y}{\sqrt{4\beta(t-s)}})^a}\,dx \leqslant C(t-s)^{-k/2}(1+t-s)^{a/2}, \quad k = 1, 2. \tag{2.63}
$$

Using $(2.15)_1$, it holds that

$$
\begin{aligned}
\int_0^\infty y^a \left| H\left(\bar{v}(y,s) - v_+\right) \right| dy &\leqslant C \left\| \bar{v}(s) - v_+ \right\|_{L^\infty} \left\| \bar{v}(s) - v_+ \right\|_{1,a} \\
&\leqslant C\delta_{0\bar{v}}(1+s)^{-1/2} \left\| \bar{v}(s) - v_+ \right\|_{1,a}, \\
\int_0^\infty y^a \left| \partial_y H\left(\bar{v}(y,s) - v_+\right) \right| dy &\leqslant C \left\| \partial_x \bar{v}(s) \right\|_{L^\infty} \left\| \bar{v}(s) - v_+ \right\|_{1,a} \\
&\leqslant C\delta_{0\bar{v}}(1+s)^{-1} \left\| \bar{v}(s) - v_+ \right\|_{1,a}.
\end{aligned} \tag{2.64}
$$

From (2.62)–(2.64), for $0 < a < 1$, we have

$$
\begin{aligned}
L_2(t) &\leqslant C\delta_{0\bar{v}} \int_0^{t/2} (t-s)^{-a/2-1}(1+t-s)^{a/2}(1+s)^{-1/2}\left\|\bar{v}(s)-v_+\right\|_{1,a}\,ds \\
&\quad + C\delta_{0\bar{v}} \int_{t/2}^{t} (t-s)^{-a/2-1/2}(1+t-s)^{a/2}(1+s)^{-1}\left\|\bar{v}(s)-v_+\right\|_{1,a}\,ds \\
&\leqslant C\delta_{0\bar{v}} t^{-(a+2)/2}(1+t)^{a/2} \int_0^{t/2} (1+s)^{-1/2-(1-a)/2}\,ds \times \sup_{s\geqslant 0}\left\{(1+s)^{(1-a)/2}\left\|\bar{v}(s)-v_+\right\|_{1,a}\right\} \\
&\quad + C\delta_{0\bar{v}}(1+t)^{-(3-a)/2}(1+t)^{a/2} \int_{t/2}^{t} (t-s)^{-(1+a)/2}\,ds \times \sup_{s\geqslant 0}\left\{(1+s)^{(1-a)/2}\left\|\bar{v}(s)-v_+\right\|_{1,a}\right\} \\
&\leqslant C\delta_{0\bar{v}}(1+t)^{-(2-a)/2} \sup_{s\geqslant 0}\left\{(1+s)^{(1-a)/2}\left\|\bar{v}(s)-v_+\right\|_{1,a}\right\}. \qquad (2.65)
\end{aligned}
$$

When $t < 2$, for $a \in (0,1)$, we write $L_2(t)$ as

$$
\begin{aligned}
L_2(t) &= \int_0^t \left(\sqrt{4\beta(t-s)}\right)^{-a} \left\{ \int_0^\infty y^a \left|\partial_y H\left(\bar{v}(y,s)-v_+\right)\right| dy \times \int_0^\infty (1+x)^a \frac{|\partial_x K(x,y;t-s)|}{(\frac{y}{\sqrt{4\beta(t-s)}})^a}\,dx \right\} ds \\
&\leqslant C\delta_{0\bar{v}} \int_0^t (t-s)^{-(a+1)/2}(1+t-s)^{a/2}(1+s)^{-(3-a)/2} \times \sup_{s\geqslant 0}\left\{(1+s)^{(1-a)/2}\left\|\bar{v}(s)-v_+\right\|_{1,a}\right\} \\
&\leqslant Ct^{(1-a)/2} \sup_{s\geqslant 0}\left\{(1+s)^{(1-a)/2}\left\|\bar{v}(s)-v_+\right\|_{1,a}\right\} \\
&\leqslant C(1+t)^{-(2-a)/2} \sup_{s\geqslant 0}\left\{(1+s)^{(1-a)/2}\left\|\bar{v}(s)-v_+\right\|_{1,a}\right\}. \qquad (2.66)
\end{aligned}
$$

Then, (2.65) and (2.66) give that

$$
L_2(t) \leqslant C\delta_{0\bar{v}}(1+t)^{-(2-a)/2} \sup_{s\geqslant 0}\left\{(1+s)^{(1-a)/2}\left\|\bar{v}(s)-v_+\right\|_{1,a}\right\}, \quad t \geqslant 0. \qquad (2.67)
$$

Substituting (2.61) and (2.67) into (2.54) yields

$$
\begin{aligned}
\left\|\bar{v}(t)-v_+\right\|_{1,a} &\leqslant (1+t)^{-\frac{1-a}{2}}\|\bar{v}_0-v_+\|_{1,1} \\
&\quad + C\delta_{0\bar{v}}(1+t)^{-(2-a)/2} \sup_{s\geqslant 0}\left\{(1+s)^{(1-a)/2}\left\|\bar{v}(s)-v_+\right\|_{1,a}\right\}, \quad t \geqslant 0, \qquad (2.68)
\end{aligned}
$$

which implies

$$\sup_{t\geqslant 0}(1+t)^{\frac{1-a}{2}}\left\|\bar v(t)-v_+\right\|_{1,a}\leqslant C\|\bar v_0-v_+\|_{1,1}$$
$$+C\delta_{0\bar v}\sup_{t\geqslant 0}\left\{(1+t)^{(1-a)/2}\left\|\bar v(t)-v_+\right\|_{1,a}\right\}. \tag{2.69}$$

Choose $\delta_{0\bar v}$ small enough, we can get

$$\sup_{t\geqslant 0}(1+t)^{\frac{1-a}{2}}\left\|\bar v(t)-v_+\right\|_{1,a}\leqslant C\|\bar v_0-v_+\|_{1,1},$$

that is,

$$\left\|\bar v(t)-v_+\right\|_{1,a}\leqslant C(1+t)^{-\frac{1-a}{2}}\|\bar v_0-v_+\|_{1,1}.$$

The cases for $j=0$, $1\leqslant k\leqslant l-1$ can be proved similarly. Thus we omit the details. For the case $j=1$, $k+2j\leqslant l-1$, noticing that $\bar v_t=O(1)p(\bar v)_{xx}$, using the results obtained for the case $j=0$, we have

$$\left\|\partial_t\big(\bar v(t)-v_+\big)\right\|_{1,a}\leqslant C(1+t)^{-1/2-(2-a)/2}.$$

Hence, we finish the proof of Lemma 2.4. □

2.3. *Proof of Theorem 2.2*

The system (2.11) and the fact that $(v-\bar v-\hat v)|_{x=0}=0$ give the following boundary conditions for the lower and higher order derivatives:

$$V(0,t)=V_{xx}(0,t)=V_t(0,t)=V_{txx}=0,\quad \text{etc.} \tag{2.70}$$

The inequalities (2.17)–(2.19) can be proved similarly as in [21,26], so we omit the details. In what follows, we will try to prove inequality (2.20).

From (2.13), $V(x,t)$ satisfies the following initial boundary value problem,

$$\begin{cases} V_t-\beta V_{xx}=-\dfrac{1}{\alpha}[V_{tt}+F_1+F_3],\\ (V,V_t)|_{t=0}=\big(0,U_0(x)\big),\\ V(0,t)=0,\end{cases}\qquad (x,t)\in\mathbb{R}^+\times\mathbb{R}^+, \tag{2.71}$$

where $\beta=\frac{-p'(v_+)}{\alpha}$. Using the Green function of heat equation, we can rewrite (2.71) in the integral form of

$$V(x,t)=-\frac{1}{\alpha}\int_0^t\int_0^\infty K(x,y;t-s)(V_{ss}+F_1+F_3)(y,s)\,dy\,ds, \tag{2.72}$$

where $K(x,y;t)=G(x-y,t)-G(x+y,t)$ as defined before. Note that

$$|F_1+F_3|=O\big(|\bar v_t|+|\hat v|+|V_{xx}|^2+|\bar v-v_+||V_{xx}|\big). \tag{2.73}$$

As in [24], since

$$\int_0^{\frac{t}{2}}\int_0^{\infty} K(x,y;t-s)(-V_{ss})(y,s)\,dy\,ds$$

$$=\int_0^{\frac{t}{2}}\left(-\frac{d}{ds}\int_0^{\infty} K(x,y;t-s)V_s(y,s)\,dy+\int_0^{\infty} K_s(x,y;t-s)V_s(y,s)\,dy\right)ds$$

$$=\int_0^{\infty} K(x,y;t)U_0(y)\,dy-\int_0^{\infty} K\left(x,y;\frac{t}{2}\right)V_t\left(y,\frac{t}{2}\right)dy-\int_0^{\frac{t}{2}}\int_0^{\infty} K_t(x,y;t-s)V_s(y,s)\,dy\,ds,$$

then we have the following expression for V,

$$\begin{aligned}
V(x,t) &= \frac{1}{\alpha}\Biggl\{\int_0^{\infty} K(x,y;t)U_0(y)\,dy-\int_0^{\infty} K\left(x,y;\frac{t}{2}\right)V_t\left(y,\frac{t}{2}\right)dy\\
&\quad-\int_0^{\frac{t}{2}}\int_0^{\infty} K_t(x,y;t-s)V_s(y,s)\,dy\,ds-\int_{\frac{t}{2}}^{t}\int_0^{\infty} K(x,y;t-s)V_{ss}(y,s)\,dy\,ds\\
&\quad-\int_0^{t}\int_0^{\infty} K(x,y;t-s)(F_1+F_3)(y,s)\,dy\,ds\Biggr\}\\
&=: \frac{1}{\alpha}\{I_0+I_1+I_2+I_3+I_4\},
\end{aligned}\tag{2.74}$$

with $I_4=I_{41}+I_{42}+I_{43}+I_{44}+I_{45}$ from (2.73).

From (2.17), we have

$$\left\|\partial_x^k\partial_t^j V(t)\right\| \leqslant C(1+t)^{-\frac{k+2j}{2}}, \quad t\geqslant 0,\ j=0,1,2,3,\ 0\leqslant k\leqslant 5-j. \tag{2.75}$$

Using the estimates, for $t\geqslant 2$, we can estimates I_0, I_1, I_2, I_3 as follows,

$$\begin{aligned}
\|I_0\|_{L^\infty} &\leqslant \left\|K(x,\cdot;t)\right\|_{L^\infty}\|U_0\|_{L^1}\leqslant C\|U_0\|_{L^1}(1+t)^{-1/2};\\
\|I_1\|_{L^\infty} &\leqslant \left\|K(x,\cdot;t/2)\right\|_{L^2}\left\|V_t(t/2)\right\|_{L^2}\\
&\leqslant C(1+t)^{-1/4}(1+t)^{-1}\leqslant (1+t)^{-5/4};\\
\|I_2\|_{L^\infty} &\leqslant \int_0^{t/2}\left\|K_t(x,\cdot;t-s)\right\|_{L^2}\left\|V_s(s)\right\|_{L^2}ds\\
&\leqslant Ct^{-5/4}\int_0^{\frac{t}{2}}(1+s)^{-1}\,ds\leqslant C(1+t)^{-1};
\end{aligned}$$

$$\|I_3\|_{L^\infty} \leqslant \int_{t/2}^{t} \big\|K(x,\cdot;t-s)\big\|_{L^2} \big\|V_{ss}(s)\big\|_{L^2}\,ds$$

$$\leqslant C(1+t)^{-2}\int_{t/2}^{t}(t-s)^{-1/2}\,ds \leqslant (1+t)^{-7/4}. \tag{2.76}$$

Now we estimate I_4 term by term. From (2.75), it holds that

$$\begin{aligned}
\|I_{41}\|_{L^\infty} &\leqslant C\int_0^t \big\|K(x,\cdot;t-s)\big\|_{L^\infty}\big\|\hat{v}(s)\big\|_{L^1}\,ds \\
&\leqslant Ct^{-1/2}\int_0^{t/2} e^{-\alpha s}\,ds + Ce^{-\alpha t/2}\int_{t/2}^{t}(t-s)^{-1/2}\,ds \\
&\leqslant C(1+t)^{-1/2}; \\
\|I_{42}\|_{L^\infty} &\leqslant C\int_0^t \big\|K(x,\cdot;t-s)\big\|_{L^\infty}\big\|V_{xx}(s)\big\|^2\,ds \\
&\leqslant Ct^{-1/2}\int_0^{t/2}(1+s)^{-2}\,ds + C(1+t)^{-2}\int_{t/2}^{t}(t-s)^{-1/2}\,ds \\
&\leqslant C(1+t)^{-1/2}.
\end{aligned} \tag{2.77}$$

Notice that $v_0 - v_+ \in L^{1,\gamma}(\mathbb{R}^+)$ implies that $\bar{v}_0 - v_+ \in L^{1,\gamma}(\mathbb{R}^+)$. Using (2.16), we have

$$\begin{aligned}
\|I_{44}\|_{L^\infty} &\leqslant C\int_0^t \big\|K(x,\cdot;t-s)\big\|_{L^\infty}\big\|(\bar{v}-v_+)V_{xx}(s)\big\|^2\,ds \\
&\leqslant Ct^{-1/2}\int_0^{t/2}(1+s)^{-1/4-1}\,ds + C(1+t)^{-1/4-1}\int_{t/2}^{t}(t-s)^{-1/2}\,ds \\
&\leqslant C(1+t)^{-1/2};
\end{aligned} \tag{2.78}$$

$$\begin{aligned}
\|I_{45}\|_{L^\infty} &\leqslant C\int_0^t \big\|K(x,\cdot;t-s)\big\|_{L^\infty}\|\bar{v}_s\|_{L^1}\,ds \\
&\leqslant Ct^{-1/2}\int_0^{t/2}(1+s)^{-3/2}\,ds + C(1+t)^{-\frac{3}{2}}\int_{t/2}^{t}(t-s)^{-1/2}\,ds
\end{aligned}$$

$$\leqslant Ct^{-1/2}\int_0^{t/2}(1+s)^{-3/2}\,ds+C(1+t)^{-1}$$
$$\leqslant C(1+t)^{-1/2}. \tag{2.79}$$

Substituting (2.76)–(2.79) into (2.74) gives

$$\big\|V(t)\big\|_{L^\infty}\leqslant C(1+t)^{-1/2},\quad t\geqslant 2. \tag{2.80}$$

Similarly, we can get

$$\big\|V(t)\big\|_{L^1}\leqslant C,\quad t\geqslant 2. \tag{2.81}$$

Thus for any $1\leqslant p\leqslant\infty$, we have

$$\big\|V(t)\big\|_{L^p}\leqslant C(1+t)^{-(1-1/p)/2},\quad t\geqslant 2. \tag{2.82}$$

The high order terms can also easily estimated. For example,

$$\begin{aligned}\big\|\partial_x^2 I_{45}\big\|_{L^\infty}&\leqslant\int_0^{t/2}\big\|\partial_x^2K(x,\cdot;t-s)\big\|_{L^\infty}\|\bar v_s\|_{L^1}\,ds+\int_{t/2}^{t}\big\|K(x,\cdot;t-s)\big\|_{L^\infty}\big\|\partial_x^2\bar v_s\big\|_{L^1}\,ds\\&\leqslant Ct^{-3/2}\int_0^{t/2}(1+s)^{-3/2}\,ds+C(1+t)^{-2}\int_{t/2}^{t}(t-s)^{-1/2}\,ds\\&\leqslant Ct^{-3/2}+C(1+t)^{-3/2}\\&\leqslant C(1+t)^{-3/2}.\end{aligned} \tag{2.83}$$

L^1-estimates of $\partial_x^2 I_{45}$ and L^1 and L^∞-estimates of the other higher order in x estimates are similar as above. Then using interpolation inequality, we can obtain $(2.20)_1$ for $t\geqslant 2$. On the other hand, from (2.17), we can easily get that

$$\big\|\partial_x^k V(t)\big\|_{L^2\cap L^\infty}\leqslant C\lambda_3,\quad 0\leqslant t\leqslant 2,\ k=0,1,2.$$

Thus, we complete the proof of $(2.20)_1$.

To get $(2.20)_2$, we differentiate (2.74) in t,

$$\begin{aligned}V_t(x,t)=\frac{1}{\alpha}\Bigg\{&\int_0^\infty K_t(x,y;t)U_0(y)\,dy-\int_0^\infty K_t\left(x,y;\frac{t}{2}\right)V_t\left(y,\frac{t}{2}\right)dy\\&-\int_0^\infty K\left(x,y;\frac{t}{2}\right)\left(V_{tt}\left(y,\frac{t}{2}\right)+(F_1+F_3)\left(y,\frac{t}{2}\right)\right)dy\\&-\int_0^{\frac{t}{2}}\int_0^\infty K_{tt}(x,y;t-s)V_s(y,s)\,dy\,ds\end{aligned}$$

$$
\begin{aligned}
&- \int_{\frac{t}{2}}^{t} \int_0^{\infty} K(x, y; t-s) V_{sss}(y, s)\, dy\, ds \\
&- \int_0^{t/2} \int_0^{\infty} K_t(x, y; t-s)(F_1 + F_3)(y, s)\, dy\, ds \\
&- \int_{t/2}^{t} \int_0^{\infty} K(x, y; t-s) \partial_s (F_1 + F_3)(y, s)\, dy\, ds \Bigg\}.
\end{aligned} \tag{2.84}
$$

Here we have used integration by parts in s. Notice that $U = V_t$. Taking the similar estimates as $(2.20)_1$ above, we then get $(2.20)_2$. We omit the details for brevity.

2.4. *Proof of Theorem 2.3*

From (2.12), $V(x,t)$ satisfies

$$
\begin{cases}
V_{tt} + \big(p'(\bar v) V_x\big)_x + \alpha V_t = -F_1 - F_2, \\
(V, V_t)|_{t=0} = \big(0, U_0(x)\big), \\
V(0,t) = 0,
\end{cases}
\qquad (x,t) \in \mathbb{R}^+ \times \mathbb{R}^+. \tag{2.85}
$$

Based on the decay rates obtained in Theorem 2.2. For any $a \in (0, \frac{1}{2})$, we first have the following energy estimates on the $L^{2,a}$-norm of (V, V_x, V_t).

Lemma 2.5. *It holds that*

$$
\begin{aligned}
&\big\| (V, V_x, V_t)(t) \big\|_{2,a}^2 + \int_0^t \big\| (V_x, V_s)(s) \big\|_{2,a}^2 ds \\
&\quad \leqslant C \left(1 + \sum_{k=0}^{2} \big\| \partial_x^k U_0 \big\|_{2,a}^2 \right) + C \int_0^t \big(\|V(s)\|_{L^\infty} + \|V_s(s)\|_{L^\infty} \big) \|V_{xx}(s)\|_{2,a}^2 ds.
\end{aligned} \tag{2.86}
$$

Proof. Multiplying $(2.85)_1$ by $\frac{\alpha}{2} V + V_t$, we have

$$
\big\{E_1(V, V_x, V_t)\big\}_t + E_2(V_x, V_t) + \big\{E_3(V, V_x, V_t)\big\}_x = -(F_1 + F_2)\left(\frac{\alpha}{2} V + V_t\right), \tag{2.87}
$$

where

$$
\begin{aligned}
E_1(V, V_x, V_t) &:= \frac{1}{2} V_t^2 + \frac{\alpha}{2} V V_t + \frac{\alpha^2}{4} V^2 - \frac{1}{2} p'(\bar v) V_x^2, \\
E_2(V_x, V_t) &:= \frac{\alpha}{2} V_t^2 + \left[-\frac{\alpha}{2} p'(\bar v) + \frac{1}{2} p''(\bar v) \bar v_t \right] V_x^2, \\
E_3(V, V_x, V_t) &:= p'(\bar v) V_x \left(\frac{\alpha}{2} V + V_t \right).
\end{aligned} \tag{2.88}
$$

Notice that $-p'(\bar v) \geqslant C_0$ for some positive constant C_0, when $\delta_{0\bar v} \ll 1$, the following estimates hold

$$C_1\big(V^2+V_x^2+V_t^2\big)\leqslant E_1(V,V_x,V_t)\leqslant C_2\big(V^2+V_x^2+V_t^2\big),$$
$$C_3\big(V_x^2+V_t^2\big)\leqslant E_2(V_x,V_t),\tag{2.89}$$

for some positive constants C_i $(i=1,2,3)$. Multiplying (2.87) by $(1+x)^a$ and then integrating it over $\mathbb{R}^+\times[0,t]$ with respect to x and t, we have

$$\begin{aligned}
&\big\|(V,V_x,V_t)(t)\big\|_{2,a}^2+\int_0^t\big\|(V_x,V_s)(s)\big\|_{2,a}^2\,ds\\
&\leqslant C\|U_0\|_{2,a}^2+\int_0^t\int_0^\infty a(1+x)^{a-1}E_3(V,V_x,V_s)(x,s)\,dx\,ds\\
&\quad+\int_0^t\int_0^\infty(1+x)^a(F_1+F_2)(x,s)\bigg(\frac{\alpha}{2}|V|+|V_s|\bigg)(x,s)\,dx\,ds.
\end{aligned}\tag{2.90}$$

For $a\in(0,\frac{1}{2})$, we have $\|(1+x)^{a-1}\|\leqslant C$. From (2.20), for $t\geqslant 2$, we have $\|\partial_t^jV(s)\|_{L^\infty}\leqslant\sqrt{2}\|\partial_t^jV(s)\|^{1/2}\|\partial_t^jV_x(s)\|^{1/2}\leqslant C(1+s)^{-1/2-j}$, $j=0,1$, and $\|V_x(s)\|\leqslant C(1+s)^{-3/4}$, we can estimates the second term on the R.H.S. of (2.90) as

$$\begin{aligned}
\int_0^t\int_0^\infty a(1+x)^{a-1}E_3(V,V_x,V_s)(x,s)\,dx\,ds&\leqslant a\int_0^t\big\|(1+x)^{a-1}\big\|\big\|(V,V_s)(s)\big\|_{L^\infty}\big\|V_x(s)\big\|\,ds\\
&\leqslant C\int_2^t\big((1+s)^{-1/2}+(1+s)^{-3/2}\big)(1+s)^{-3/4}\,ds\\
&\quad+C\int_0^2\big((1+s)^{-1/4}+(1+s)^{-5/4}\big)(1+s)^{-1/2}\,ds\\
&\leqslant C.
\end{aligned}\tag{2.91}$$

For $t<2$, we can easily get that

$$\int_0^t\int_0^\infty a(1+x)^{a-1}E_3(V,V_x,V_s)(x,s)\,dx\,ds\leqslant C.$$

Similarly, for the third term on the R.H.S. of (2.90), it holds that

$$\begin{aligned}
&\int_0^t\int_0^\infty(1+x)^a(F_1+F_2)(x,s)\bigg(\frac{\alpha}{2}|V|+|V_s|\bigg)(x,s)\,dx\,ds\\
&\leqslant C\int_0^t\int_0^\infty(1+x)^a\big(|V|+|V_s|\big)(x,s)\big(|\bar v_s|+|\hat v|+|V_{xx}|^2+|\bar v_xV_x|\big)(x,s)\,dx\,ds
\end{aligned}$$

$$
\begin{aligned}
&\leqslant C\int_0^t \big\|V(s)\big\|_{L^\infty}\big(\|\bar v_s\|_{1,a}+\|\hat v\|_{1,a}+\|\bar v_x\|_{1,a}\|V_x\|_{L^\infty}\big)(s)\,ds\\
&\quad+C\int_0^t \big\|V_s(s)\big\|_{L^\infty}\big(\|\bar v_s\|_{1,a}+\|\hat v\|_{1,a}+\|\bar v_x\|_{1,a}\|V_x\|_{L^\infty}\big)(s)\,ds\\
&\quad+C\int_0^t \big\{\big\|V(s)\big\|_{L^\infty}+\big\|V_s(s)\big\|_{L^\infty}\big\}\big\|V_{xx}(s)\big\|_{2,a}^2\,ds\\
&\leqslant C\int_0^t (1+s)^{-\frac14}\big\{(1+s)^{-\frac{3-a}{2}}+e^{-\alpha t}+(1+s)^{-\frac{2-a}{2}-\frac34}\big\}\,ds\\
&\quad+C\int_0^t (1+s)^{-\frac54}\big\{(1+s)^{-\frac{3-a}{2}}+e^{-\alpha t}+(1+s)^{-\frac{2-a}{2}-\frac34}\big\}\,ds\\
&\quad+C\int_0^t \big\{\big\|V(s)\big\|_{L^\infty}+\big\|V_s(s)\big\|_{L^\infty}\big\}\big\|V_{xx}(s)\big\|_{2,a}^2(s)\,ds\\
&\leqslant C+C\int_0^t \big(\big\|V(s)\big\|_{L^\infty}+\big\|V_s(s)\big\|_{L^\infty}\big)\big\|V_{xx}(s)\big\|_{2,a}^2\,ds. \qquad (2.92)
\end{aligned}
$$

Substituting (2.91) and (2.92) into (2.90) yields

$$
\begin{aligned}
&\big\|(V,V_x,V_t)(t)\big\|_{2,a}^2+\int_0^t \big\|(V_x,V_s)(s)\big\|_{2,a}^2\,ds\\
&\quad\leqslant C\big(\|U_0\|_{2,a}^2+1\big)+C\int_0^t \big(\big\|V(s)\big\|_{L^\infty}+\big\|V_s(s)\big\|_{L^\infty}\big)\big\|V_{xx}(s)\big\|_{2,a}^2\,ds. \qquad (2.93)
\end{aligned}
$$

The proof is complete. □

Lemma 2.6. *It holds that*

$$
\begin{aligned}
&(1+t)\big\|(V_x,V_t)(t)\big\|_{2,a}^2+\int_0^t (1+s)\big\|V_s(s)\big\|_{2,a}^2\,ds\\
&\quad\leqslant C\big(\|U_0\|_{2,a}^2+1\big)+C\int_0^t \big(\big\|V(s)\big\|_{L^\infty}+(1+s)\big\|V_s(s)\big\|_{L^\infty}\big)\big\|V_{xx}(s)\big\|_{2,a}^2\,ds. \qquad (2.94)
\end{aligned}
$$

Proof. Multiplying (2.86) by $(1+t)(1+x)^a V_t$ and then integrating the resultant equation over $\mathbb{R}^+ \times [0,t]$, we have

$$\int_0^t \int_0^\infty (1+x)^a (1+s) \frac{\partial}{\partial_s}\left(\frac{V_s^2}{2}\right) dx\,ds + \alpha \int_0^t (1+s) \left\| V_s(s) \right\|_{2,a}^2 ds$$
$$+ \int_0^t \int_0^\infty (1+x)^a (1+s) \big(V_s p'(\bar{v}) V_x\big)_x \, dx\,ds$$
$$= \int_0^t \int_0^\infty (1+x)^a (1+s) V_s (-F_1 - F_2) \, dx\,ds. \tag{2.95}$$

Integrating by parts, we can get

$$\int_0^t \int_0^\infty (1+x)^a (1+s) \frac{\partial}{\partial_s}\left(\frac{V_s^2}{2}\right) dx\,ds$$
$$= \int_0^t \int_0^\infty (1+x)^a \frac{\partial}{\partial_s}\left((1+s)\frac{V_s^2}{2}\right) dx\,ds - \frac{1}{2}\int_0^t \int_0^\infty (1+x)^a V_s^2 \, dx\,ds$$
$$= \frac{(1+t)}{2} \left\| V_t(t) \right\|_{2,a}^2 - \frac{1}{2}\|U_0\|_{2,a}^2 - \frac{1}{2}\int_0^t \left\| V_s(s) \right\|_{2,a}^2 ds, \tag{2.96}$$

and

$$\int_0^t \int_0^\infty (1+x)^a (1+s) V_s \big(p'(\bar{v}) V_x\big)_x \, dx\,ds = -\int_0^t \int_0^\infty (1+x)^a (1+s) V_{xs} p'(\bar{v}) V_x \, dx\,ds$$
$$- \int_0^t \int_0^\infty a(1+x)^{a-1} (1+s) V_s p'(\bar{v}) V_x \, dx\,ds$$
$$=: J_1 - J_2. \tag{2.97}$$

Using Theorem 2.1, we have

$$J_1 = -\int_0^t \int_0^\infty (1+x)^a \frac{\partial}{\partial_s}\left\{(1+s) p'(\bar{v}) \frac{V_x^2}{2}\right\} dx\,ds$$
$$+ \int_0^t \int_0^\infty (1+x)^a (1+s) \big[p''(\bar{v}) \bar{v}_s\big] \frac{V_x^2}{2} \, dx\,ds + \int_0^t \int_0^\infty (1+x)^a p'(\bar{v}) \frac{V_x^2}{2} \, dx\,ds$$

$$\geqslant \frac{-p'(\bar{v})}{2}(1+t)\big\|V_x(t)\big\|_{2,a}^2 - C\int_0^t \big\{(1+s)\big\|\bar{v}_s(s)\big\|_{L^1}+1\big\}\big\|V_x(s)\big\|_{2,a}^2\,ds$$

$$\geqslant \frac{-p'(\bar{v})}{2}(1+t)\big\|V_x(t)\big\|_{2,a}^2 - C\int_0^t \big\|V_x(s)\big\|_{2,a}^2\,ds. \tag{2.98}$$

For $t \leqslant 2$, it easy to get $J_2 > C$. For $t \geqslant 2$,

$$J_2 \leqslant a\int_0^t (1+s)\big\|(1+x)^{a-1}\big\|\|V_s\|\|V_x\|_{L^\infty}\,dx\,ds$$

$$\leqslant C\int_2^t (1+s)(1+s)^{-\frac{5}{4}}(1+s)^{-1}\,ds + a\int_0^2 (1+s)\big\|(1+x)^{a-1}\big\|\|V_s\|\|V_x\|_{L^\infty}\,dx\,ds \leqslant C. \tag{2.99}$$

For the term on the L.H.S. of (2.95), it holds that

$$\int_0^t\int_0^\infty (1+x)^a(1+s)V_s(x,s)(-F_1-F_2)(x,s)\,dx\,ds$$

$$\leqslant \int_0^t\int_0^\infty (1+x)^a(1+s)\big|(F_1+F_2)(x,s)\big|\big|V_s(x,s)\big|\,dx\,ds$$

$$\leqslant C\int_0^t\int_0^\infty (1+x)^a(1+s)\big|V_s(x,s)\big|\big(|\bar{v}_s|+|\hat{v}|+|V_{xx}|^2+|\bar{v}_xV_x|\big)(x,s)\,dx\,ds$$

$$\leqslant C\int_0^t (1+s)\big\|V_s(s)\big\|_{L^\infty}\big(\|\bar{v}_s\|_{1,a}+\|\hat{v}\|_{1,a}\big)(s)\,ds$$

$$+ C\int_0^t (1+s)\big\|V_s(s)\big\|_{L^\infty}\Big(\big\|V_{xx}(s)\big\|_{2,a}^2+\big\|\bar{v}_x(s)\big\|_{1,a}\big\|V_x(s)\big\|_{L^\infty}\Big)\,ds$$

$$\leqslant C\int_0^t (1+s)^{-\frac{1}{4}}\Big\{(1+s)^{-\frac{3-a}{2}}+e^{-\alpha t}+(1+s)^{-\frac{2-a}{2}-\frac{3}{4}}\Big\}\,ds$$

$$+ C\int_0^t (1+s)\big\|V_s(s)\big\|_{L^\infty}\big\|V_{xx}(s)\big\|_{2,a}^2\,ds$$

$$\leqslant C + C\int_0^t (1+s)\big\|V_s(s)\big\|_{L^\infty}\big\|V_{xx}(s)\big\|_{2,a}^2\,ds. \tag{2.100}$$

Substituting (2.96)–(2.100) into (2.95) and using (2.93), we can obtain (2.94). This completes the proof of the lemma. □

Lemma 2.7. *It holds that*

$$\left\|(V_x, V_{xx}, V_{xt})(t)\right\|_{2,a}^2 + \int_0^t \left\|(V_{xx}, V_{xs})(s)\right\|_{2,a}^2 ds \leqslant C\left(1 + \sum_{k=0}^{1} \left\|\partial_x^k U_0\right\|_{2,a}^2\right). \tag{2.101}$$

Proof. Differentiating (2.86) with respect to x and multiplying $(2.85)_1$ for $(1+x)^a(\frac{\alpha}{2}V_x + V_{xt})$, and integrating the resultant equation over $\mathbb{R}^+ \times [0,t]$ with respect to x and t, we have

$$\begin{aligned}
&\int_0^t \int_0^\infty \left(V_{xss} + \left(p'(\bar v)V_x\right)_{xx} + \alpha V_{xs}\right)(x,s)(1+x)^a\left(\frac{\alpha}{2}V_x + V_{xs}\right)(x,s)\,dx\,ds \\
&\quad = \int_0^t \int_0^\infty (-F_1 - F_2)(1+x)^a\left(\frac{\alpha}{2}V_x + V_{xs}\right) dx\,ds.
\end{aligned} \tag{2.102}$$

Integrating it by parts, we have

$$\begin{aligned}
&\int_0^t \int_0^\infty V_{xss}(1+x)^a\left(\frac{\alpha}{2}V_x + V_{xs}\right) dx\,ds \\
&\quad = \int_0^t \int_0^\infty (1+x)^a \frac{\partial}{\partial_s}\left\{\frac{\alpha}{2}V_xV_{xs} + \frac{V_{xs}^2}{2}\right\} dx\,ds - \frac{\alpha}{4}\int_0^t \int_0^\infty (1+x)^a V_{xs}^2\,dx\,ds \\
&\quad = \int_0^\infty (1+x)^a\left\{\frac{\alpha}{2}V_xV_{xt} + \frac{V_{xt}^2}{2}\right\} dx - \frac{1}{2}\|\partial_x U_0\|_{2,a}^2 - \frac{\alpha}{4}\int_0^t \left\|V_{xs}(s)\right\|_{2,a}^2 ds
\end{aligned} \tag{2.103}$$

and

$$\begin{aligned}
&\int_0^t \int_0^\infty \left(p'(\bar v)V_x\right)_{xx}(1+x)^a(\lambda V_x + V_{xs})\,dx\,ds \\
&\quad = \int_0^t \int_0^\infty \left\{\left(p'(\bar v)V_{xx}\right)_x + \left(p''(\bar v)\bar v_x V_x\right)_x\right\}(1+x)^a\left(\frac{\alpha}{2}V_x + V_{xs}\right) dx\,ds \\
&\quad = \int_0^t \int_0^\infty \left(-p'(\bar v)\right)V_{xx}(1+x)^a\frac{\alpha}{2}V_{xx}\,dx\,ds - \frac{a\alpha}{2}\int_0^t \int_0^\infty p'(\bar v)V_{xx}(1+x)^{a-1}V_x\,dx\,ds \\
&\qquad - \int_0^t \int_0^\infty p'(\bar v)V_{xx}(1+x)^a V_{xxs}\,dx\,ds - a\int_0^t \int_0^\infty p'(\bar v)V_{xx}(1+x)^{a-1}V_{xs}\,dx\,ds
\end{aligned}$$

$$
\begin{aligned}
&- \int_0^t \int_0^\infty \bigl\{-p'''(\bar v)\bar v_x \bar v_x V_x - p''(\bar v)\bar v_{xx} V_x - p''(\bar v)\bar v_x V_{xx}\bigr\}(1+x)^a \left(\frac{\alpha}{2} V_x + V_{xs}\right) dx\,ds \\
&=: J_3 - J_4 + J_5 - J_6 - J_7. \qquad (2.104)
\end{aligned}
$$

Now we estimate J_i $(i = 3, 4, 5, 6, 7)$. Note that $-p'(\bar v) \geqslant C_0$ for some positive constant C_0, we have

$$
J_3 \geqslant \frac{C_0 \alpha}{2} \int_0^t \bigl\| V_{xx}(s) \bigr\|_{2,a}^2 \, ds. \qquad (2.105)
$$

As showed in Lemma 2.6, we can similarly prove

$$
J_4 \leqslant C, \qquad J_6 \leqslant C, \qquad (2.106)
$$

and

$$
\begin{aligned}
J_5 &\geqslant \frac{C_0}{2} \bigl\| V_{xx}(t) \bigr\|_{2,a}^2 - \frac{1}{2} \int_0^t \int_0^\infty p''(\bar v)\bar v_t (1+x)^a V_{xx}^2 \, dx\,ds \geqslant \frac{C_0}{2} \bigl\| V_{xx}(s) \bigr\|_{2,a}^2 - C, \\
J_7 &\leqslant C \int_0^t \bigl(\|\bar v_x\|_{1,a} \|\bar v_x\|_{L^\infty} + \|\bar v_{xx}\|_{1,a}\bigr) \|V_x\|_{L^\infty} \left(\frac{\alpha}{2} \|V_x\|_{L^\infty} + \|V_{xs}\|_{L^\infty}\right) ds \\
&\quad + C \int_0^t \|\bar v_x\|_{1,a} \|V_{xx}\|_{L^\infty} \left(\frac{\alpha}{2} \|V_x\|_{L^\infty} + \|V_{xs}\|_{L^\infty}\right) ds \\
&\leqslant C \int_0^t \bigl\{(1+s)^{-\frac{2-a}{2} - \frac{3}{2} - \frac{3}{4}} + (1+s)^{-\frac{3-a}{2} - \frac{3}{4}}\bigr\} \bigl\{(1+s)^{-\frac{3}{4}} + (1+s)^{-2}\bigr\} \, ds \\
&\quad + C \int_0^t (1+s)^{-\frac{2-a}{2} - \frac{5}{4}} \bigl\{(1+s)^{-\frac{3}{4}} + (1+s)^{-\frac{7}{4}}\bigr\} \, ds \\
&\leqslant C. \qquad (2.107)
\end{aligned}
$$

Substituting (2.105)–(2.108) into (2.104), we have

$$
\begin{aligned}
&\int_0^t \int_0^\infty \bigl(p'(\bar v) V_x\bigr)_{xx} (1+x)^a \left(\frac{\alpha}{2} V_x + V_{xs}\right) dx\,ds \\
&\qquad \geqslant \frac{C_0 \alpha}{2} \int_0^t \bigl\| V_{xx}(s) \bigr\|_{2,a}^2 \, ds + \frac{c_0}{2} \bigl\| V_{xx}(t) \bigr\|_{2,a}^2 - C. \qquad (2.108)
\end{aligned}
$$

On the other hand, it holds that

$$\int_0^t \int_0^\infty \alpha V_{xs}(1+x)^a\left(\frac{\alpha}{2}V_x + V_{xs}\right)dx\,ds = \alpha\int_0^t \left\|V_{xs}(s)\right\|_{2,a}^2 ds + \frac{\alpha^2}{4}\left\|V_x(t)\right\|_{2,a}^2. \tag{2.109}$$

From (2.103)–(2.109), we have

$$\begin{aligned}
&\int_0^t \int_0^\infty \left[V_{xss} + \left(p'(\bar{v})V_x\right)_{xx} + \alpha V_{xs}\right](1+x)^a(\lambda V_x + V_{xs})\,dx\,ds \\
&\quad \geqslant c_1\left\|(V_x, V_{xx}, V_{xt})(t)\right\|_{2,a}^2 + c_2\int_0^t \left\|(V_{xx}, V_{xs})(s)\right\|_{2,a}^2 ds - C\left(1 + \|\partial_x U_0\|_{2,a}^2\right),
\end{aligned} \tag{2.110}$$

for some positive constants c_1, c_2. Similarly, as showed in (2.92), we have

$$\begin{aligned}
&\int_0^t \int_0^\infty (-F_1 - F_2)_x(1+x)^a(\lambda V_x + V_{xs})\,dx\,ds \\
&\quad \leqslant C + C\int_0^t \|V_{xx}\|_{2,a}^2\left(\frac{\alpha}{2}\|V_{xx}\|_{L^\infty} + \|V_{xxs}\|_{L^\infty}\right)ds \\
&\quad \leqslant C + C\left(\delta_0 + \delta_{0v} + \|U_0\|_3^2\right)\int_0^t (1+s)^{-3/2}\|V_{xx}\|_{2,a}^2\,ds.
\end{aligned} \tag{2.111}$$

Substituting (2.110) and (2.111) into (2.102) and choosing $\delta_0 + \delta_{0v} + \|U_0\|_3^2$ small enough, we can obtain (2.101). □

Similarly, multiplying (2.86) by $(1+t)(\frac{\alpha}{2}V_x + V_{xt})$ and using (2.87) and (2.101), we can further obtain the following energy estimate.

Lemma 2.8. *It holds that*

$$(1+t)\left\|(V_x, V_{xx}, V_{xt})(t)\right\|_{2,a}^2 + \int_0^t \left\|(1+s)(V_{xx}, V_{xs})(s)\right\|_{2,a}^2 ds \leqslant C\left(1 + \|\partial_x U_0\|_{2,a}^2\right). \tag{2.112}$$

Again, by taking $\int_0^t \int_0^\infty \partial_x(2.86)\cdot(1+s)^k V_{xs}\,dx\,ds$ $(k = 0, 1, 2)$, and using (2.94), (2.101) and (2.112), we can obtain the decay rate for $\|V_{xx}\|_{2,a}^2$ as follows.

Lemma 2.9. *It holds that*

$$(1+t)^2\left\|(V_{xx}, V_{xt})(t)\right\|_{2,a}^2 + \int_0^t (1+s)^2\left\|V_{xs}(s)\right\|_{2,a}^2 ds \leqslant C\left(1 + \|\partial_x U_0\|_{2,a}^2\right). \tag{2.113}$$

Combining (2.86), (2.94), (2.112) and (2.113) yields (2.21). This completes the proof of Theorem 2.3.

2.5. Proof of Theorem 2.4

Now we are going to prove Theorem 2.4. For the system (2.11), by substituting the first equation $U = V_t$ into the second one, we obtain

$$\begin{cases} V_{tt} + \alpha V_t + p'(v_+)V_{xx} = -F_1 - F_3, \\ (V,U)|_{t=0} = \big(0, U_0(x)\big), \qquad (x,t) \in \mathbb{R}^+ \times \mathbb{R}^+. \\ V(0,t) = 0, \end{cases} \tag{2.114}$$

As in [17], since $V(x,0) = 0$, the IBVP (2.114) can be rewritten equivalently in the integral form

$$\begin{aligned} V(x,t) = {} & \int_0^\infty \big[\bar{K}(x-y,t) - \bar{K}(x+y,t)\big]U_0(y)\,dy \\ & + \int_0^t \int_0^\infty \big[\bar{K}(x-y,t-\tau) - \bar{K}(x+y,t-\tau)\big](-F_1 - F_3)(y,\tau)\,dy\,d\tau, \end{aligned} \tag{2.115}$$

where $\bar{K}(x,t)$ is given by

$$\bar{K}(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ix\xi} R(\xi,t)\,d\xi,$$

and $R(x,t)$ is defined by

$$R(x,t) = \begin{cases} \frac{2e^{\frac{-\alpha t}{2}}}{\sqrt{\alpha^2 - 4\alpha\beta\xi^2}} \sinh(\frac{\sqrt{\alpha^2 - 4\alpha\beta\xi^2}}{2}t), & |\xi| < 2\sqrt{\frac{\alpha}{\beta}}, \\ te^{-\frac{\alpha t}{2}}, & |\xi| = 2\sqrt{\frac{\alpha}{\beta}}, \\ \frac{2e^{\frac{-\alpha t}{2}}}{\sqrt{4\alpha\beta\xi^2 - \alpha^2}} \sinh(\frac{\sqrt{4\alpha\beta\xi^2 - \alpha^2}}{2}t), & |\xi| > 2\sqrt{\frac{\alpha}{\beta}}, \end{cases}$$

with $\beta = \frac{-p'(v_+)}{\alpha}$.

For the kernel $\bar{K}(x,t)$, we have the following energy estimates, cf. [19,17,28].

Lemma 2.10. *Let $a \in [0,1]$. If $f \in L^{1,a}(\mathbb{R}^+) \cap H^{j+k}(\mathbb{R}^+)$ and $f(0) = 0$, then*

$$\begin{aligned} & \left\| \partial_t^j \partial_x^k \int_0^\infty \big[\bar{K}(x-y,t) - \bar{K}(x+y,t)\big] f(y)\,dy \right\| \\ & \quad \leqslant C(1+t)^{-j-\frac{2k+1}{4}-\frac{a}{2}} \big\{ \|f\|_{1,a} + \|f\|_{j+k-1} \big\}, \end{aligned} \tag{2.116}$$

for $0 \leqslant j+k \leqslant 4$.

We also state another auxiliary lemma which is useful to prove the decay rates, see [19].

Lemma 2.11. *If $r_1 > 1$ and $r_2 \in [0, r_1]$, then it holds that*

$$\int_0^t (1+t-s)^{-r_1}(1+s)^{-r_2}\,ds \leqslant C(1+t)^{-r_2}. \tag{2.117}$$

Now we are going to prove the optimal decay rates for $\|\partial_x^k V(t)\|$ $(k = 0, 1, 2)$.

Differentiating (2.115) k-times $(k = 0, 1, 2)$ with respect to x and taking its L^2-norm, noticing that $U_0(0) = (F_1 + F_3)|_{x=0} = 0$, we obtain

$$\begin{aligned}
\left\|\partial_x^k V(t)\right\| &= \left\|\partial_x^k \int_0^\infty \big[K(x-y,t) - K(x+y,t)\big]U_0(y)\,dy\right\| \\
&\quad + \int_0^t \left\|\partial_x^k \int_0^\infty \big[K(x-y,t-\tau) - K(x+y,t-\tau)\big](F_1+F_3)(y,\tau)\,dy\right\| d\tau \\
&\leqslant C(1+t)^{-\frac{2k+1}{4}-\frac{a}{2}}\big\{\|U_0\|_{1,a} + \|U_0\|_{k-1}\big\} \quad \text{[by Lemma 2.10]} \\
&\quad + C\int_0^t (1+t-\tau)^{-\frac{2k+1}{4}-\frac{a}{2}}\big\{\big\|(F_1+F_3)(\tau)\big\|_{L^{1,a}} + \big\|(F_1+F_3)(\tau)\big\|_{k-1}\big\}\,d\tau.
\end{aligned} \tag{2.118}$$

From (2.14), using (2.16), (2.20) and (2.21), the $L^{1,a}$-norm for $(F_1 + F_3)$ can be estimated as

$$\begin{aligned}
\big\|(F_1+F_3)(t)\big\|_{L^{1,a}} &\leqslant C\big(\|\hat{v}\|_{1,a} + \big\|V_{xx}^2\big\|_{1,a} + \big\|(\bar{v}-v_+)V_{xx}\big\|_{1,a} + \|\bar{v}_t\|_{1,a}\big) \\
&\leqslant C\big(\|\hat{v}\|_{1,a} + \|V_{xx}\|_{2,a}^2 + \|V_{xx}\|_{L^\infty}\big\|(\bar{v}-v_+)\big\|_{1,a} + \|\bar{v}_t\|_{1,a}\big) \\
&\leqslant C\big(e^{-\alpha t} + (1+t)^{-2} + (1+t)^{-(4-a)/2} + (1+t)^{-\frac{3-a}{2}}\big) \\
&\leqslant C(1+t)^{-\frac{3}{2}+\frac{a}{2}}.
\end{aligned} \tag{2.119}$$

Similarly, we can also prove

$$\big\|(F_1+F_3)(\tau)\big\|_{k-1} \leqslant C(1+t)^{-\frac{7}{4}}. \tag{2.120}$$

Choose $a > 0$ such that

$$\frac{3}{2} - \frac{a}{2} > 1 \quad \text{and} \quad \frac{2\times 2+1}{4} + \frac{a}{2} \leqslant \frac{3}{2} - \frac{a}{2} < \frac{7}{4},$$

which gives

$$0 < a \leqslant \frac{1}{4}. \tag{2.121}$$

From (2.118) and (2.119), using Lemma 2.11, we can get

$$\big\|\partial_x^k V(t)\big\| \leqslant C(1+t)^{-\frac{2k+1}{4}-\frac{a}{2}}, \quad k = 0, 1, 2. \tag{2.122}$$

This completes the proof of Theorem 2.4.

Remark 2.3. From the procedure of our proof, we can find that, the decay rates for the terms $\|\partial_x^k V(t)\|$ with $k \geqslant 3$ are the same as $(1+t)^{-\frac{3-a}{2}}$, and the decay rates for the terms $\|\partial_x^k \partial_t V(t)\|$ with $k \geqslant 1$ are also same as $(1+t)^{-\frac{3-a}{2}}$. Thus, using this method, we cannot get better decay rates for the terms $\|\partial_x^k V(t)\|$ with $k \geqslant 3$ and $\|\partial_x \partial_t V(t)\|$ with $k \geqslant 1$ than those obtained in Theorem 2.2.

2.6. *Proof of Corollary 2.1*

Thanks to Theorems 2.2 and 2.4, noticing that $V_{xx} = v - \bar{v} - \hat{v}$, $U_x = u - \bar{u} - \hat{u}$, and $|\partial_x^k \hat{v}(x,t)|$, $|\partial_x^k \hat{u}(x,t)|$ decay like $e^{-\alpha t}$, we have

$$\begin{aligned} \big\|(v-\bar{v}-\hat{v})(t)\big\| &= \big\|V_{xx}(t)\big\| \leqslant C(1+t)^{-\frac{5+2a}{4}}, \\ \big\|(u-\bar{u}-\hat{u})(t)\big\| &= \big\|U_x(t)\big\| \leqslant C(1+t)^{-\frac{7}{4}}, \end{aligned} \tag{2.123}$$

and

$$\begin{aligned} \big\|(v-\bar{v})(t)\big\|_{L^\infty} &= \big\|(V_{xx}+\hat{v})(t)\big\|_{L^\infty} \\ &\leqslant \big\|V_{xx}(t)\big\|_{L^\infty} + \big\|\hat{v}(t)\big\|_{L^\infty} \\ &\leqslant C\big\|V_{xx}(t)\big\|^{1/2}\big\|V_{xxx}(t)\big\|^{1/2} + Ce^{-\alpha t} \\ &\leqslant C(1+t)^{-\frac{5}{8}-\frac{a}{4}-\frac{7}{8}} + Ce^{-\alpha t} \\ &\leqslant C(1+t)^{-\frac{6+a}{4}}, \\ \big\|(u-\bar{u})(t)\big\|_{L^\infty} &\leqslant \big\|(U_x+\hat{u})(t)\big\|_{L^\infty} \\ &\leqslant \big\|U_x(t)\big\|_{L^\infty} + \big\|\hat{u}(t)\big\|_{L^\infty} \\ &\leqslant C(1+t)^{-2} + Ce^{-\alpha t} \\ &\leqslant C(1+t)^{-2}. \end{aligned} \tag{2.124}$$

This completes the proof of Corollary 2.1.

3. Initial boundary value problem (1.3)

In this section, we study the initial boundary problem (1.3). The main goal is to improve the previous stability of diffusion waves and show the best convergence rates. The best asymptotic profile for the CP of hyperbolic p-system with damping was studied by Mei [21] and Nishihara [24]. The main idea of arriving our goal is to change the IBVP to CP, then we can obtain the best convergence rates by making use of the known results in [21,24].

By an odd extension to $u(x,t), u_0(x)$ and an even extension to $v(x,t), v_0(x)$ in the above IBVP,

$$\tilde{u}(x,t) := \begin{cases} u(x,t), & x \geqslant 0, \\ -u(-x,t), & x < 0, \end{cases} \qquad \tilde{u}_0(x) := \begin{cases} u_0(x), & x \geqslant 0, \\ -u_0(-x), & x < 0, \end{cases} \tag{3.1}$$

and

$$\tilde{v}(x,t) := \begin{cases} v(x,t), & x \geqslant 0, \\ v(-x,t), & x < 0, \end{cases} \qquad \tilde{v}_0(x) := \begin{cases} v_0(x), & x \geqslant 0, \\ v_0(-x), & x < 0. \end{cases} \tag{3.2}$$

We consider its corresponding CP for $(\tilde v(x,t),\tilde u(x,t))$ with the initial data $(\tilde v_0(x),\tilde u_0(x))$ as follows

$$\begin{cases} \tilde v_t-\tilde u_x=0, \quad (x,t)\in\mathbb{R}\times\mathbb{R}^+,\\ \tilde u_t+p(\tilde v)_x=-\alpha\tilde u,\\ (\tilde v,\tilde u)(x,0)=(\tilde v_0,\tilde u_0)(x)\to(v_+,\pm u_+) \quad \text{as } x\to\pm\infty. \end{cases} \tag{3.3}$$

Claim. *If the solution of the system* (3.3) *uniquely exists, then the solution* $(\tilde v(x,t),\tilde u(x,t))$ *satisfies*: $\tilde v(x,t)$ *is an even function and* $\tilde u(x,t)$ *is an odd function.*

In fact, suppose that $(\tilde v_1(x,t),\tilde u_1(x,t))$ is the unique solution to the system (3.3). Set

$$\tilde u_2(x,t):=\begin{cases}\tilde u_1(x,t), & x\geqslant 0,\\ -\tilde u_1(-x,t), & x<0,\end{cases}\qquad \tilde v_2(x,t):=\begin{cases}\tilde v_1(x,t), & x\geqslant 0,\\ \tilde v_1(-x,t), & x<0.\end{cases} \tag{3.4}$$

Then we can easily check that $(\tilde v_2(x,t),\tilde u_2(x,t))$ is also the solution to the system (3.3). Since $\tilde v_2(x,t)$ is even, $\tilde u_2(x,t)$ is odd, by the uniqueness, we know that $(\tilde v_2(x,t),\tilde u_2(x,t))=(\tilde v_1(x,t),\tilde u_1(x,t))$.

As in [21], we construct function $m(x)$ as

$$m(x)=-u_++2u_+\int_{-\infty}^{x} m_0(y)\,dy, \quad x\in\mathbb{R},$$

where $m_0(x)$ is an even function and satisfies

$$m_0(x)\in C_0^\infty(\mathbb{R}), \quad \text{with} \int_{-\infty}^{\infty} m_0(x)\,dx=1.$$

Then, we have, $\forall x\in\mathbb{R}$,

$$\begin{aligned} m(x)+m(-x)&=-2u_++2u_+\left(\int_{-\infty}^{x} m_0(y)\,dy+\int_{-\infty}^{-x} m_0(y)\,dy\right)\\ &=-2u_++2u_+\left(\int_{-\infty}^{x} m_0(y)\,dy+\int_{x}^{\infty} m_0(-y)\,dy\right)\\ &=-2u_++2u_+\left(\int_{-\infty}^{x} m_0(y)\,dy+\int_{x}^{\infty} m_0(y)\,dy\right)\\ &=-2u_++2u_+=0. \end{aligned} \tag{3.5}$$

This means that $m(x)$ is an odd function. Let $(\tilde{\tilde v},\tilde{\tilde u})(x,t)$ be the solution of the CP

$$\begin{cases} \tilde{\tilde v}_t-\tilde{\tilde u}_x=0,\\ p(\tilde{\tilde v})_x=-\alpha\tilde{\tilde u}, \quad (x,t)\in\mathbb{R}\times\mathbb{R}^+,\\ \tilde{\tilde v}(x,0)=\tilde v_0(x)+\dfrac{2u_+}{\alpha}m_0(x)\to v_+ \quad \text{as } x\to\pm\infty. \end{cases} \tag{3.6}$$

As in [24], if $|\int_{-\infty}^{\infty}(\tilde{\tilde v}_0(x)-v_+)\,dx|$ suitably small, we can similarly prove that there exists a unique global solution $(\tilde{\tilde v},\tilde{\tilde u})(x,t)$ to (3.6) satisfying that $\tilde{\tilde v}(x,t)$ is even and $\tilde{\tilde u}(x,t)$ is odd.

The function $(\tilde{\hat{v}}, \tilde{\hat{u}})(x,t)$ is defined by

$$(\tilde{\hat{v}}, \tilde{\hat{u}})(x,t) = \left(-\frac{2u_+}{\alpha} m_0(x) e^{-\alpha t}, m(x) e^{-\alpha t}\right), \quad (x,t) \in \mathbb{R} \times \mathbb{R}^+.$$

Then $(\tilde{v} - \tilde{\bar{v}} - \tilde{\hat{v}})(x,t)$ is even, $(\tilde{u} - \tilde{\bar{u}} - \tilde{\hat{v}})(x,t)$ is odd. Define

$$\tilde{V}(x,t) := \int_{-\infty}^{x} \int_{-\infty}^{y} (\tilde{v} - \tilde{\bar{v}} - \tilde{\hat{v}})(z,t)\, dz\, dy,$$

$$\tilde{U}(x,t) := \int_{-\infty}^{x} (\tilde{u} - \tilde{\bar{u}} - \tilde{\hat{u}})(y,t)\, dy,$$

$$\tilde{V}_0(x) := \int_{-\infty}^{x} \int_{-\infty}^{y} \bigl(\tilde{v}_0(z) - \tilde{\bar{v}}_0(z) - \tilde{\hat{v}}(z,0)\bigr)\, dz\, dy,$$

$$\tilde{U}_0(x) := \int_{-\infty}^{x} \bigl(\tilde{u}_0(y) - \tilde{\bar{u}}(y,0) - \tilde{\hat{u}}(y,0)\bigr)\, dy,$$

namely

$$\tilde{V}_{xx} = \tilde{v} - \tilde{\bar{v}} - \tilde{\hat{v}}, \qquad \tilde{U}_x = \tilde{u} - \tilde{\bar{u}} - \tilde{\hat{u}}, \tag{3.7}$$

we can check that $\tilde{V}_0(x) = 0$ and $\tilde{U}(\infty, t) = 0$, then we can finally establish a new working system of equations

$$\begin{cases} \tilde{V}_t - \tilde{U} = 0, \\ \tilde{U}_t + p(\tilde{\bar{v}} + \tilde{\hat{v}} + \tilde{V}_{xx}) - p(\tilde{\bar{v}}) = -\alpha \tilde{U} + \dfrac{1}{\alpha} p(\tilde{\bar{v}})_t, \quad (x,t) \in \mathbb{R}^+ \times \mathbb{R}^+, \\ (\tilde{V}, \tilde{U})(x,0) = \bigl(0, \tilde{U}_0(x)\bigr). \end{cases} \tag{3.8}$$

Our present goal is to obtain a global solution $(\tilde{V}, \tilde{U})(x,t)$ to the system (3.8) and its behavior as $t \to \infty$, provided that $\tilde{U}_0$ is suitably small. To do this, it is necessary to know the behavior of $(\tilde{\bar{v}}, \tilde{\bar{u}})(x,t)$. The diffusion wave $(\tilde{\bar{v}}, \tilde{\bar{u}})(x,t)$ defined in (3.6) enjoys the following properties, cf. [24].

Lemma 3.1. *Let $l \geqslant 3$. Suppose $\tilde{\bar{v}}_0 - v_+ \in L^1(\mathbb{R}^+) \cap H^l(\mathbb{R}^+) \cap W^{l-1,1}(\mathbb{R}^+)$, let $\delta_{1v} = |\int_0^\infty (\tilde{\bar{v}}_0(x) - v_+)\, dx|$ be suitably small. Then there exists a unique (weak) solution $(\tilde{\bar{v}}, \tilde{\bar{u}})(x,t)$ to (3.6) satisfying the decay properties*

$$\begin{aligned} &\bigl\| \partial_t^j \partial_x^k (\tilde{\bar{v}} - v_+)(t) \bigr\|_{L^p} \leqslant C \delta_{1v} (1+t)^{-(1-1/p)/2-(k+2j)/2}, \\ &\quad t \geqslant 0,\ 1 \leqslant p \leqslant \infty,\ j,k \geqslant 0,\ 2j+k \leqslant l-1, \\ &\qquad \bigl\| \partial_x^k \tilde{\bar{u}}(t) \bigr\|_{L^p} \leqslant C \delta_{1v} (1+t)^{-(1-1/p)/2-(1+k)/2}, \\ &\qquad t \geqslant 0,\ 1 \leqslant p \leqslant \infty,\ 0 \leqslant k \leqslant l-2. \end{aligned} \tag{3.9}$$

Hence, as in [21,24], the same estimates for the solution $(\tilde{V}, \tilde{U})$ to (3.8) are obtained. Thus, we have the following theorem.

Theorem 3.1. *Let $l \geqslant 3$ and $\tilde{U}_0(x) \in H^{l-1}(\mathbb{R})$ and $\delta_{0l} = |u_+| + \delta_{1v} + \|\tilde{U}_0\|_{l-1}$ be suitably small. Then the global solution $(\tilde{V}, \tilde{U})(x,t)$ of (3.8) uniquely exists and satisfies*

$$\tilde{V}(x,t) \in C^k\big(0,\infty; H^{3-k}(\mathbb{R})\big), \quad k = 0,1,2,3, \qquad \tilde{U}(x,t) \in C^k\big(0,\infty; H^{2-k}(\mathbb{R})\big), \quad k = 0,1,2,$$

and

$$\begin{aligned} &\sum_{k=0}^{3}(1+t)^k \big\|\partial_x^k \tilde{V}(t)\big\|^2_{L^2(\mathbb{R})} + \sum_{k=0}^{2}(1+t)^{k+2}\big\|\partial_x^k \tilde{U}(t)\big\|^2_{L^2(\mathbb{R})} \\ &\qquad + \int_0^t \left\{ \sum_{k=0}^{3}(1+s)^{k-1}\big\|\partial_x^k \tilde{V}(s)\big\|^2_{L^2(\mathbb{R})} + \sum_{k=0}^{2}(1+s)^{k+1}\big\|\partial_x^k \tilde{U}(s)\big\|^2_{L^2(\mathbb{R})} \right\} ds \\ &\quad \leqslant C\delta_{0l}. \end{aligned} \tag{3.10}$$

Moreover, if $l = 4$ and $\tilde{U}_0(x) \in L^1(\mathbb{R})$, then for $2 \leqslant p \leqslant \infty$, the convergence rates can be further improved as follows

$$\begin{aligned} &\big\|\partial_x^k \tilde{V}(t)\big\|_{L^p} \leqslant C\big(\delta_{04} + \|\tilde{U}_0\|_{L^1}\big)(1+t)^{-(1-1/p)/2-k/2}\ln(2+t), \quad k = 0,1,2, \\ &\big\|\partial_x^k \tilde{U}(t)\big\|_{L^p} \leqslant C\big(\delta_{04} + \|\tilde{U}_0\|_{L^1}\big)(1+t)^{-(1-1/p)/2-(k+2)/2}\ln(2+t), \quad k = 0,1. \end{aligned} \tag{3.11}$$

From (3.7), for $2 \leqslant p \leqslant \infty$, we can immediately obtain that

$$\begin{aligned} &\big\|(\tilde{v} - \tilde{\bar{v}} - \tilde{\hat{v}})(t)\big\|_{L^p} \leqslant C(1+t)^{-(1-1/p)/2-1}\ln(2+t), \\ &\big\|(\tilde{u} - \tilde{\bar{u}} - \tilde{\hat{u}})(t)\big\|_{L^p} \leqslant C(1+t)^{-(1-1/p)/2-3/2}\ln(2+t). \end{aligned} \tag{3.12}$$

Let

$$\begin{aligned} &\bar{v}(x,t) = \tilde{\bar{v}}(x,t), \qquad \hat{v}(x,t) = \tilde{\hat{v}}(x,t), \\ &\bar{u}(x,t) = \tilde{\bar{u}}(x,t), \qquad \hat{u}(x,t) = \tilde{\hat{u}}(x,t), \end{aligned} \qquad x \in \mathbb{R}^+. \tag{3.13}$$

Notice that

$$\begin{aligned} &\big\|(v - \bar{v} - \hat{v})(t)\big\|_{L^p(\mathbb{R}^+)} \leqslant \big\|(\tilde{v} - \tilde{\bar{v}} - \tilde{\hat{v}})(t)\big\|_{L^p(\mathbb{R})}, \\ &\big\|(u - \bar{u} - \hat{u})(t)\big\|_{L^p(\mathbb{R}^+)} \leqslant \big\|(\tilde{u} - \tilde{\bar{u}} - \tilde{\hat{u}})(t)\big\|_{L^p(\mathbb{R})}, \end{aligned} \tag{3.14}$$

and for any nonnegative integer k, $|\partial_x^k \hat{v}(x,t)|, |\partial_x^k \hat{u}(x,t)| \sim O(1)e^{-\alpha t}$, we can easily get

$$\begin{aligned} \big\|(v - \bar{v} - \hat{v})(t)\big\|_{L^2(\mathbb{R}^+)} &\leqslant C(1+t)^{-\frac{5}{4}}\ln(2+t), \\ \|v - \bar{v}\|_{L^\infty(\mathbb{R}^+)} &\leqslant C(1+t)^{-3/2}\ln(2+t), \\ \big\|(u - \bar{u} - \hat{u})(t)\big\|_{L^2(\mathbb{R}^+)} &\leqslant C(1+t)^{-\frac{7}{4}}\ln(2+t), \\ \|u - \bar{u}\|_{L^\infty(\mathbb{R}^+)} &\leqslant C(1+t)^{-2}\ln(2+t). \end{aligned} \tag{3.15}$$

Remark. The reason why we have an additional term $\ln(2+t)$ is that, in this case, we can only obtain that $\|\bar{v}_t(t)\|_{L^1}$ behaves like $(1+t)^{-1}$ and cannot improve its decay rate since the boundary condition is Neumann boundary condition.

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